Believe it or not,

AJM's games model is a model of classical Linear Logic.

Patrick Baillot
IML-CNRS Marseille

Vincent Danos
Université Paris 7-CNRS

Thomas Ehrhard
IML-CNRS Marseille

Laurent Regnier
IML-CNRS Marseille

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Abstract

A general category of games is constructed, adapting and extending [1, 2]. Then, a subcategory of *saturated* strategies, closed under all possible codings in copy games, is shown to model reduction in classical Linear Logic.

1 Introduction

The expectations of games semantics. Game theory long ago brought to the fore the concept of *strategy*. Many developments happened since Von Neumann's minimax theorem, but in the eye of the computer scientist, strategies have a striking property which long stayed unexploited.

Strategies can either be thought of as single decisions in so-called matrix games:

$$\begin{pmatrix} 1,1 & 2,-5 \\ -5,2 & -9,-9 \end{pmatrix}$$

where a strategy simply is a column or a row and interaction is resolved by taking the intersection of columns and rows.[1]

Or they can be thought of as subtrees in arboreal games and interaction then consists in intersecting subtrees: a process usually visualized as a communication between players where they announce successive moves in their subtrees.

Or they can be seen as procedures by which a player, given the position of the game, produces a next move. And there, interaction unmistakably seems a real computational interaction between agents, following a protocol described by the rules of the game.

The strength of games semantics which interprets programs as strategies lies in this threefold nature of them. It unifies the operational and the extensional.

Many recent results testify for the efficiency of the approach. Games categories were built for lazy lambda-calculus, classical PCF and imperative languages (respectively [3], [21, 18] and [4]). Probably many more would be necessary to really bridge the gap in between.

An account of the present contribution. Our own contribution is to break the intuitionistic circle in which games semantics seemed trapped.

More precisely, we give a proof that a slight but effective variant of AJM's celebrated PCF games model turns it into a model of *classical* Linear Logic (thereafter simply LL).

The model first is simplified in that we get rid of the mathematically unnecessary commitment that strategies always wait for Opponent, to be thought of as the context, to begin. There is more here

[1] In this game, greedy players both lose.

than just a convenience since concrete interaction in LL as embodied in Girard's geometry of interaction (thereafter simply GoI) is symmetric. It is then natural to ask for a model with a full duality, that is where the context is on the same foot as agents are. Technically, our model makes it now possible for strategies in a game A to interact with strategies in the dual game $A^{\perp}$.

Note that in the intuitionistic case the issue is somewhat less vital since agents are themselves not symmetric: there is the function and there are the arguments and one never needs a full duality.

The second progress reported here is in the finding out of the right notion of equivalence between strategies which can cope with the classical case. The trick is to forget about narrow determinism, and saturate strategies until they are able to manage copy-tags in *all* possible AJM equivalent ways. That is, whatever the encoding some other strategy, willing to communicate through a channel, is using to make copies, a saturated strategy will have it.

In the case of minimal intuitionistic types, this saturation of strategies can be described by using [11]'s 'pointifixion' embedding of AJM's model into HO's model, another games model defined in [19]. In the multiplicative case, no copy game is played, and all strategies are already saturated. So saturation saves determinism in the large.

In [22], Lamarche, who by the way was one of the first promoter of games in semantics, took advantage of a polarization scheme, derived from [12], to circumnavigate around problems inherent to the classical case, artfully reducing it to the intuitionistic one. This paper allows for a more straightforward navigation.

Exploitation of the model. As was shown in [5] the translation of PCF-terms, or even intuitionistic LL proof-nets, as history-free strategies in AJM's model is carried over by a typed version of the GoI interpretation (see [16, 17] for the original). So we already knew that GoI could be turned into a real invariant of computation in ILL (except for the additives).

The first subject of satisfaction is that we now have a model in which the same holds for classical LL (again without additives). This has even consequences on the concrete side of affairs since conjectural variations on the GoI are seen to be correct. Namely a variant of the Interaction Abstract Machine where operators might depend on the current B-stack is validated by the model (see [14, 23]).

But this first matter of satisfaction admittedly is for the GoI and/or optimal reductions fan (-in of course). (Let us recall that Lamping's algorithm is just a graphical computation of Girard's GoI formula. Others are given in [13].)

A second and more vague satisfaction is that this