Second-order arithmetic, comprehension scheme and bar recursion

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Introduction

Bar recursion vs. System F

System F

- Girard (1971), Reynolds (1974)

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- Polymorphic types
- Datatypes can be encoded: $\mathbb{N} \equiv \forall X\,(X \to (X \to X) \to X)$
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Bar recursion

- Spector (1962)
- Simple types
- Primitive natural numbers
- Normalization:
  - dependent choice or Zorn's lemma
  - arbitrary functions on $\mathbb{N}$ + continuity
    $\rightsquigarrow$ CPO-model + computational adequacy

Realizability interpretations of $HA2$

- Second-order arithmetic ($HA2$):
  - Quantification on $\mathbb{N}$: $\forall n$
  - Quantification on $\mathcal{P}(\mathbb{N})$: $\forall X$
  - Induction
  - Comprehension: $\exists X \forall n (n \in X \Leftrightarrow A(n))$

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  - in system T + bar recursion (simply-typed)
    - Spector, Kohlenbach, Berardi-Bezem-Coquand, Berger-Oliva

$$\texttt{brec} \quad \Vdash \quad \text{countable choice}$$

$$\begin{array}{r} \text{countable choice} \\ + \text{ classical logic} \end{array} \quad \vdash \quad \text{comprehension}$$

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Parametric polymorphism:

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- Example: $\texttt{cons} : \forall\alpha\,(\alpha \rightarrow \texttt{List}\ \alpha \rightarrow \texttt{List}\ \alpha)$

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  $\forall\alpha\,(\alpha \to \alpha)$ is instantiated with itself
- **Termination proofs** are **impredicative**

Bar induction

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any $\varphi : \mathbb{N} \to A$ has a finite approximation $f \sqsubset \varphi$ in R

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Bar induction

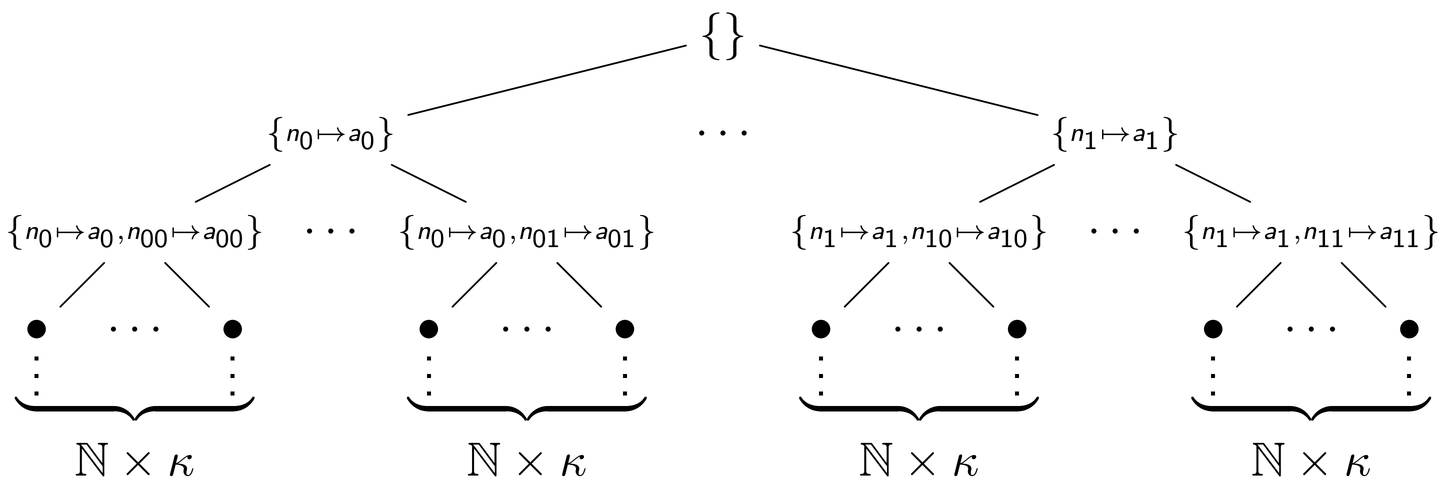
If P predicate on $\mathbb{N} \rightharpoonup A$ satisfies:

- base case:
  $P(f)$ for each $f \in R$
- inductive case:
  if $\forall n \in \mathbb{N}, \forall x \in A, P(f \cup \{n \mapsto x\})$, then $P(f)$

then $P(\{\})$

Bar induction

Complete $\mathbb{N} \times \kappa$-branching tree:



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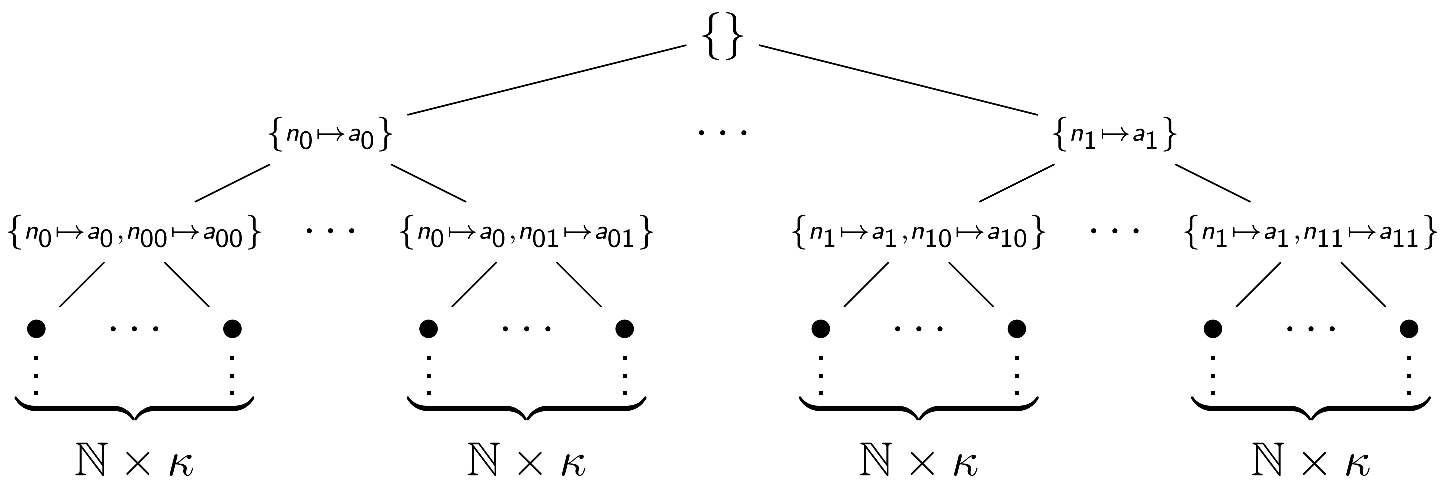

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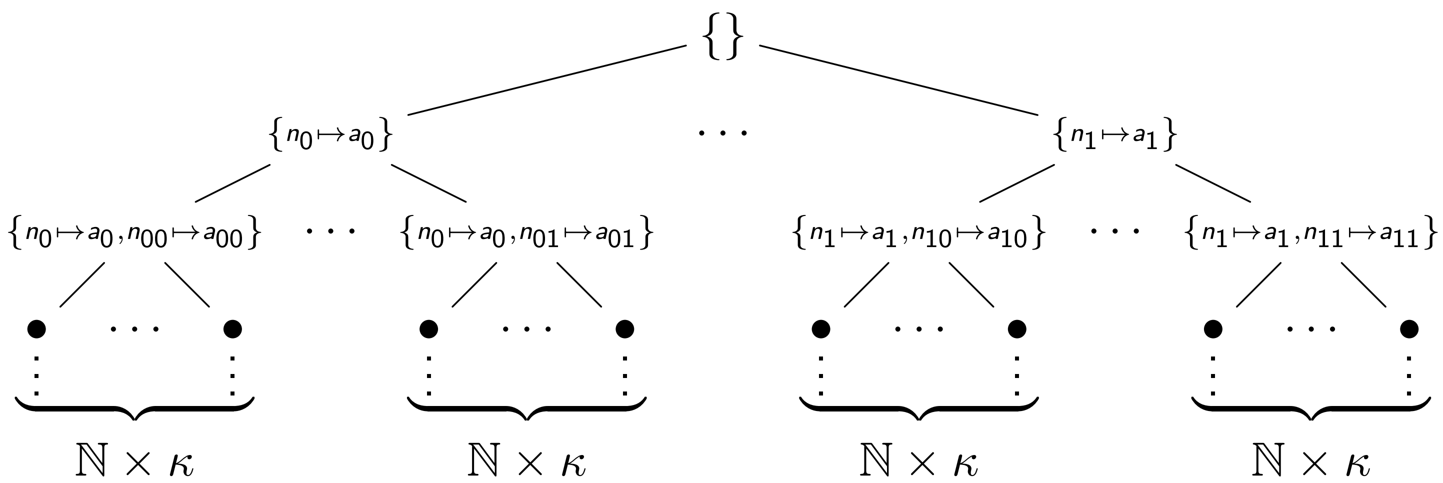

Let $R \subseteq \mathbb{N} \rightharpoonup \kappa$ and define subtree $T \subseteq \mathbb{N} \rightharpoonup \kappa$
by cutting childrens of every branch that is in R.

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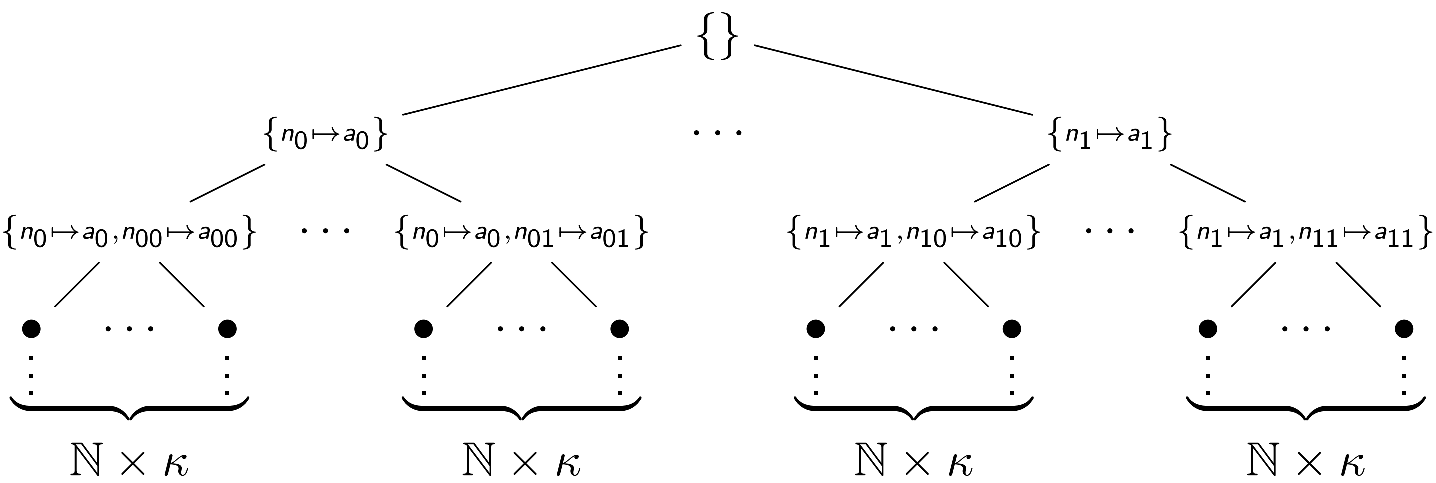

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by cutting childrens of every branch that is in R.

Bar induction says:

$$R \text{ bar} \implies T \text{ well-founded}$$

Bar recursion

base case: $F : (\mathbb{N} \to \kappa) \to \mathbb{N}$

inductive case: $G : (\mathbb{N} \rightharpoonup \kappa) \to (\kappa \to \mathbb{N}) \to \kappa$

$$\texttt{barrec} : (\mathbb{N} \rightharpoonup \kappa) \to \mathbb{N}$$

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$$\texttt{barrec} : (\mathbb{N} \rightharpoonup \kappa) \to \mathbb{N}$$

$$\texttt{barrec}\,(f) = F\,(f\,@\,\lambda n. G\,f\,(\lambda x.\texttt{barrec}\,(s \cup \{n \mapsto x\})))$$

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Compute F:

- If F only uses f where it is defined: done (base case)
- Otherwise F needs some undefined $f\,(n)$
  - Let $h = \lambda x.\texttt{barrec}\,(s \cup \{n \mapsto x\})$ (inductive case)
  - Give G f h for $f\,(n)$

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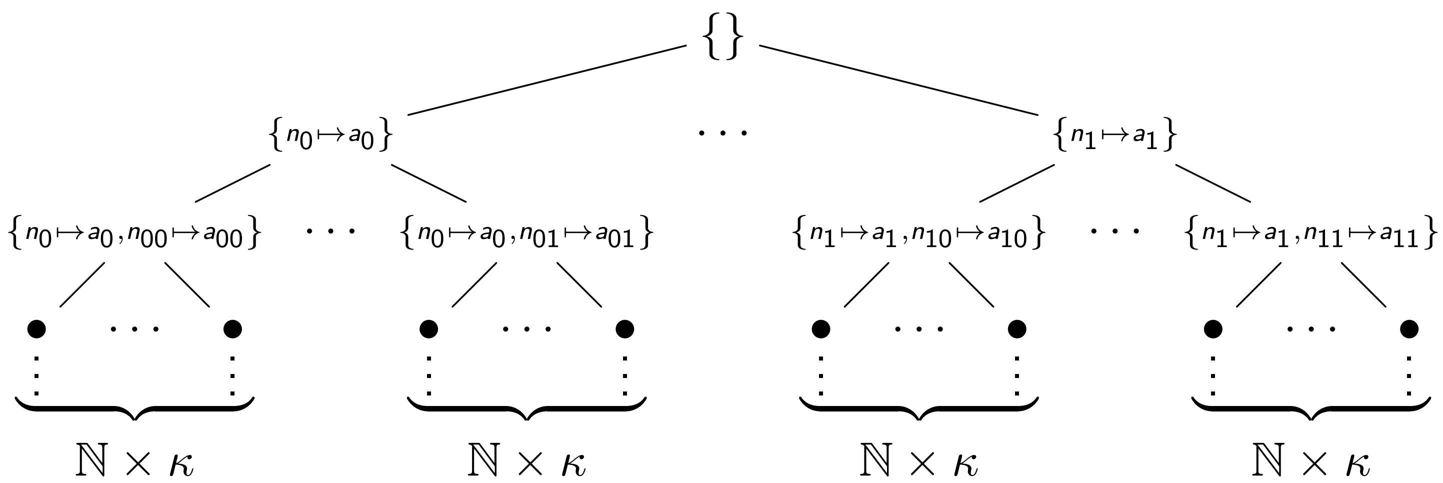

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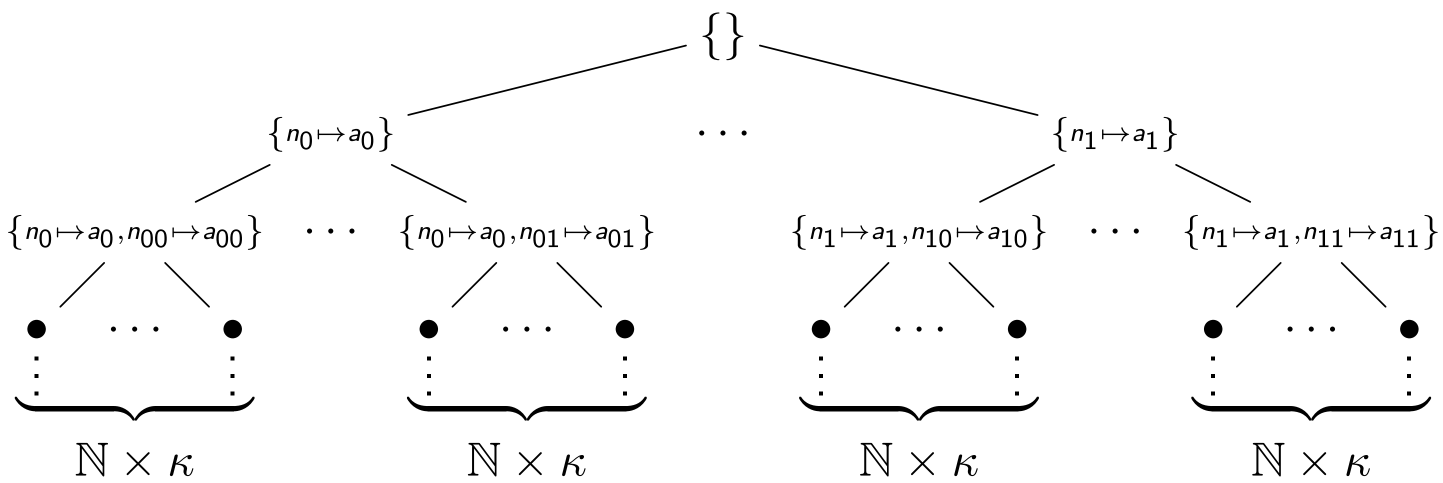

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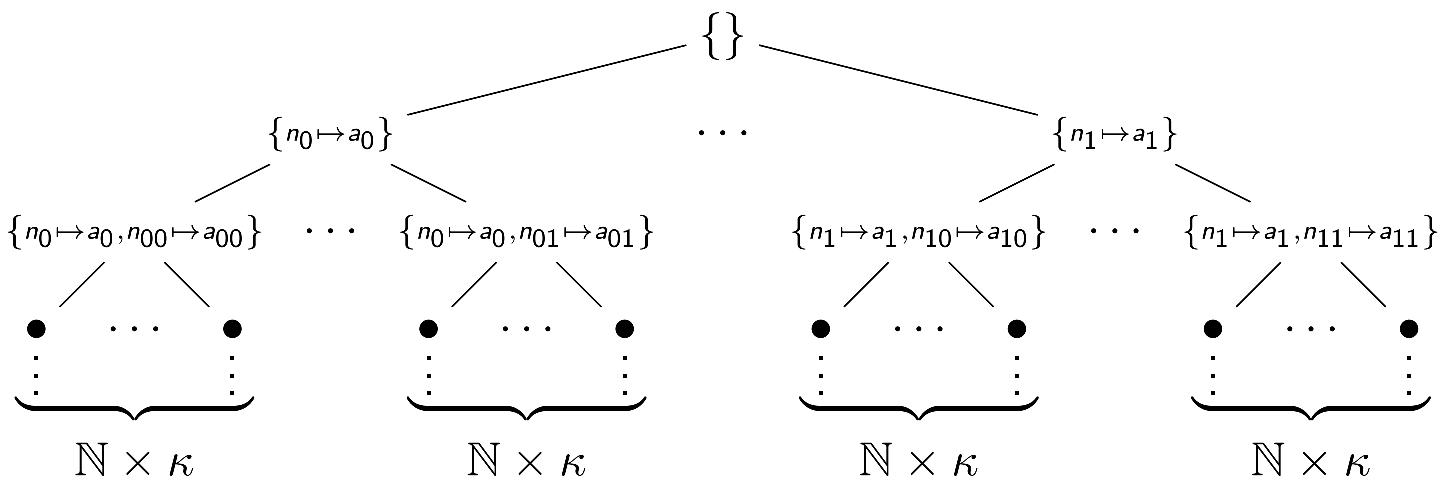

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$$f \in T \qquad \text{iff} \qquad \text{there exists extensions } \varphi, \psi \in \mathbb{N} \to \kappa \text{ s.t. } F(\varphi) \neq F(\psi)$$

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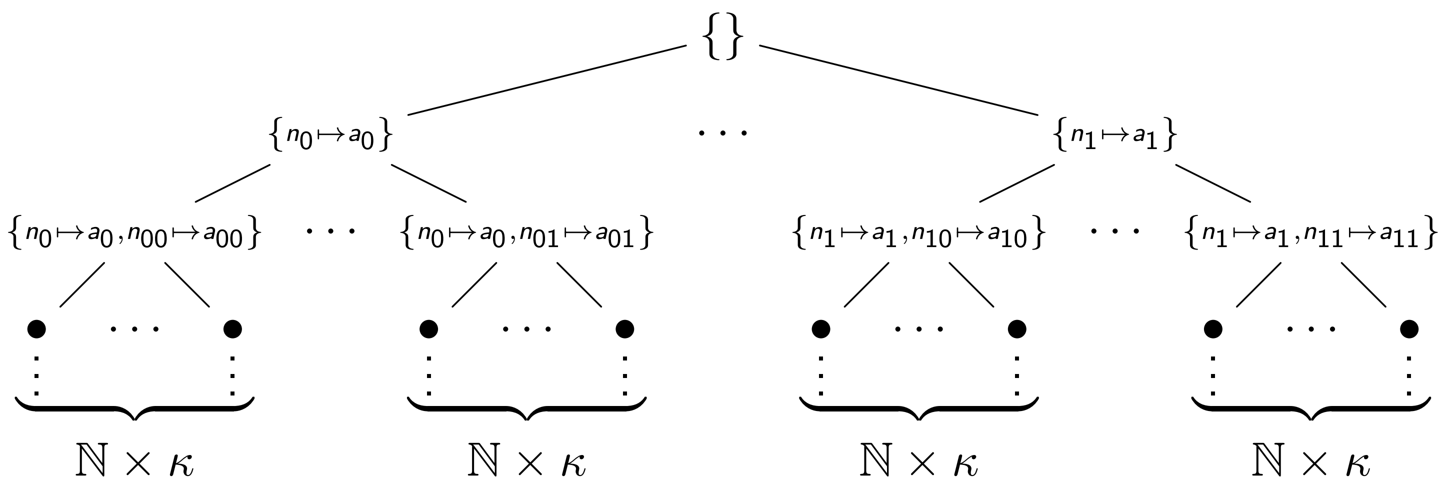

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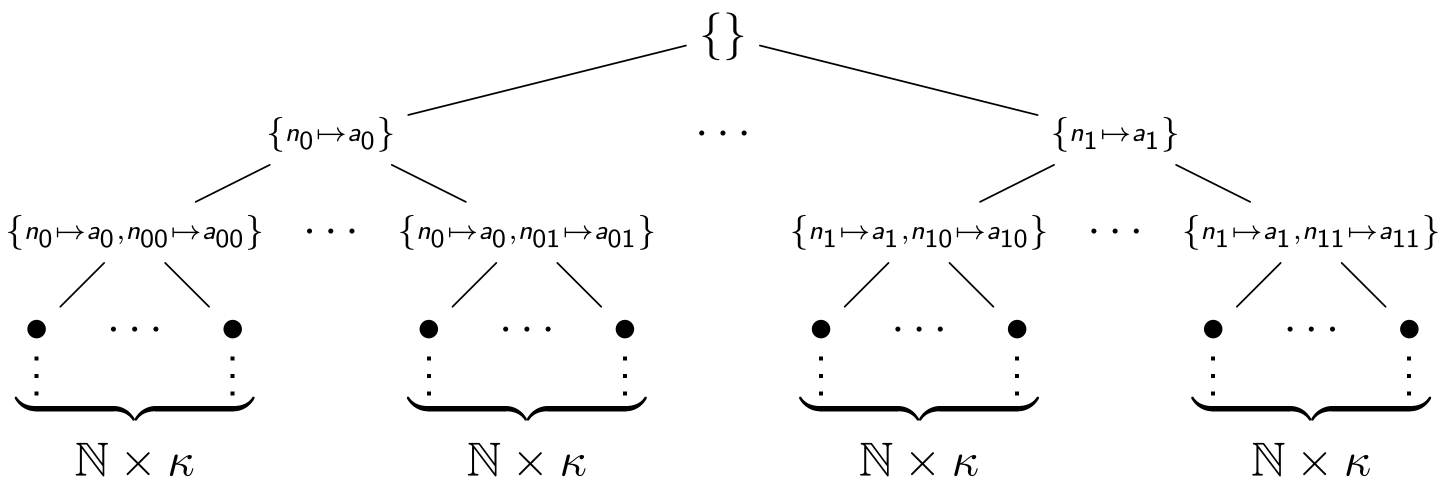

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bar recursion takes F as input and performs recursion over T

Interpreting HA2 into System T with bar recursion

From *HA*2 to $PA^{\omega} + AC$

Translation of formulas

Sets encoded as their characteristic functions:

	*HA*2	$PA^{\omega} + AC$
1st-order var.	x	$x^{\mathbb{N}}$
2nd-order var.	X	$X^{\mathbb{N}\to\mathbb{N}}$
2nd-order atoms	$t \in X$	$X(t) = 1$

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Translation of proofs

Only one non-trivial case: $\forall X\, A \Rightarrow A(B)$

- $\forall x^{\mathbb{N}} (C(x) \Leftrightarrow B(x)) \quad \Rightarrow \quad (A(C) \Leftrightarrow A(B))$

  proof defined by induction on A

- $\exists X^{\mathbb{N}\to\mathbb{N}} \forall x^{\mathbb{N}} (X(x) = 1 \Leftrightarrow B(x))$ (comprehension)

  comes from AC and classical logic

Comprehension in $PA^{\omega} + AC$

- Instance of AC:

$$\forall x^{\mathbb{N}} \exists y^{\mathbb{N}} (y = 1 \Leftrightarrow B(x)) \implies \exists X^{\mathbb{N} \to \mathbb{N}} \forall x^{\mathbb{N}} (X(x) = 1 \Leftrightarrow B(x))$$

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- Provable in classical logic (case analysis on $B\left(x\right)$):

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- We obtain the axiom schema of comprehension:

$$\exists X^{\mathbb{N} \to \mathbb{N}} \forall x^{\mathbb{N}} (X(x) = 1 \Leftrightarrow B(x))$$

Realizability: $PA^\omega + AC \longrightarrow$ System T + bar recursion

A two steps interpretation

$$PA^\omega + AC \xrightarrow[\text{translation}]{\text{negative}} HA^\omega + AC + DNS \xrightarrow[\text{interpretation}]{\text{realizability}} \begin{array}{c}\text{System T}\\ +\ \text{bar recursion}\end{array}$$

where $DNS \equiv \forall x\, \neg\neg A \Rightarrow \neg\neg\forall x\, A$

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Extraction

From a proof of $\exists x^{\mathbb{N}}\, t(x) = u(x)$ in $PA^\omega + AC$ one can extract a program in System T + bar recursion that normalizes to some $\mathtt{n}$ such that $t(\mathtt{n}) = u(\mathtt{n})$

Realizing comprehension with bar recursion

Realizability in a model

For bar recursion, the realizers must satisfy:

- Sequence internalization:

  if $(\varphi_n)_{n\in\mathbb{N}}$ realizers, then there exists φ s.t.:

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  if $\varphi\left(\psi\right) = \mathtt{m}$ there exists $n \in \mathbb{N}$ s.t.:

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$\rightsquigarrow$ Realizability in a CPO model of System T + bar recursion

Realizability in a model

This means:

- Realizers are elements of the CPO model rather than programs:

$$\mathfrak{a} \Vdash A \text{ where } \mathfrak{a} \text{ element of the CPO model}$$

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- Extraction is still correct by computational adequacy:

$$[\varphi] = n \Rightarrow \varphi \succ^* \mathtt{n}$$

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Let Emily $\Vdash \forall n\,(A\,(n) \vee \neg A\,(n))$

$$\exists X \;\; \forall n\,(n \in X \Leftrightarrow A\,(n))$$

Realizing comprehension with bar recursion

Let Emily $\Vdash \forall n\,(A\,(n) \vee \neg A\,(n))$

$$\underbrace{\forall X\,(\forall n\,(n \in X \Leftrightarrow A\,(n)) \Rightarrow \bot)}_{George} \Rightarrow \bot$$

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Me: George, I have some X and $\forall n\,(n \in X \Leftrightarrow A\,(n))$, give me $\bot$.

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Emily: It is $\neg A\,(n)$

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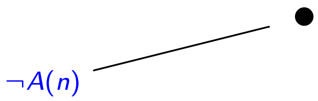

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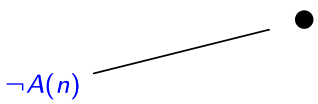

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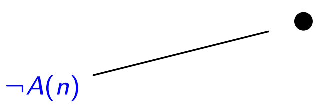

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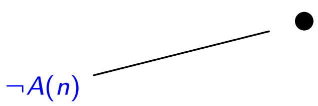

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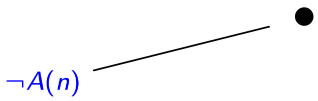

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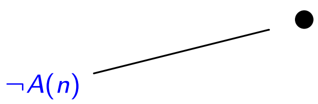

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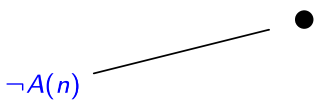

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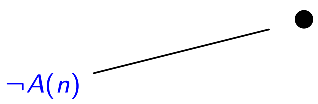

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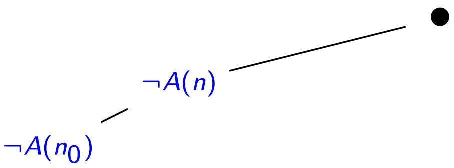

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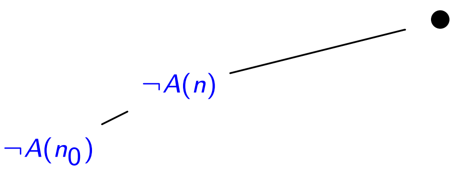

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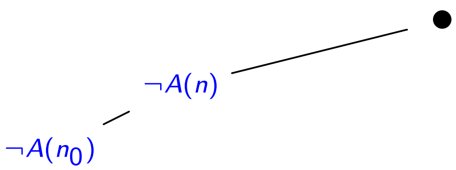

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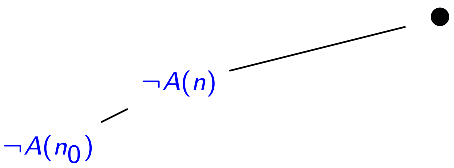

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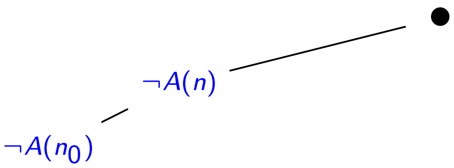

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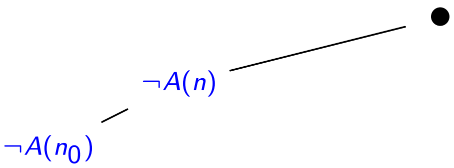

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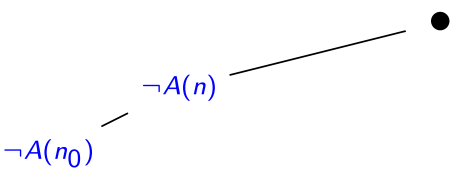

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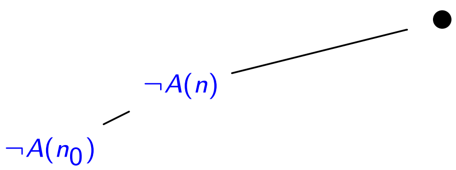

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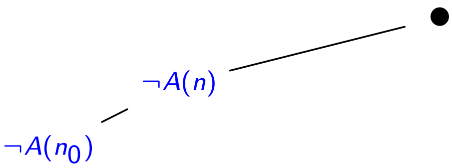

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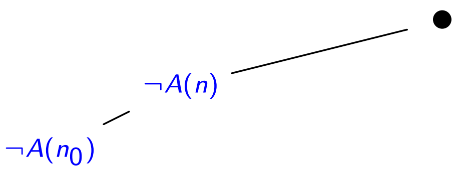

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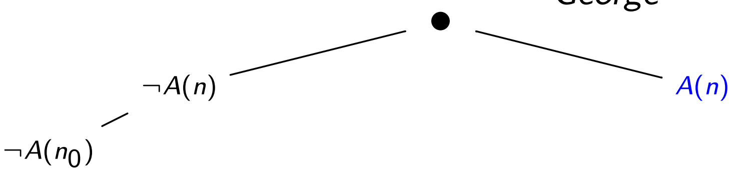

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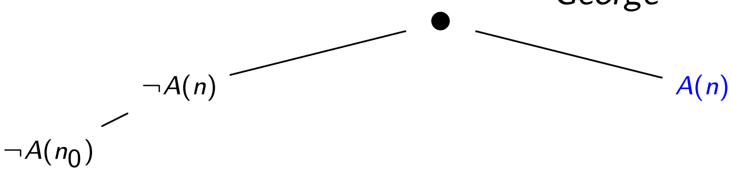

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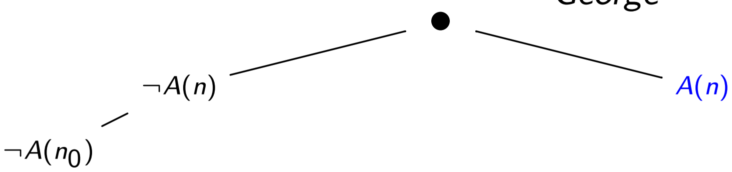

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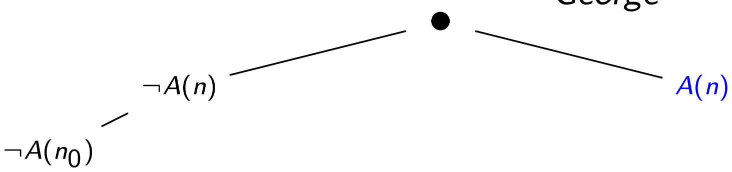

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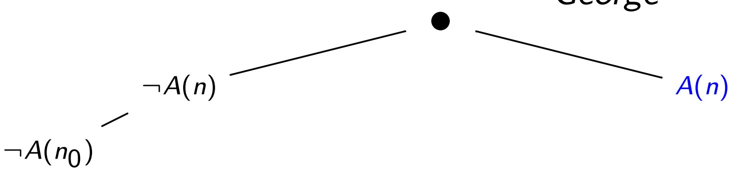

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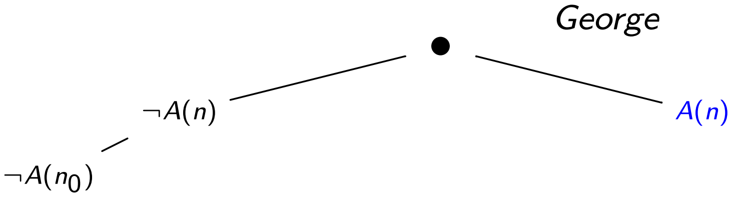

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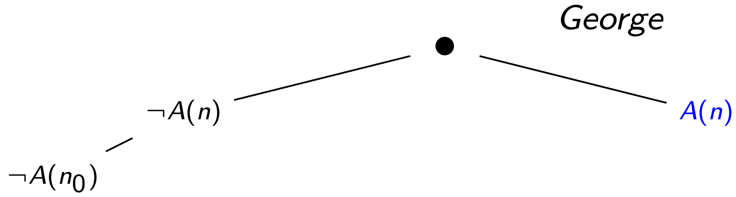

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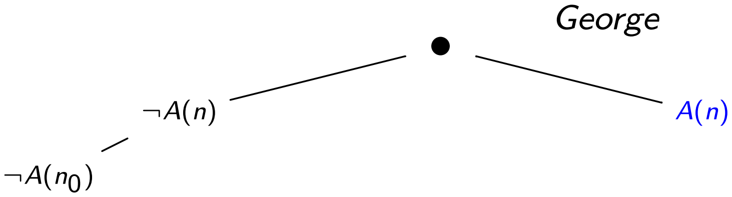

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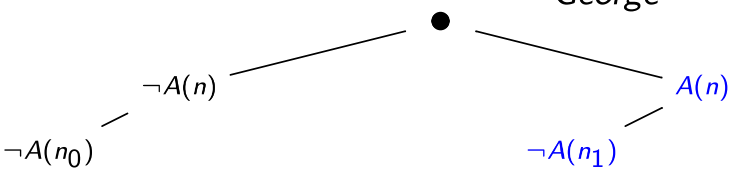

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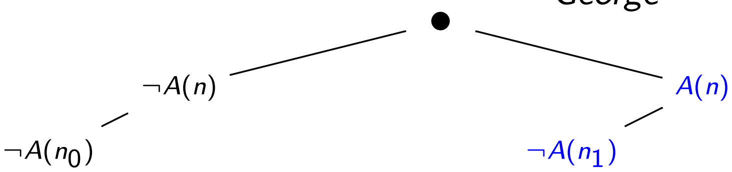

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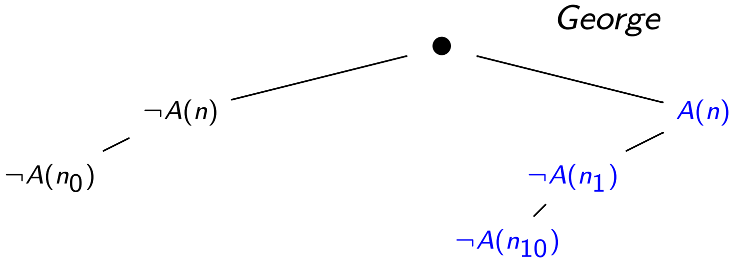

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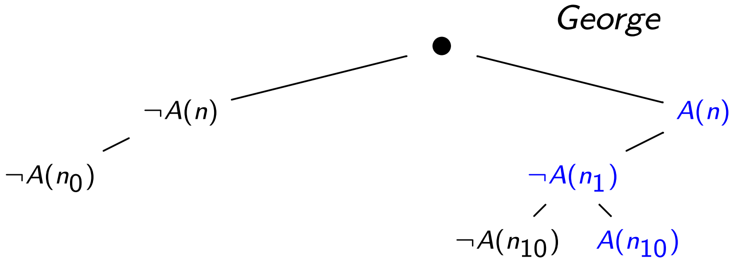

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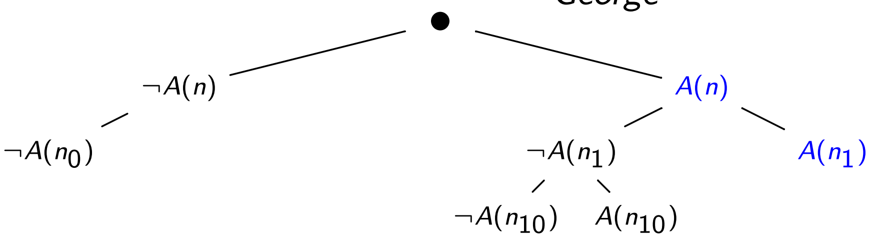

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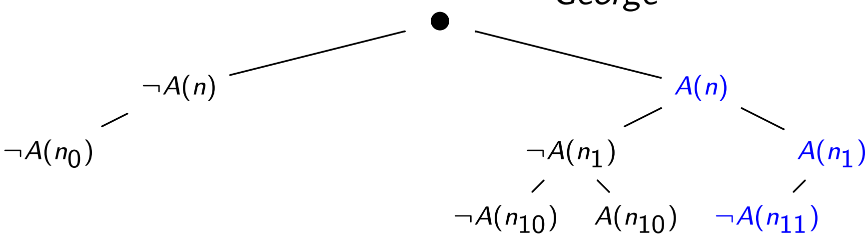

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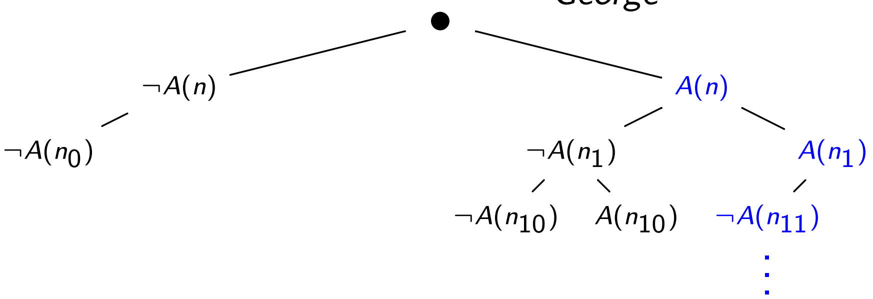

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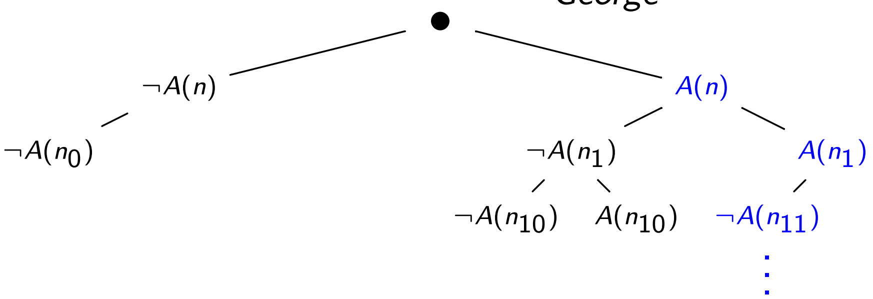

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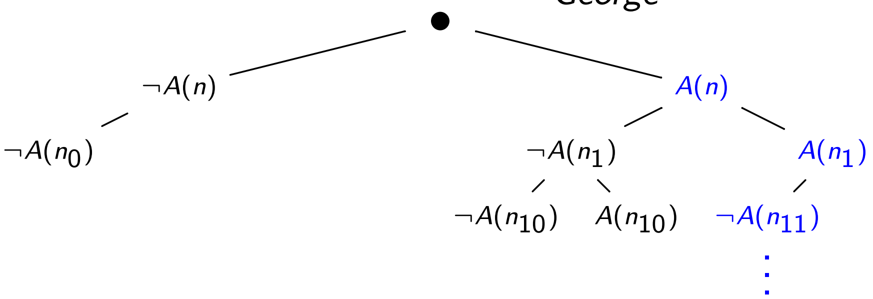

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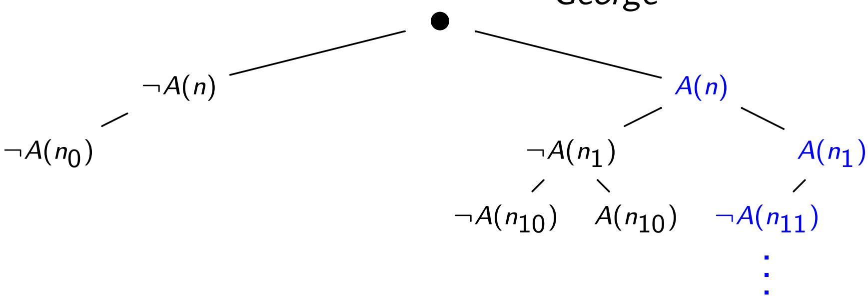

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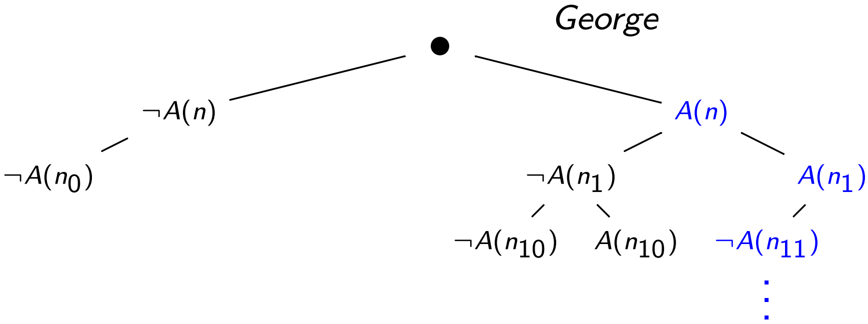

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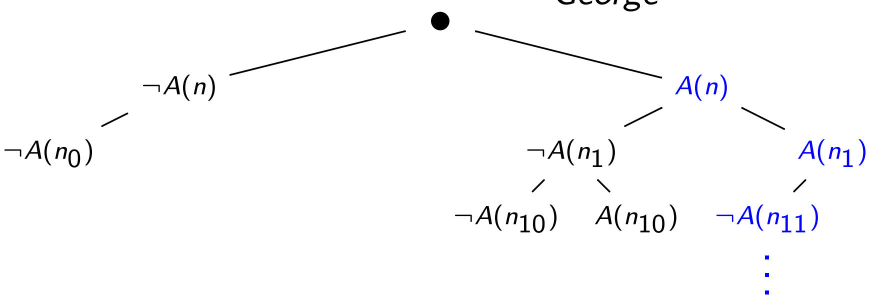

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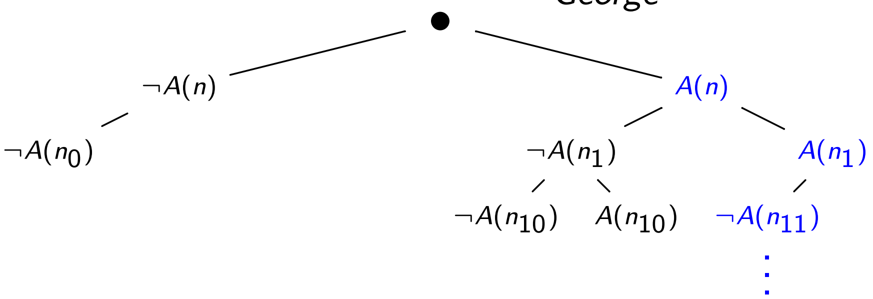

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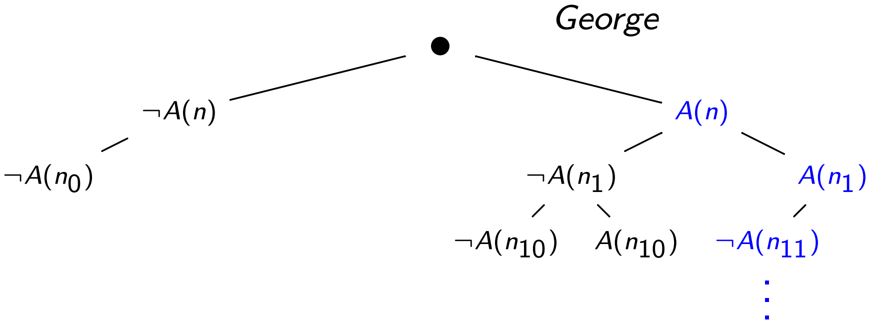

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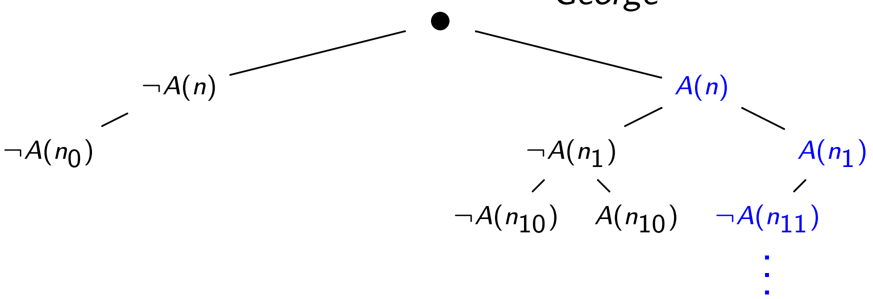

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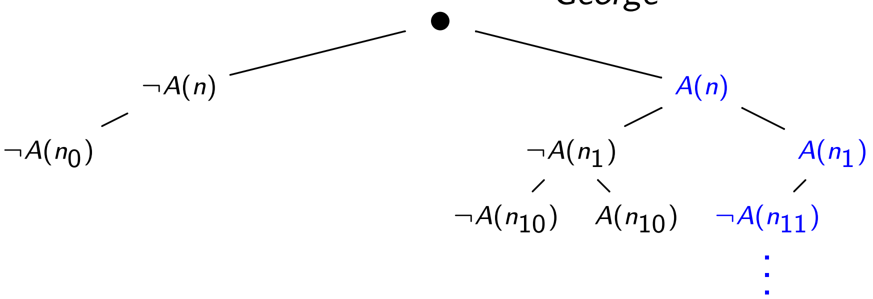

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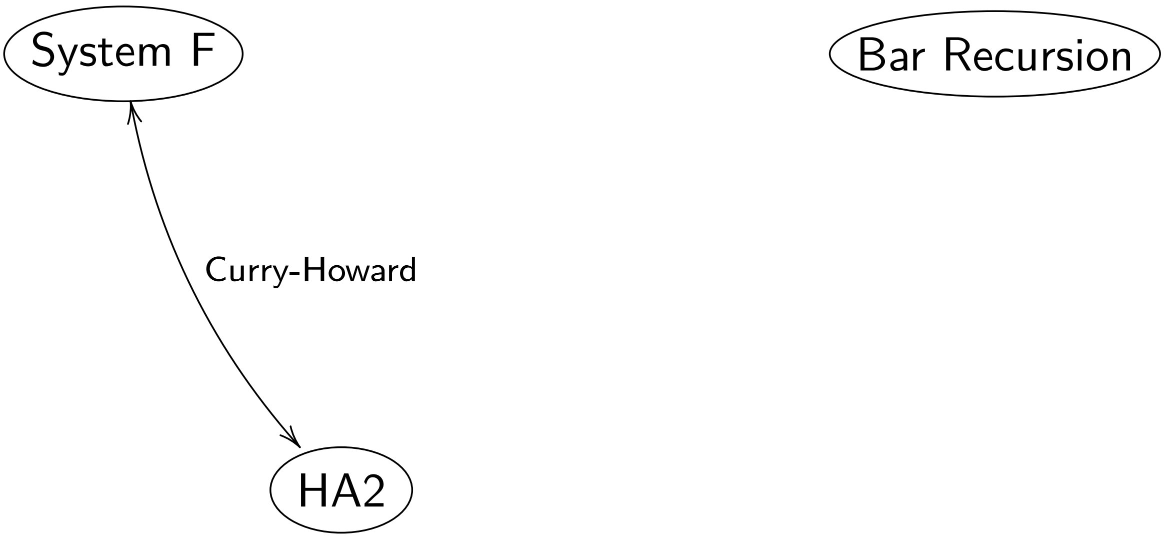
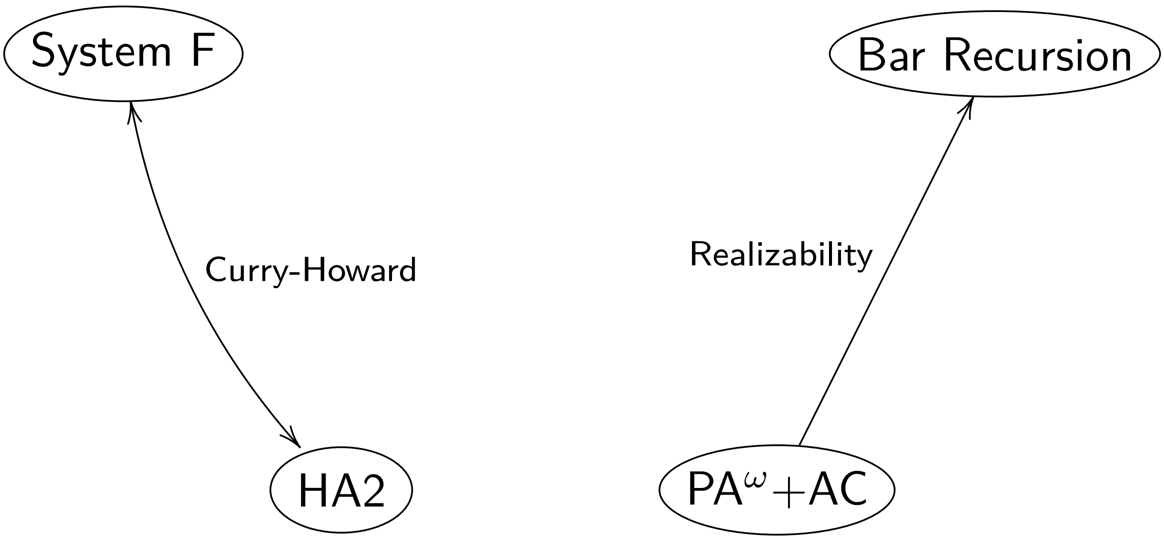
. . .
George: Thanks! Here is $\bot$ $\longleftarrow$ George is continuous

Translating System F to System T with bar recursion


System F
Bar Recursion


System F
Bar Recursion
Curry-Howard
HA2


System F
Curry-Howard
HA2
Bar Recursion
Realizability
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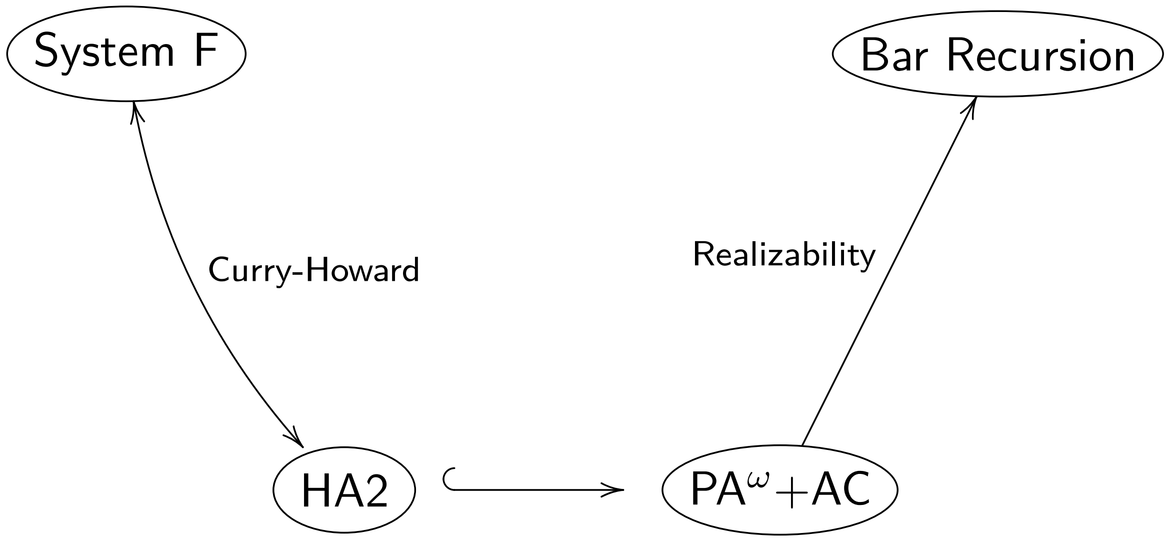
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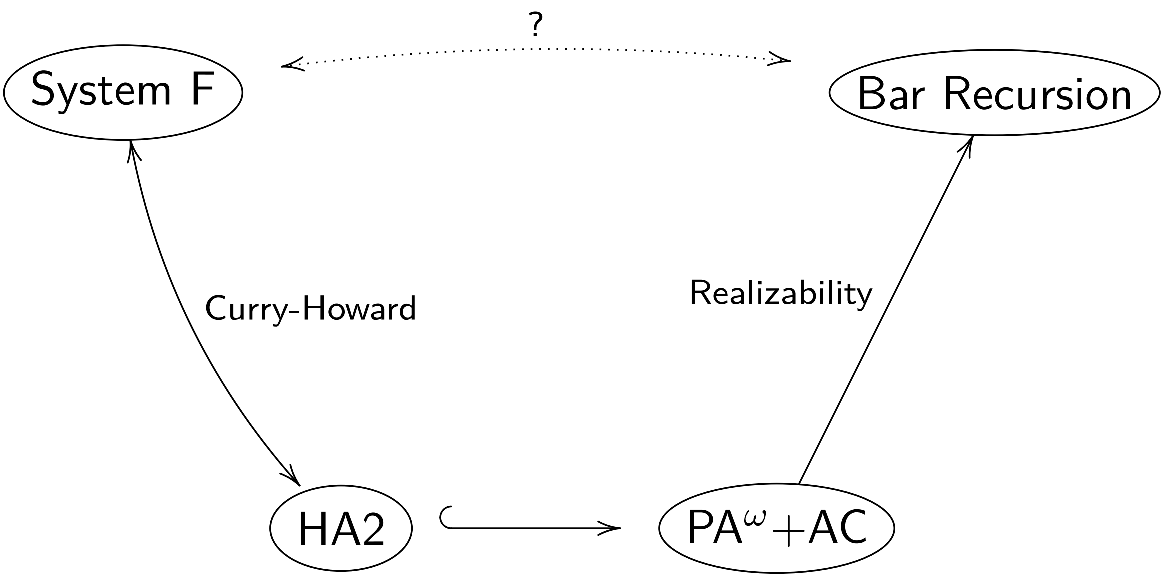
?
System F
Bar Recursion
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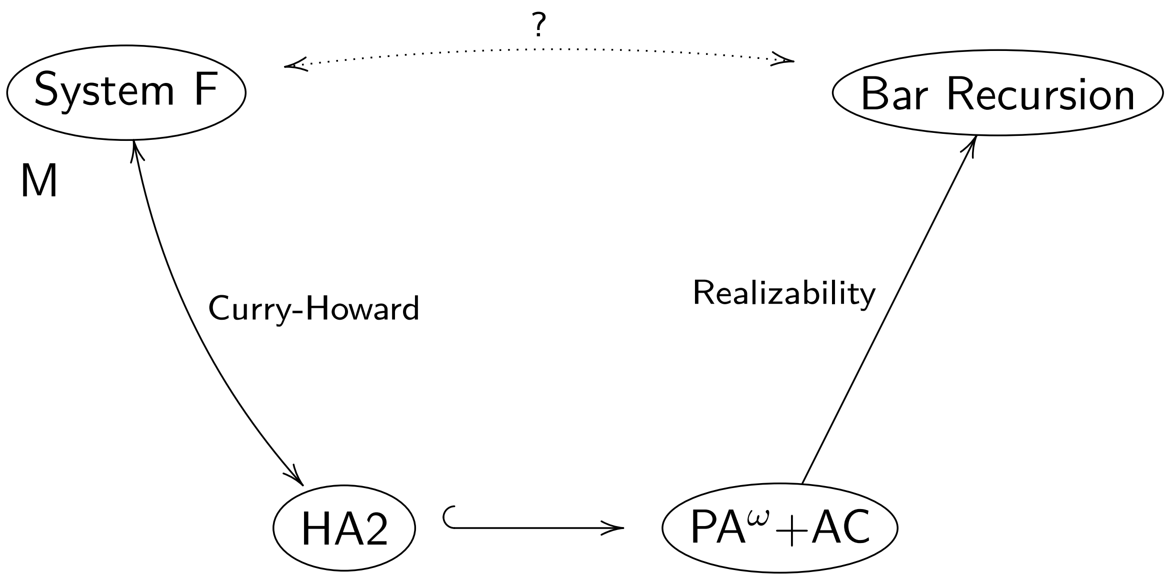
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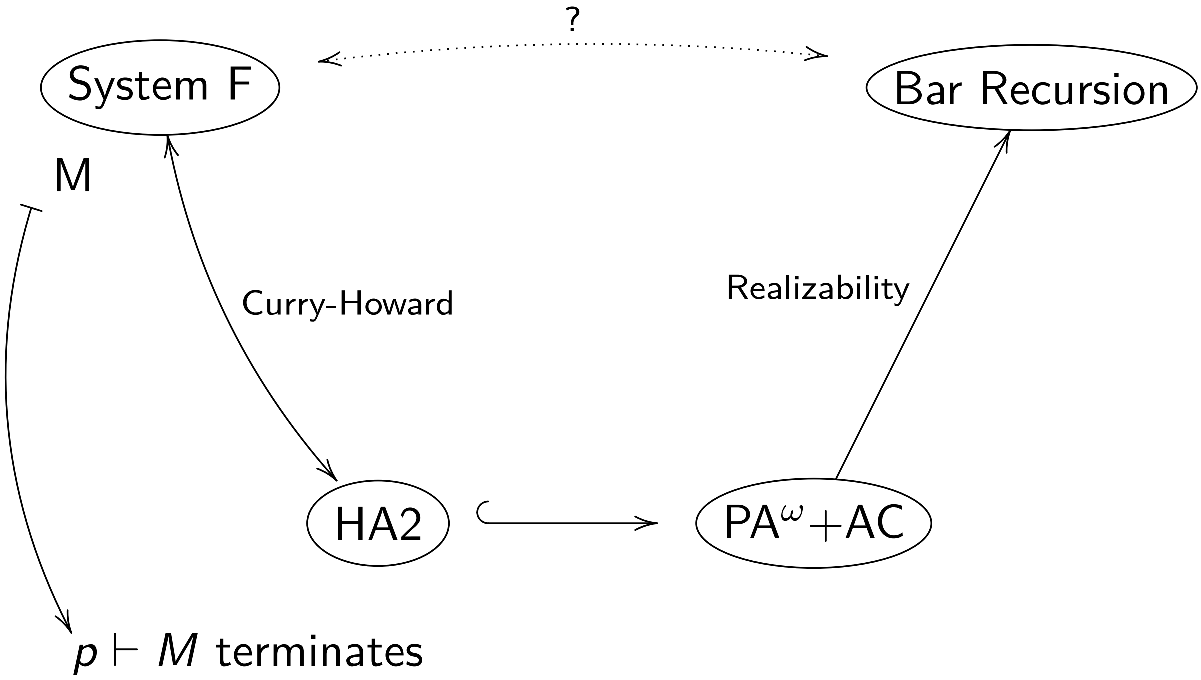
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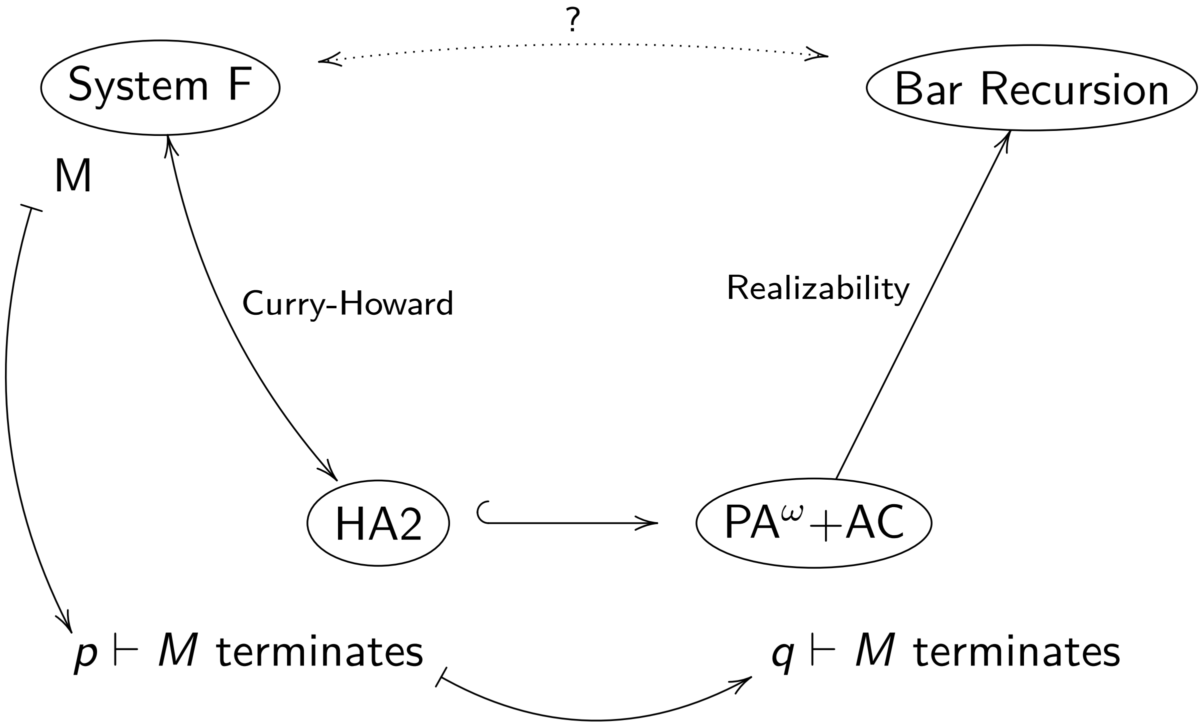
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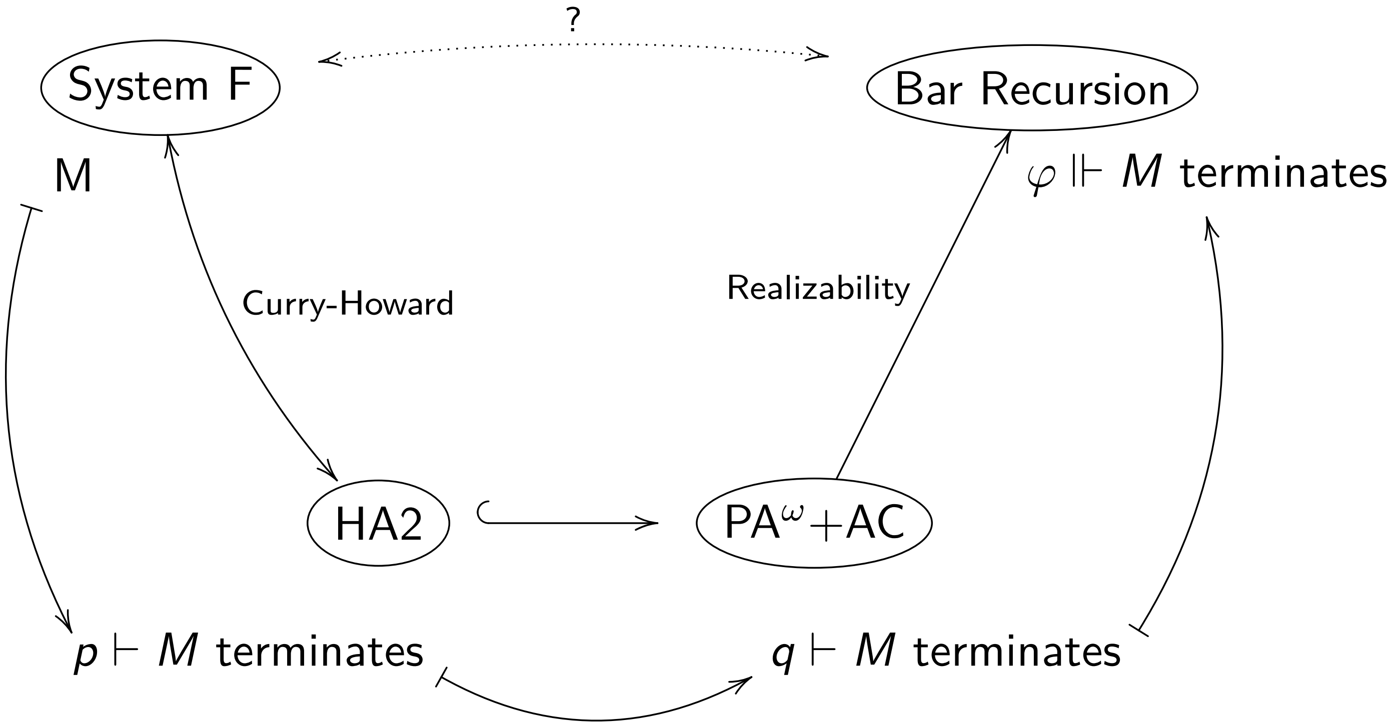
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Normalization of System F: definitions

Weak head reduction

$$(\lambda x.t)\, u\, v_1\, \dots\, v_n \succ t\,[u/x]\; v_1\, \dots\, v_n$$

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Reducibility candidates

For $X \subseteq \Lambda$ define property $\mathcal{R}ed\mathcal{C}and\,(X)$ as:

- X is closed by anti-evaluation
- every $(x\, t_1\, \ldots\, t_n)$ is in X
- every element of X normalizes

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Set-interpretation of types

For T type of system F, define $RC_T \subseteq \Lambda$ as:

- RC_X is a parameter such that $\mathcal{RedCand}\,(RC_X)$
- $RC_{T\to U} = \{t \in \Lambda \mid \forall u \in RC_T, t\ u \in RC_U\}$
- $RC_{\forall X\ T} = \bigcap \{RC_T \mid RC_X \subseteq \Lambda \quad \text{verifies} \quad \mathcal{RedCand}\,(RC_X)\}$

Normalization of System F: steps of the proof

2nd-order adequacy

For any type T, $\mathcal{R}ed\mathcal{C}and\,(RC_T)$

Proof: By induction on T.

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1st-order adequacy

If:

- $x_1 : T_1, \ldots, x_n : T_n \vdash M : U$ in system F
- $N_i \in RC_{T_i}$

Then:

$$M\,[N_1/x_1, \ldots, N_n/x_n] \in RC_U$$

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If $\vdash M : T$ in system F then $M \in RC_T$
moreover $\mathcal{R}ed\mathcal{C}and\,(RC_T)$
$\Longrightarrow M$ normalizes

Formalizing the normalization proof in HA2

- Gödel's incompleteness:

$$HA2 \nvdash \forall n \begin{pmatrix} \text{“}n \text{ is the code of} & \\ \text{a typing derivation of} & \Rightarrow \text{“}M \text{ normalizes”} \\ \text{term } M \text{ in system F”} & \end{pmatrix}$$

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$$f : \mathbb{N} \to \mathbb{N} \text{ representable in System F}$$
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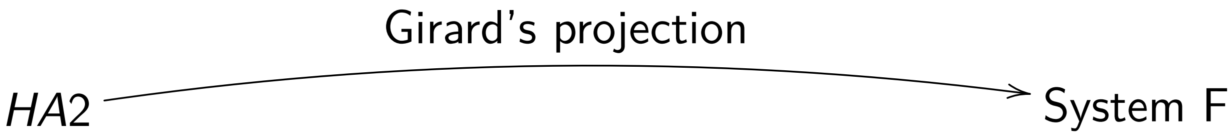
The translation of $M : T$
is the bar recursive interpretation of $HA2 \vdash$ “M normalizes”

The big picture

*HA*2 System F

The big picture



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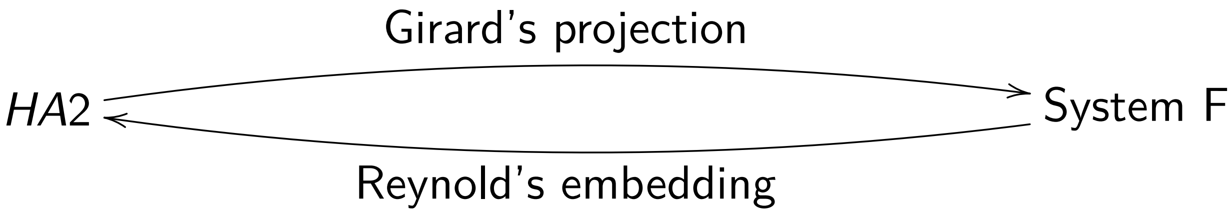

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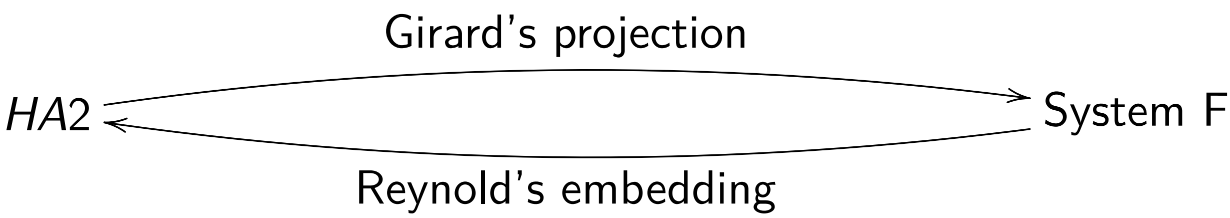
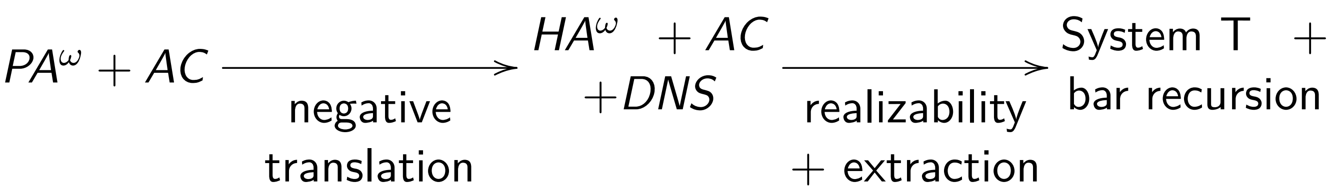
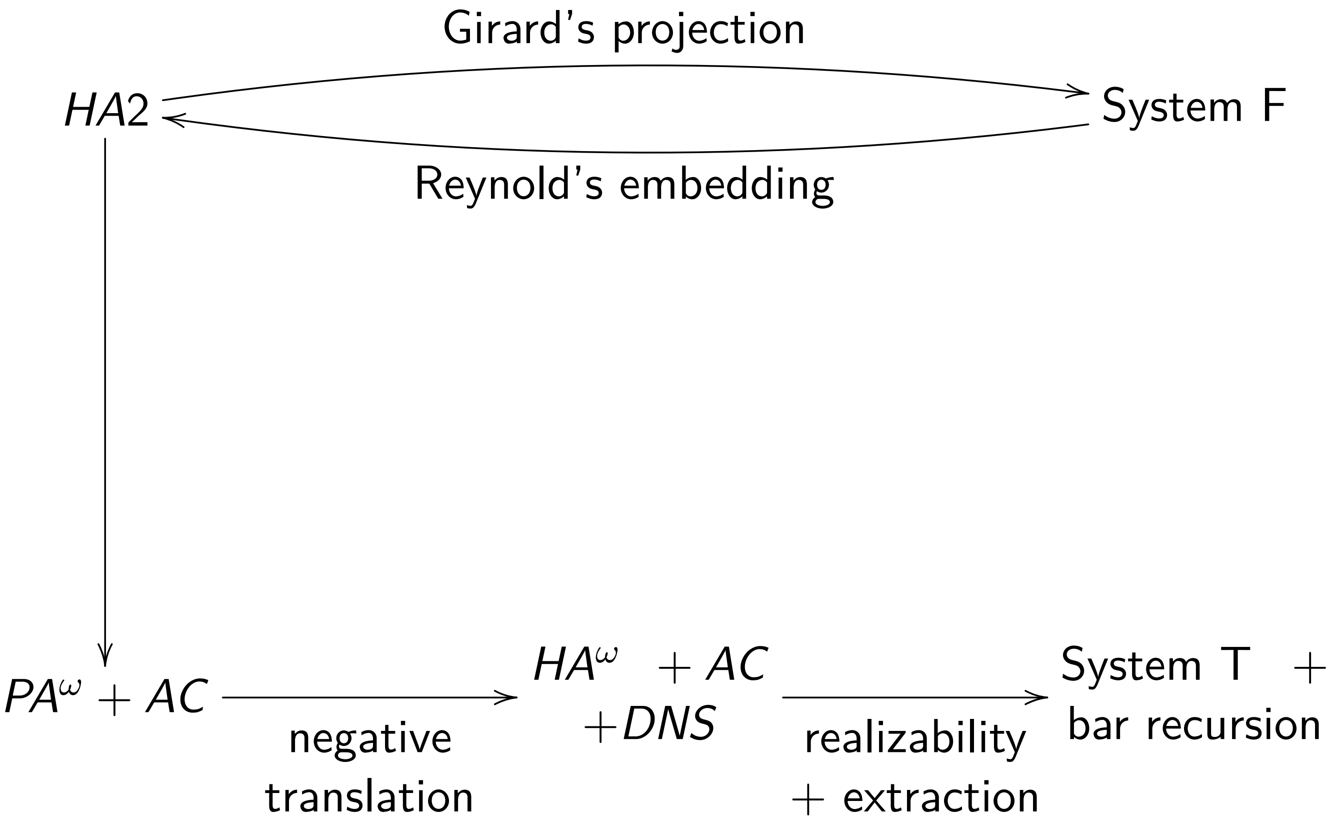




AC = axiom of countable choice

The big picture



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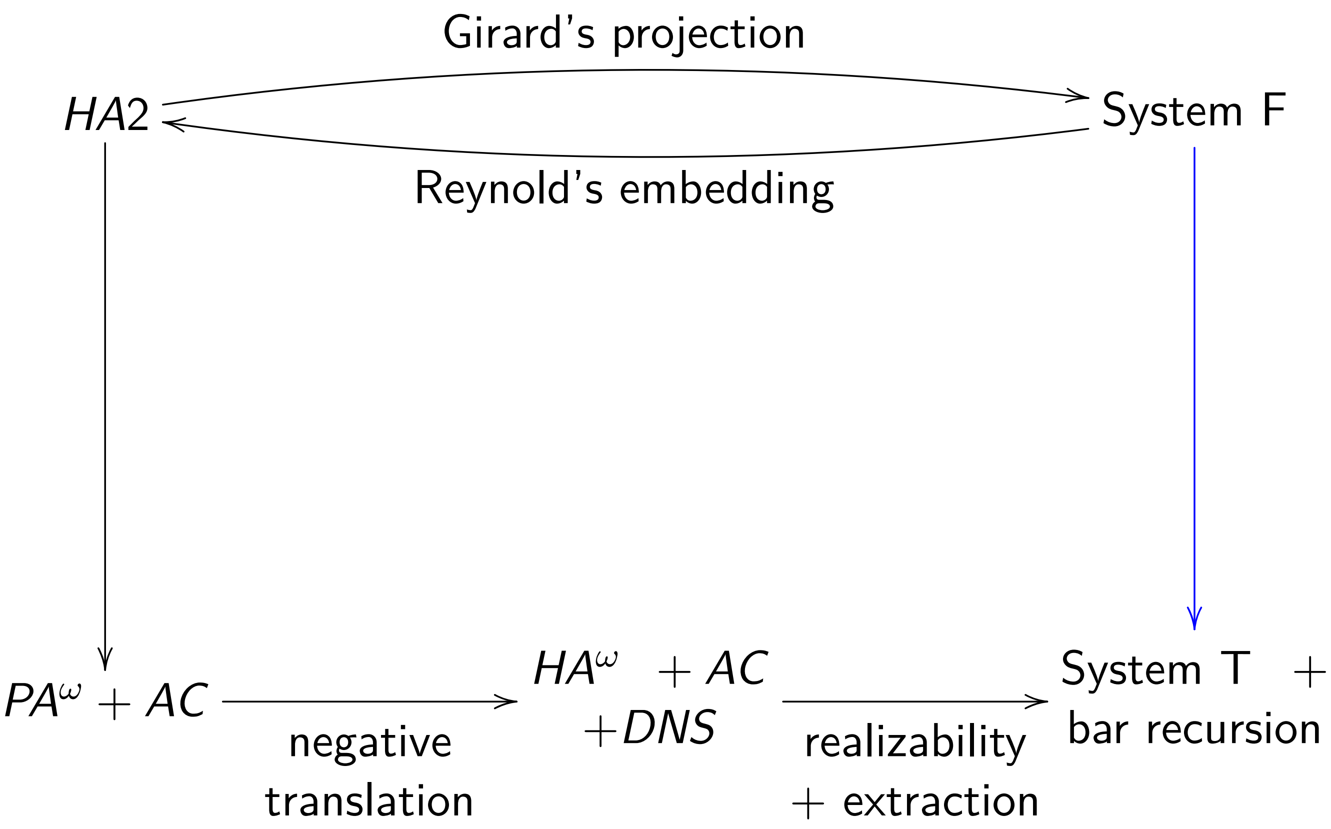

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Reynold's embedding

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$$\frac{\vdots}{x : X \vdash M : \forall Y\,(Y \to Z)}$$

$$\rightsquigarrow$$

HA2

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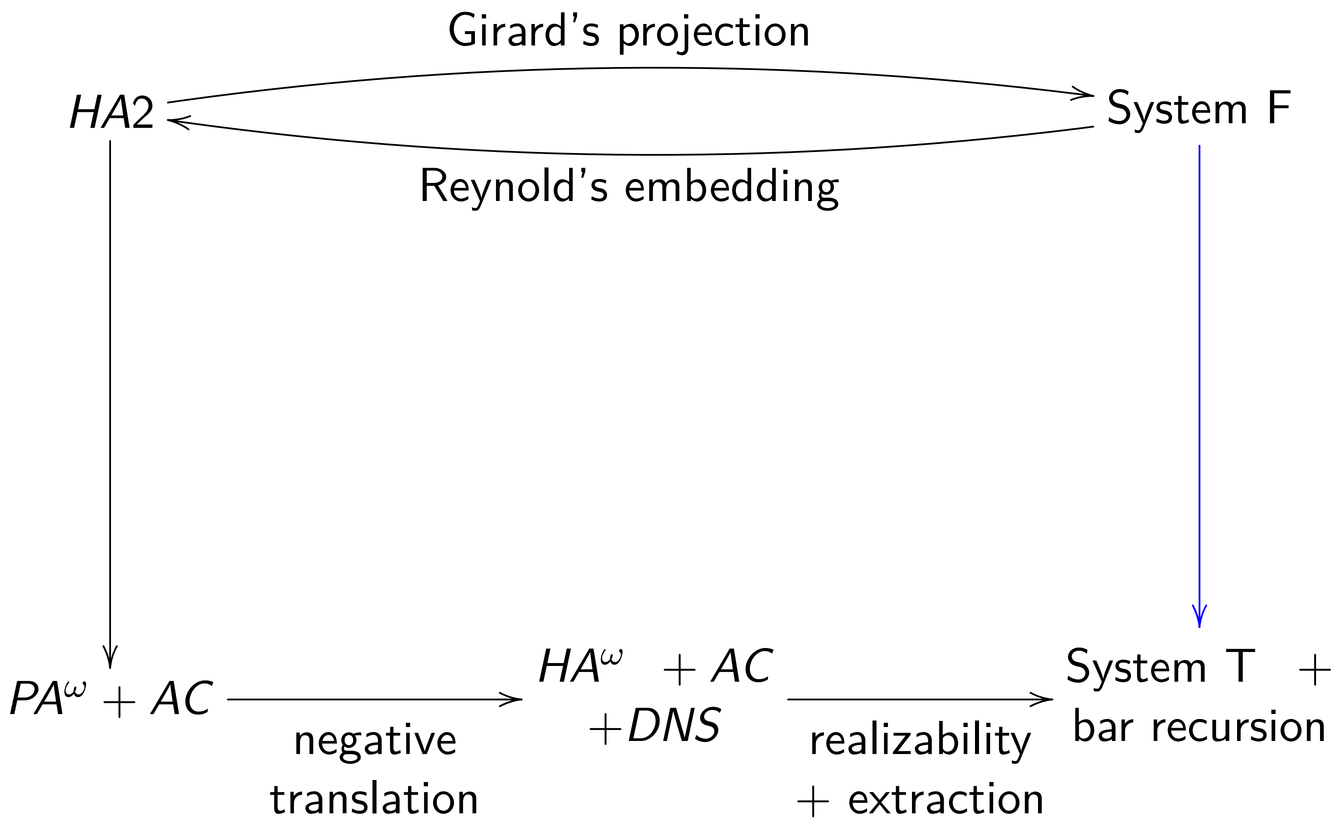
More generally:

$\vdash M : T$ in Syst. F $\Longrightarrow$ $\vdash RC_T\,(M)$ in $HA2$

where $RC_T\,(M)$ inductively-defined formula representing $RC_T \subseteq \Lambda$

The big picture



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- Negative logic (target of a negative translation):
  - no $\vee$
  - $\exists$ encoded as $\neg\forall\neg$

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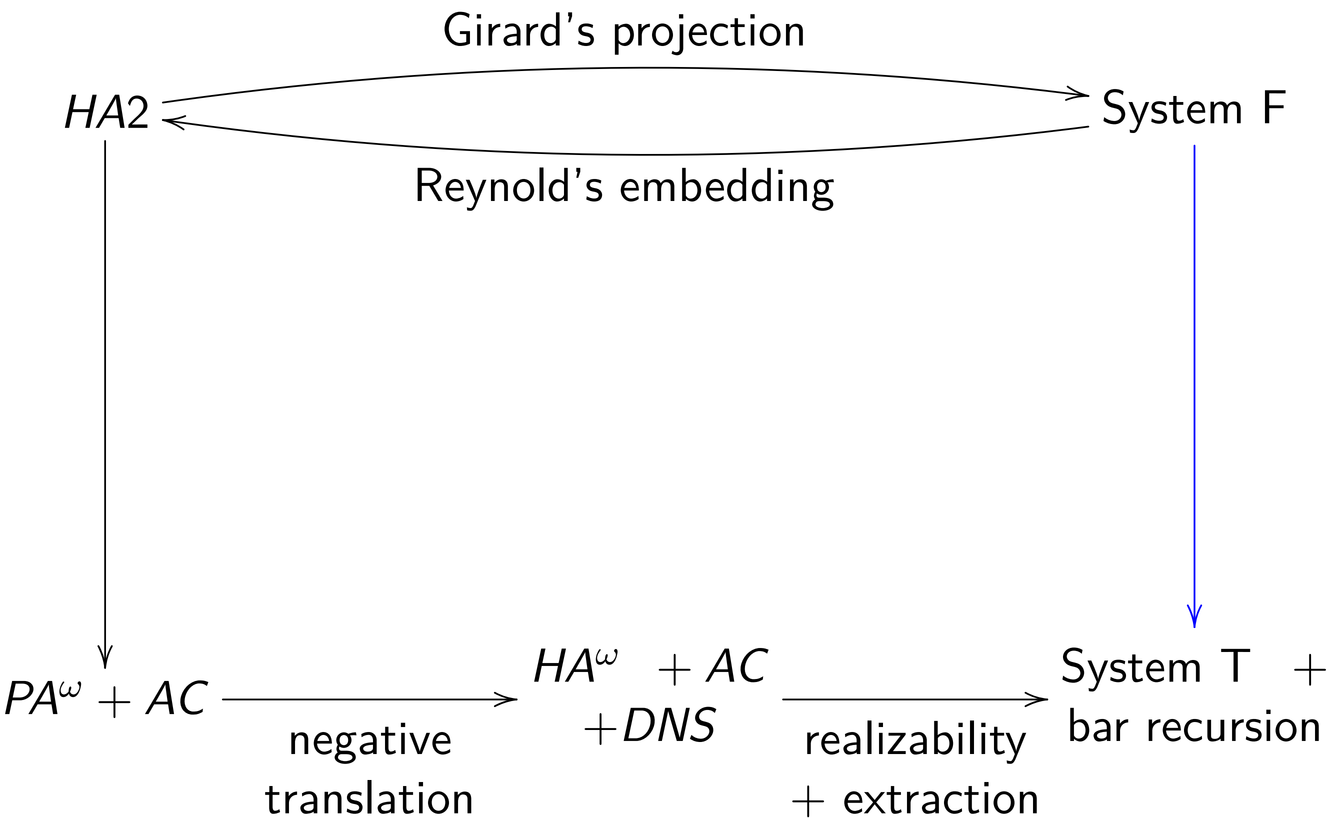
We can implement substitution in this language

The big picture



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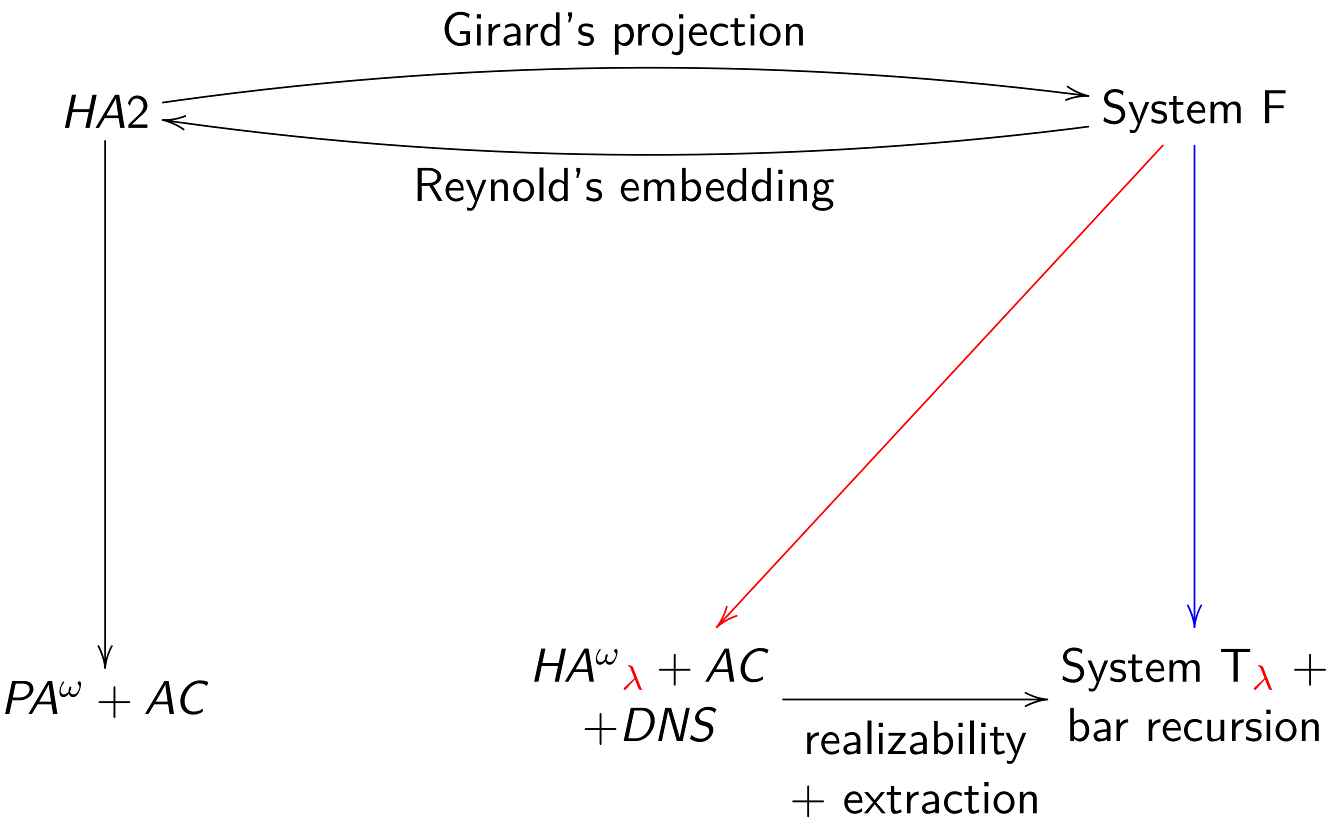

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Summary

System F

$HA^{\omega}_{\lambda} + AC + DNS$

System T_{λ}
$+$ bar recursion

$\mathcal{R}ed\mathcal{C}and\,(RC_T)$
$\Rightarrow \forall N\,(RC_T\,(N) \Rightarrow \exists n\, N \downarrow^{n})$

$M : T \longrightarrow RC_T\,(M) \longrightarrow \varphi \Vdash \exists n\, M \downarrow^{n}$

$\mathcal{R}ed\mathcal{C}and\,(RC_T)$

With the extraction theorem we get:
$\varphi\,(\lambda x.x) \succ^{*} \mathtt{n}$ such that M normalizes in n steps

The concrete translation

$$\mathtt{dne}_{\Phi} = \lambda x.x\,(\lambda y.y) \qquad \mathtt{dne}_{\forall i\,A} = \lambda xi.\mathtt{dne}_A\,(\lambda y.x\,(\lambda z.y\,(z\,i)))$$
$$\mathtt{dne}_{\forall b\,A} = \mathtt{dne}_A \qquad \mathtt{dne}_{\forall t\,A} = \lambda xt.\mathtt{dne}_A\,(\lambda y.x\,(\lambda z.y\,(z\,t)))$$
$$\mathtt{dne}_{\forall X\,A} = \mathtt{dne}_A \qquad \mathtt{dne}_{\forall \pi\,A} = \lambda x\pi.\mathtt{dne}_A\,(\lambda y.x\,(\lambda z.y\,(z\,\pi)))$$
$$\mathtt{dne}_{A\wedge B} = \lambda x.\,\langle \mathtt{dne}_A\,(\lambda y.x\,(\lambda z.y\,(\mathrm{p}_1\,z)))\,, \mathtt{dne}_B\,(\lambda y.x\,(\lambda z.y\,(\mathrm{p}_2\,z)))\rangle$$
$$\mathtt{dne}_{A\Rightarrow B} = \lambda xy.\mathtt{dne}_B\,(\lambda z.x\,(\lambda u.z\,(u\,y))) \qquad \mathtt{dne}_A \Vdash \neg\neg A \Rightarrow A$$

$$\mathtt{exf}_A = \lambda x.\mathtt{dne}_A\,(\lambda\_.x) \Vdash f\!\!f \Rightarrow A$$

$$\mathtt{normrc} = \langle\langle \lambda\pi x.x\,\mathrm{z}, \lambda tx.x\rangle\,, \lambda tu\pi xy.x\,(\lambda i.y\,(\mathrm{s}\,i))\rangle \Vdash \mathcal{R}ed\mathcal{C}and\,(\Downarrow)$$

$$\mathtt{repl}'_{\overline{X}\mapsto M\in X} = x\,M^{\diamond} \qquad \mathtt{repl}'_{\overline{X}\mapsto\Phi} = \langle \lambda y.y, \lambda y.y\rangle \text{ if } \Phi \not\equiv M \in X \qquad \mathtt{repl}'_{\forall X\,A} = \mathtt{repl}'_A \qquad \mathtt{repl}'_{\forall b\,A} = \mathtt{repl}'_A$$
$$\mathtt{repl}'_{\forall i\,A} = \langle \lambda yi.\mathrm{p}_1\,\mathtt{repl}'_A\,(y\,i)\,, \lambda yi.\mathrm{p}_2\,\mathtt{repl}'_A\,(y\,i)\rangle \qquad \mathtt{repl}'_{\forall t\,A} = \langle \lambda yt.\mathrm{p}_1\,\mathtt{repl}'_A\,(y\,t)\,, \lambda yt.\mathrm{p}_2\,\mathtt{repl}'_A\,(y\,t)\rangle$$
$$\mathtt{repl}'_{\forall\pi\,A} = \langle \lambda y\pi.\mathrm{p}_1\,\mathtt{repl}'_A\,(y\,\pi)\,, \lambda y\pi.\mathrm{p}_2\,\mathtt{repl}'_A\,(y\,\pi)\rangle$$
$$\mathtt{repl}'_{A_1\Rightarrow A_2} = \left\langle \lambda yz.\mathrm{p}_1\,\mathtt{repl}'_{A_2}\left(y\left(\mathrm{p}_2\,\mathtt{repl}'_{A_1}\,z\right)\right), \lambda yz.\mathrm{p}_2\,\mathtt{repl}'_{A_2}\left(y\left(\mathrm{p}_1\,\mathtt{repl}'_{A_1}\,z\right)\right)\right\rangle$$
$$\mathtt{repl}'_{A_1\wedge A_2} = \left\langle \lambda y.\left\langle \mathrm{p}_1\,\mathtt{repl}'_{A_1}\,(\mathrm{p}_1\,y)\,, \mathrm{p}_1\,\mathtt{repl}'_{A_2}\,(\mathrm{p}_2\,y)\right\rangle, \lambda y.\left\langle \mathrm{p}_2\,\mathtt{repl}'_{A_1}\,(\mathrm{p}_1\,y)\,, \mathrm{p}_2\,\mathtt{repl}'_{A_2}\,(\mathrm{p}_2\,y)\right\rangle\right\rangle$$
$$\mathtt{repl}_A = \lambda x.\mathtt{repl}'_A \Vdash \forall t\,(B\,(t) \Leftrightarrow C\,(t)) \Rightarrow (A\,(B) \Leftrightarrow A\,(C))$$

$$\mathtt{comp}_A = \lambda x.\mathtt{brec}\,(\lambda y.\mathtt{exf}_A\,(y\,\langle \mathtt{exf}_A, \lambda u.y\,\langle \lambda\_.u, \lambda\_.\mathrm{z}\rangle\rangle))\,x\,\{\} \Vdash \exists X\,\forall t\,(t \in X \Leftrightarrow A\,(t))$$

$$\mathtt{elim}_{A,B} = \lambda x.\mathtt{dne}_{A(B)}\,(\lambda y.\mathtt{comp}_B\,(\lambda z.y\,(\mathrm{p}_1\,(\mathtt{repl}_A\,z)\,x))) \Vdash \forall X\,A(\overline{X}) \Rightarrow A\,(B)$$

$$\mathtt{isrc}^{(1)}_X = \mathrm{p}_1\,(\mathrm{p}_1\,x_X) \qquad \mathtt{isrc}^{(2)}_X = \mathrm{p}_2\,(\mathrm{p}_1\,x_X) \qquad \mathtt{isrc}^{(3)}_X = \mathrm{p}_2\,x_X \qquad \mathtt{isrc}^{(1)}_{T\to U} = \lambda\pi tx.\mathtt{isrc}^{(1)}_U\,(\mathtt{cons}\,\pi\,t)$$
$$\mathtt{isrc}^{(2)}_{T\to U} = \lambda tx.\mathtt{isrc}^{(2)}_U\,(\mathtt{app}\,t\,(\mathtt{var}\,\mathrm{z}))\left(x\,(\mathtt{var}\,\mathrm{z})\left(\mathtt{isrc}^{(1)}_T\,\mathtt{nil}\right)\right) \qquad \mathtt{isrc}^{(3)}_{T\to U} = \lambda tu\pi xvy.\mathtt{isrc}^{(3)}_U\,t\,u\,(\mathtt{cons}\,\pi\,v)\,(x\,v\,y)$$
$$\mathtt{isrc}^{(2)}_{\forall X\,T} = \lambda tx.\mathtt{elim}_{\overline{X}\mapsto\mathcal{R}ed\mathcal{C}and(\overline{X})\Rightarrow\forall t(RC_T(t)\Rightarrow t\downarrow),\Downarrow}\left(\lambda x_X\mathtt{isrc}^{(2)}_T\right)\mathtt{normrc}\,t\left(\mathtt{elim}_{\overline{X}\mapsto\mathcal{R}ed\mathcal{C}and(\overline{X})\Rightarrow RC_T(t),\Downarrow}\,x\,\mathtt{normrc}\right)$$
$$\mathtt{isrc}^{(1)}_{\forall X\,T} = \lambda\pi x_X.\mathtt{isrc}^{(1)}_T\pi \qquad \mathtt{isrc}^{(3)}_{\forall X\,T} = \lambda tu\pi yx_X.\mathtt{isrc}^{(3)}_T\,t\,u\,\pi\,(y\,x_X) \qquad \mathtt{isrc}_T = \left\langle\left\langle \mathtt{isrc}^{(1)}_T, \mathtt{isrc}^{(2)}_T\right\rangle, \mathtt{isrc}^{(3)}_T\right\rangle \Vdash \mathcal{R}ed\mathcal{C}and\,(RC_T)$$

$$\mathtt{adeq}_{\Gamma\vdash\underline{m}:U} = y_U \qquad \mathtt{adeq}_{\Gamma\vdash\lambda.M:U\to T} = \lambda t_U y_U.\mathtt{isrc}^{(3)}_T\left(\mathtt{subst}\,M^{\diamond}\,(\mathrm{s}\,\mathrm{z})\left(\mathtt{shift}^{*}\,\vec{t_\Gamma}\right)\right)t_U\,\mathtt{nil}\,\mathtt{adeq}_{\Gamma\vdash M:T}$$
$$\mathtt{adeq}_{\Gamma\vdash\mathtt{app}\,M\,N:T} = \mathtt{adeq}_{\Gamma\vdash M:U\to T}\left(\mathtt{subst}\,N^{\diamond}\,\mathrm{z}\,\vec{t_\Gamma}\right)\mathtt{adeq}_{\Gamma\vdash N:U} \qquad \mathtt{adeq}_{\Gamma\vdash M:\forall X\,T} = \lambda x_X.\mathtt{adeq}_{\Gamma\vdash M:T}$$
$$\mathtt{adeq}_{\Gamma\vdash M:T\{U/X\}} = \mathtt{elim}_{\overline{X}\mapsto\mathcal{R}ed\mathcal{C}and(\overline{X})\Rightarrow RC_T(M[\vec{t_\Gamma}]),RC_U}\,\mathtt{adeq}_{\Gamma\vdash M:\forall X\,T}\,\mathtt{isrc}_U \qquad \mathtt{adeq}_{\Gamma\vdash M:T} \Vdash RC_T\left(M\left[\vec{t_\Gamma}\right]\right)$$

$$\mathtt{norm}_{\vdash M:T} = \mathtt{isrc}^{(2)}_T\,M^{\diamond}\,\mathtt{adeq}_{\vdash M:T} \Vdash M\!\downarrow \quad \equiv \quad \exists n\,M\downarrow^{n}$$

The concrete translation

$$\mathtt{dne}_{\Phi} = \lambda x.x\,(\lambda y.y) \qquad \mathtt{dne}_{\forall i\,A} = \lambda xi.\mathtt{dne}_A\,(\lambda y.x\,(\lambda z.y\,(z\,i)))$$
$$\mathtt{dne}_{\forall b\,A} = \mathtt{dne}_A \qquad \mathtt{dne}_{\forall t\,A} = \lambda xt.\mathtt{dne}_A\,(\lambda y.x\,(\lambda z.y\,(z\,t)))$$
$$\mathtt{dne}_{\forall X\,A} = \mathtt{dne}_A \qquad \mathtt{dne}_{\forall \pi\,A} = \lambda x\pi.\mathtt{dne}_A\,(\lambda y.x\,(\lambda z.y\,(z\,\pi)))$$
$$\mathtt{dne}_{A\wedge B} = \lambda x.\,\langle \mathtt{dne}_A\,(\lambda y.x\,(\lambda z.y\,(\mathrm{p}_1\,z)))\,,\mathtt{dne}_B\,(\lambda y.x\,(\lambda z.y\,(\mathrm{p}_2\,z)))\rangle$$
$$\mathtt{dne}_{A\Rightarrow B} = \lambda xy.\mathtt{dne}_B\,(\lambda z.x\,(\lambda u.z\,(u\,y))) \qquad \mathtt{dne}_A \Vdash \neg\neg A \Rightarrow A$$

$$\mathtt{exf}_A = \lambda x.\mathtt{dne}_A\,(\lambda\_.x) \Vdash f\!f \Rightarrow A$$

$$\mathtt{normrc} = \langle\langle \lambda\pi x.x\,\mathrm{z}, \lambda tx.x\rangle\,,\lambda tu\pi xy.x\,(\lambda i.y\,(\mathrm{s}\,i))\rangle \Vdash \mathcal{R}ed\mathcal{C}and\,(\Downarrow)$$

$$\mathtt{repl}'_{\overline{X}\mapsto M\in X} = x\,M^{\diamond} \qquad \mathtt{repl}'_{\overline{X}\mapsto\Phi} = \langle \lambda y.y, \lambda y.y\rangle \text{ if } \Phi \not\equiv M\in X \qquad \mathtt{repl}'_{\forall X\,A} = \mathtt{repl}'_A \qquad \mathtt{repl}'_{\forall b\,A} = \mathtt{repl}'_A$$
$$\mathtt{repl}'_{\forall i\,A} = \langle \lambda yi.\mathrm{p}_1\,\mathtt{repl}'_A\,(y\,i)\,,\lambda yi.\mathrm{p}_2\,\mathtt{repl}'_A\,(y\,i)\rangle \qquad \mathtt{repl}'_{\forall t\,A} = \langle \lambda yt.\mathrm{p}_1\,\mathtt{repl}'_A\,(y\,t)\,,\lambda yt.\mathrm{p}_2\,\mathtt{repl}'_A\,(y\,t)\rangle$$
$$\mathtt{repl}'_{\forall \pi\,A} = \langle \lambda y\pi.\mathrm{p}_1\,\mathtt{repl}'_A\,(y\,\pi)\,,\lambda y\pi.\mathrm{p}_2\,\mathtt{repl}'_A\,(y\,\pi)\rangle$$
$$\mathtt{repl}'_{A_1\Rightarrow A_2} = \left\langle \lambda yz.\mathrm{p}_1\,\mathtt{repl}'_{A_2}\left(y\left(\mathrm{p}_2\,\mathtt{repl}'_{A_1}\,z\right)\right), \lambda yz.\mathrm{p}_2\,\mathtt{repl}'_{A_2}\left(y\left(\mathrm{p}_1\,\mathtt{repl}'_{A_1}\,z\right)\right)\right\rangle$$

$$\mathtt{repl}_A \Vdash \forall t\,(B\,(t) \Leftrightarrow C\,(t)) \Rightarrow (A\,(B) \Leftrightarrow A\,(C))$$

$$\mathtt{com} \quad \mathtt{comp}_A \Vdash \exists X\,\forall t\,(t\in X \Leftrightarrow A\,(t)) \quad \in X \Leftrightarrow A\,(t))$$

$$\mathtt{elim}_{A,B} = \lambda x.\mathtt{dne}_{A(B)}\,(\lambda y.\mathtt{comp}_B\,(\lambda z.y\,(\mathrm{p}_1\,(\mathtt{repl}_A\,z)\,x))) \Vdash \forall X\,A(\overline{X}) \Rightarrow A\,(B)$$

$$\mathtt{isrc}^{(1)}_X = \mathrm{p}_1\,(\mathrm{p}_1\,x_X) \qquad \mathtt{isrc}^{(2)}_X = \mathrm{p}_2\,(\mathrm{p}_1\,x_X) \qquad \mathtt{isrc}^{(3)}_X = \mathrm{p}_2\,x_X \qquad \mathtt{isrc}^{(1)}_{T\to U} = \lambda\pi tx.\mathtt{isrc}^{(1)}_U\,(\mathtt{cons}\,\pi\,t)$$
$$\mathtt{isrc}^{(2)}_{T\to U} = \lambda tx.\mathtt{isrc}^{(2)}_U\,(\mathtt{app}\,t\,(\mathtt{var}\,\mathrm{z}))\left(x\,(\mathtt{var}\,\mathrm{z})\left(\mathtt{isrc}^{(1)}_T\,\mathtt{nil}\right)\right) \qquad \mathtt{isrc}^{(3)}_{T\to U} = \lambda tu\pi xvy.\mathtt{isrc}^{(3)}_U\,t\,u\,(\mathtt{cons}\,\pi\,v)\,(x\,v\,y)$$
$$\mathtt{isrc}^{(2)}_{\forall X\,T} = \lambda tx.\mathtt{elim}_{\overline{X}\mapsto\mathcal{R}ed\mathcal{C}and(\overline{X})\Rightarrow\forall t(RC_T(t)\Rightarrow t\Downarrow),\Downarrow}\left(\lambda x_X\mathtt{isrc}^{(2)}_T\right)\mathtt{normrc}\,t\left(\mathtt{elim}_{\overline{X}\mapsto\mathcal{R}ed\mathcal{C}and(\overline{X})\Rightarrow RC_T(t),\Downarrow}\,x\,\mathtt{normrc}\right)$$
$$\mathtt{isrc}^{(1)}_{\forall X\,T} = \lambda\pi x_X.\mathtt{isrc}^{(1)}_T\,\pi \qquad \mathtt{isrc}^{(3)}_{\forall X\,T} = \lambda tu\pi yx_X.\mathtt{isrc}^{(3)}_T\,t\,u\,\pi\,(y\,x_X) \qquad \mathtt{isrc}_T = \left\langle\left\langle \mathtt{isrc}^{(1)}_T, \mathtt{isrc}^{(2)}_T\right\rangle, \mathtt{isrc}^{(3)}_T\right\rangle \Vdash \mathcal{R}ed\mathcal{C}and\,(RC_T)$$

$$\mathtt{adeq}_{\Gamma\vdash \underline{m}:U} = y_U \qquad \mathtt{adeq}_{\Gamma\vdash\lambda.M:U\to T} = \lambda t_U y_U.\mathtt{isrc}^{(3)}_T\,(\mathtt{subst}\,M^{\diamond}\,(\mathrm{s}\,\mathrm{z})\,(\mathtt{shift}^*\,\vec{t_\Gamma}))\,t_U\,\mathtt{nil}\,\mathtt{adeq}_{\Gamma\vdash M:T}$$
$$\mathtt{adeq}_{\Gamma\vdash\mathtt{app}\,M\,N:T} = \mathtt{adeq}_{\Gamma\vdash M:U\to T}\,(\mathtt{subst}\,N^{\diamond}\,\mathrm{z}\,\vec{t_\Gamma})\,\mathtt{adeq}_{\Gamma\vdash N:U} \qquad \mathtt{adeq}_{\Gamma\vdash M:\forall X\,T} = \lambda x_X.\mathtt{adeq}_{\Gamma\vdash M:T}$$
$$\mathtt{adeq}_{\Gamma\vdash M:T\{U/X\}} = \mathtt{elim}_{\overline{X}\mapsto\mathcal{R}ed\mathcal{C}and(\overline{X})\Rightarrow RC_T(M[\vec{t_\Gamma}]),RC_U}\,\mathtt{adeq}_{\Gamma\vdash M:\forall X\,T}\,\mathtt{isrc}_U \qquad \mathtt{adeq}_{\Gamma\vdash M:T} \Vdash RC_T\,(M\,[\vec{t_\Gamma}])$$

$$\mathtt{norm}_{\vdash M:T} = \mathtt{isrc}^{(2)}_T\,M^{\diamond}\,\mathtt{adeq}_{\vdash M:T} \Vdash M\!\Downarrow \quad \equiv \quad \exists n\,M\downarrow^n$$

The concrete translation

$$\mathtt{dne}_{\Phi} = \lambda x.x\,(\lambda y.y) \qquad \mathtt{dne}_{\forall i\,A} = \lambda xi.\mathtt{dne}_A\,(\lambda y.x\,(\lambda z.y\,(z\,i)))$$
$$\mathtt{dne}_{\forall b\,A} = \mathtt{dne}_A \qquad \mathtt{dne}_{\forall t\,A} = \lambda xt.\mathtt{dne}_A\,(\lambda y.x\,(\lambda z.y\,(z\,t)))$$
$$\mathtt{dne}_{\forall X\,A} = \mathtt{dne}_A \qquad \mathtt{dne}_{\forall \pi\,A} = \lambda x\pi.\mathtt{dne}_A\,(\lambda y.x\,(\lambda z.y\,(z\,\pi)))$$
$$\mathtt{dne}_{A\wedge B} = \lambda x.\,\langle \mathtt{dne}_A\,(\lambda y.x\,(\lambda z.y\,(\mathrm{p}_1\,z))), \mathtt{dne}_B\,(\lambda y.x\,(\lambda z.y\,(\mathrm{p}_2\,z)))\rangle$$
$$\mathtt{dne}_{A\Rightarrow B} = \lambda xy.\mathtt{dne}_B\,(\lambda z.x\,(\lambda u.z\,(u\,y))) \qquad \mathtt{dne}_A \Vdash \neg\neg A \Rightarrow A$$

$$\mathtt{exf}_A = \lambda x.\mathtt{dne}_A\,(\lambda\_.x) \Vdash \mathit{ff} \Rightarrow A$$

$$\mathtt{normrc} = \langle\langle \lambda\pi x.x\,\mathrm{z}, \lambda tx.x\rangle, \lambda tu\pi xy.x\,(\lambda i.y\,(\mathrm{s}\,i))\rangle \Vdash \mathcal{R}ed\mathcal{C}and\,(\Downarrow)$$

$$\mathtt{repl'}_{\overline{X}\mapsto M\in X} = x\,M^{\diamond} \qquad \mathtt{repl'}_{\overline{X}\mapsto \Phi} = \langle \lambda y.y, \lambda y.y\rangle \text{ if } \Phi \not\equiv M \in X \qquad \mathtt{repl'}_{\forall X\,A} = \mathtt{repl'}_A \qquad \mathtt{repl'}_{\forall b\,A} = \mathtt{repl'}_A$$
$$\mathtt{repl'}_{\forall i\,A} = \langle \lambda yi.\mathrm{p}_1\,\mathtt{repl'}_A\,(y\,i), \lambda yi.\mathrm{p}_2\,\mathtt{repl'}_A\,(y\,i)\rangle \qquad \mathtt{repl'}_{\forall t\,A} = \langle \lambda yt.\mathrm{p}_1\,\mathtt{repl'}_A\,(y\,t), \lambda yt.\mathrm{p}_2\,\mathtt{repl'}_A\,(y\,t)\rangle$$
$$\mathtt{repl'}_{\forall \pi\,A} = \langle \lambda y\pi.\mathrm{p}_1\,\mathtt{repl'}_A\,(y\,\pi), \lambda y\pi.\mathrm{p}_2\,\mathtt{repl'}_A\,(y\,\pi)\rangle$$
$$\mathtt{repl'}_{A_1\Rightarrow A_2} = \left\langle \lambda yz.\mathrm{p}_1\,\mathtt{repl'}_{A_2}\left(y\left(\mathrm{p}_2\,\mathtt{repl'}_{A_1}\,z\right)\right), \lambda yz.\mathrm{p}_2\,\mathtt{repl'}_{A_2}\left(y\left(\mathrm{p}_1\,\mathtt{repl'}_{A_1}\,z\right)\right)\right\rangle$$
$$\mathtt{repl'}_{A_1\wedge A_2} = \left\langle \lambda y.\left\langle \mathrm{p}_1\,\mathtt{repl'}_{A_1}\,(\mathrm{p}_1\,y), \mathrm{p}_1\,\mathtt{repl'}_{A_2}\,(\mathrm{p}_2\,y)\right\rangle, \lambda y.\left\langle \mathrm{p}_2\,\mathtt{repl'}_{A_1}\,(\mathrm{p}_1\,y), \mathrm{p}_2\,\mathtt{repl'}_{A_2}\,(\mathrm{p}_2\,y)\right\rangle\right\rangle$$
$$\mathtt{repl}_A = \lambda x.\mathtt{repl'}_A \Vdash \forall t\,(B\,(t) \Leftrightarrow C\,(t)) \Rightarrow (A\,(B) \Leftrightarrow A\,(C))$$

$$\mathtt{comp}_A = \lambda x.\mathtt{brec}\,(\lambda y.\mathtt{exf}_A\,(y\,\langle \mathtt{exf}_A, \lambda u.y\,\langle \lambda\_.u, \lambda\_.\mathrm{z}\rangle\rangle))\,x\,\{\} \Vdash \exists X\,\forall t\,(t \in X \Leftrightarrow A\,(t))$$

$$\mathtt{elim}_{A,B} \Vdash \forall X\,A(\overline{X}) \Rightarrow A\,(B)$$

$$\mathtt{isrc}^{(1)}_X = \mathrm{p}_1\,(\mathrm{p}_1\,x_X) \qquad \mathtt{isrc}^{(2)}_X = \mathrm{p}_2\,(\mathrm{p}_1\,x_X) \qquad \mathtt{isrc}^{(3)}_X = \mathrm{p}_2\,x_X \qquad \mathtt{isrc}^{(1)}_{T\to U} = \lambda\pi tx.\mathtt{isrc}^{(1)}_U\,(\mathtt{cons}\,\pi\,t)$$
$$\mathtt{isrc}^{(2)}_{T\to U} = \lambda tx.\mathtt{isrc}^{(2)}_U\,(\mathtt{app}\,t\,(\mathtt{var}\,\mathrm{z}))\left(x\,(\mathtt{var}\,\mathrm{z})\left(\mathtt{isrc}^{(1)}_T\,\mathtt{nil}\right)\right) \qquad \mathtt{isrc}^{(3)}_{T\to U} = \lambda tu\pi xvy.\mathtt{isrc}^{(3)}_U\,t\,u\,(\mathtt{cons}\,\pi\,v)\,(x\,v\,y)$$
$$\mathtt{isrc}^{(2)}_{\forall X\,T} = \lambda tx.\mathtt{elim}_{\overline{X}\mapsto \mathcal{R}ed\mathcal{C}and(\overline{X})\Rightarrow\forall t(RC_T(t)\Rightarrow t\Downarrow),\Downarrow}\left(\lambda x_X \mathtt{isrc}^{(2)}_T\right)\mathtt{normrc}\,t\left(\mathtt{elim}_{\overline{X}\mapsto \mathcal{R}ed\mathcal{C}and(\overline{X})\Rightarrow RC_T(t),\Downarrow}\,x\,\mathtt{normrc}\right)$$
$$\mathtt{isrc}^{(1)}_{\forall X\,T} = \lambda\pi x_X.\mathtt{isrc}^{(1)}_T\,\pi \qquad \mathtt{isrc}^{(3)}_{\forall X\,T} = \lambda tu\pi y x_X.\mathtt{isrc}^{(3)}_T\,t\,u\,\pi\,(y\,x_X) \qquad \mathtt{isrc}_T = \left\langle\left\langle \mathtt{isrc}^{(1)}_T, \mathtt{isrc}^{(2)}_T\right\rangle, \mathtt{isrc}^{(3)}_T\right\rangle \Vdash \mathcal{R}ed\mathcal{C}and\,(RC_T)$$

$$\mathtt{adeq}_{\Gamma\vdash \underline{m}:U} = y_U \qquad \mathtt{adeq}_{\Gamma\vdash\lambda.M:U\to T} = \lambda t_U y_U.\mathtt{isrc}^{(3)}_T\,(\mathtt{subst}\,M^{\diamond}\,(\mathrm{s}\,\mathrm{z})\,(\mathtt{shift}^*\,\vec{t_\Gamma}))\,t_U\,\mathtt{nil}\,\mathtt{adeq}_{\Gamma\vdash M:T}$$
$$\mathtt{adeq}_{\Gamma\vdash \mathrm{app}\,M\,N:T} = \mathtt{adeq}_{\Gamma\vdash M:U\to T}\left(\mathtt{subst}\,N^{\diamond}\,\mathrm{z}\,\vec{t_\Gamma}\right)\mathtt{adeq}_{\Gamma\vdash N:U} \qquad \mathtt{adeq}_{\Gamma\vdash M:\forall X\,T} = \lambda x_X.\mathtt{adeq}_{\Gamma\vdash M:T}$$
$$\mathtt{adeq}_{\Gamma\vdash M:T\{U/X\}} = \mathtt{elim}_{\overline{X}\mapsto \mathcal{R}ed\mathcal{C}and(\overline{X})\Rightarrow RC_T(M[\vec{t_\Gamma}]),RC_U}\,\mathtt{adeq}_{\Gamma\vdash M:\forall X\,T}\,\mathtt{isrc}_U \qquad \mathtt{adeq}_{\Gamma\vdash M:T} \Vdash RC_T\,(M\,[\vec{t_\Gamma}])$$

$$\mathtt{norm}_{\vdash M:T} = \mathtt{isrc}^{(2)}_T\,M^{\diamond}\,\mathtt{adeq}_{\vdash M:T} \Vdash M{\downarrow} \quad\equiv\quad \exists n\,M\downarrow^n$$

The concrete translation

$$\mathtt{dne}_{\Phi} = \lambda x.x\,(\lambda y.y) \qquad \mathtt{dne}_{\forall i\,A} = \lambda xi.\mathtt{dne}_A\,(\lambda y.x\,(\lambda z.y\,(z\,i)))$$
$$\mathtt{dne}_{\forall b\,A} = \mathtt{dne}_A \qquad \mathtt{dne}_{\forall t\,A} = \lambda xt.\mathtt{dne}_A\,(\lambda y.x\,(\lambda z.y\,(z\,t)))$$
$$\mathtt{dne}_{\forall X\,A} = \mathtt{dne}_A \qquad \mathtt{dne}_{\forall \pi\,A} = \lambda x\pi.\mathtt{dne}_A\,(\lambda y.x\,(\lambda z.y\,(z\,\pi)))$$
$$\mathtt{dne}_{A\wedge B} = \lambda x.\left\langle \mathtt{dne}_A\,(\lambda y.x\,(\lambda z.y\,(\mathrm{p}_1\,z)))\,,\mathtt{dne}_B\,(\lambda y.x\,(\lambda z.y\,(\mathrm{p}_2\,z)))\right\rangle$$
$$\mathtt{dne}_{A\Rightarrow B} = \lambda xy.\mathtt{dne}_B\,(\lambda z.x\,(\lambda u.z\,(u\,y))) \qquad \mathtt{dne}_A \Vdash \neg\neg A \Rightarrow A$$

$$\mathtt{exf}_A = \lambda x.\mathtt{dne}_A\,(\lambda\_.x) \Vdash \mathit{ff} \Rightarrow A$$

$$\mathtt{normrc} = \langle\langle \lambda\pi x.x\,\mathrm{z}, \lambda tx.x\rangle\,, \lambda tu\pi xy.x\,(\lambda i.y\,(\mathrm{s}\,i))\rangle \Vdash \mathcal{R}ed\mathcal{C}and\,(\Downarrow)$$

$$\mathtt{repl}'_{\overline{X}\mapsto M\in X} = x\,M^{\diamond} \qquad \mathtt{repl}'_{\overline{X}\mapsto \Phi} = \langle \lambda y.y, \lambda y.y\rangle \text{ if } \Phi \not\equiv M\in X \qquad \mathtt{repl}'_{\forall X\,A} = \mathtt{repl}'_A \qquad \mathtt{repl}'_{\forall b\,A} = \mathtt{repl}'_A$$
$$\mathtt{repl}'_{\forall i\,A} = \langle \lambda yi.\mathrm{p}_1\,\mathtt{repl}'_A\,(y\,i)\,, \lambda yi.\mathrm{p}_2\,\mathtt{repl}'_A\,(y\,i)\rangle \qquad \mathtt{repl}'_{\forall t\,A} = \langle \lambda yt.\mathrm{p}_1\,\mathtt{repl}'_A\,(y\,t)\,, \lambda yt.\mathrm{p}_2\,\mathtt{repl}'_A\,(y\,t)\rangle$$
$$\mathtt{repl}'_{\forall \pi\,A} = \langle \lambda y\pi.\mathrm{p}_1\,\mathtt{repl}'_A\,(y\,\pi)\,, \lambda y\pi.\mathrm{p}_2\,\mathtt{repl}'_A\,(y\,\pi)\rangle$$
$$\mathtt{repl}'_{A_1\Rightarrow A_2} = \left\langle \lambda yz.\mathrm{p}_1\,\mathtt{repl}'_{A_2}\left(y\left(\mathrm{p}_2\,\mathtt{repl}'_{A_1}\,z\right)\right), \lambda yz.\mathrm{p}_2\,\mathtt{repl}'_{A_2}\left(y\left(\mathrm{p}_1\,\mathtt{repl}'_{A_1}\,z\right)\right)\right\rangle$$
$$\mathtt{repl}'_{A_1\wedge A_2} = \left\langle \lambda y.\left\langle \mathrm{p}_1\,\mathtt{repl}'_{A_1}\,(\mathrm{p}_1\,y)\,, \mathrm{p}_1\,\mathtt{repl}'_{A_2}\,(\mathrm{p}_2\,y)\right\rangle, \lambda y.\left\langle \mathrm{p}_2\,\mathtt{repl}'_{A_1}\,(\mathrm{p}_1\,y)\,, \mathrm{p}_2\,\mathtt{repl}'_{A_2}\,(\mathrm{p}_2\,y)\right\rangle\right\rangle$$
$$\mathtt{repl}_A = \lambda x.\mathtt{repl}'_A \Vdash \forall t\,(B\,(t) \Leftrightarrow C\,(t)) \Rightarrow (A\,(B) \Leftrightarrow A\,(C))$$

$$\mathtt{comp}_A = \lambda x.\mathtt{brec}\,(\lambda y.\mathtt{exf}_A\,(y\,\langle \mathtt{exf}_A, \lambda u.y\,\langle \lambda\_.u, \lambda\_.\mathrm{z}\rangle\rangle))\,x\,\{\} \Vdash \exists X\,\forall t\,(t\in X \Leftrightarrow A\,(t))$$

$$\mathtt{elim}_{A,B} = \lambda x.\mathtt{dne}_{A(B)}\,(\lambda y.\mathtt{comp}_B\,(\lambda z.y\,(\mathrm{p}_1\,(\mathtt{repl}_A\,z)\,x))) \Vdash \forall X\,A(\overline{X}) \Rightarrow A\,(B)$$

$$\mathtt{isrc}^{(1)}_X = \mathrm{p}_1\,(\mathrm{p}_1\,x_X) \qquad \mathtt{isrc}^{(2)}_X = \mathrm{p}_2\,(\mathrm{p}_1\,x_X) \qquad \mathtt{isrc}^{(3)}_X = \mathrm{p}_2\,x_X \qquad \mathtt{isrc}^{(1)}_{T\to U} = \lambda\pi tx.\mathtt{isrc}^{(1)}_U\,(\mathtt{cons}\,\pi\,t)$$
$$\mathtt{isrc}^{(2)}_{T\to U} = \lambda tx.\mathtt{isrc}^{(2)}_U\,(\mathtt{app}\,t\,(\mathtt{var}\,\mathrm{z}))\left(x\,(\mathtt{var}\,\mathrm{z})\left(\mathtt{isrc}^{(1)}_T\,\mathtt{nil}\right)\right) \qquad \mathtt{isrc}^{(3)}_{T\to U} = \lambda tu\pi xvy.\mathtt{isrc}^{(3)}_U\,t\,u\,(\mathtt{cons}\,\pi\,v)\,(x\,v\,y)$$
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$$\mathtt{isrc}^{(1)}_{\forall X\,T} = \lambda\pi x_X.\mathtt{isrc}^{(1)}_T\,\pi \qquad \mathtt{isrc}^{(3)}_{\forall X\,T} = \lambda tu\pi y x_X.\mathtt{isrc}^{(3)}_T\,t\,u\,\pi\,(y\,x_X)$$

is $\mathtt{isrc}_T \Vdash \mathcal{R}ed\mathcal{C}and\,(RC_T)$ $RC_T)$

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$$\mathtt{adeq}_{\Gamma\vdash M:T\{U/X\}} = \mathtt{elim}_{\overline{X}\mapsto\mathcal{R}ed\mathcal{C}and(\overline{X})\Rightarrow RC_T(M[\vec{t_\Gamma}]),RC_U}\,\mathtt{adeq}_{\Gamma\vdash M:\forall X}$$

$$\mathtt{adeq}_{\Gamma\vdash M:T} \Vdash RC_T\left(M\left[\vec{t_\Gamma}\right]\right)$$

$$\mathtt{norm}_{\vdash M:T} = \mathtt{isrc}^{(2)}_T\,M^{\diamond}\,\mathtt{adeq}_{\vdash M:T} \Vdash M\!\downarrow \quad\equiv\quad \exists n\,M\downarrow^n$$

The concrete translation

$$\mathtt{dne}_{\Phi} = \lambda x.x\,(\lambda y.y) \qquad \mathtt{dne}_{\forall i\,A} = \lambda xi.\mathtt{dne}_A\,(\lambda y.x\,(\lambda z.y\,(z\,i)))$$
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$$\mathtt{repl}'_{A_1\wedge A_2} = \left\langle \lambda y.\left\langle \mathrm{p}_1\,\mathtt{repl}'_{A_1}\,(\mathrm{p}_1\,y)\,, \mathrm{p}_1\,\mathtt{repl}'_{A_2}\,(\mathrm{p}_2\,y)\right\rangle, \lambda y.\left\langle \mathrm{p}_2\,\mathtt{repl}'_{A_1}\,(\mathrm{p}_1\,y)\,, \mathrm{p}_2\,\mathtt{repl}'_{A_2}\,(\mathrm{p}_2\,y)\right\rangle\right\rangle$$
$$\mathtt{repl}_A = \lambda x.\mathtt{repl}'_A \Vdash \forall t\,(B\,(t) \Leftrightarrow C\,(t)) \Rightarrow (A\,(B) \Leftrightarrow A\,(C))$$

$$\mathtt{comp}_A = \lambda x.\mathtt{brec}\,(\lambda y.\mathtt{exf}_A\,(y\,\langle \mathtt{exf}_A, \lambda u.y\,\langle \lambda\_.u, \lambda\_.\mathrm{z}\rangle\rangle))\,x\,\{\} \Vdash \exists X\,\forall t\,(t \in X \Leftrightarrow A\,(t))$$

$$\mathtt{elim}_{A,B} = \lambda x.\mathtt{dne}_{A(B)}\,(\lambda y.\mathtt{comp}_B\,(\lambda z.y\,(\mathrm{p}_1\,(\mathtt{repl}_A\,z)\,x))) \Vdash \forall X\,A(\overline{X}) \Rightarrow A\,(B)$$

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$\mathtt{norm}_{\vdash M:T}\,(\lambda x.x) \succ^* n$ in System T + bar recursion
where n is such that M reaches a normal form in at most n steps

Conclusion

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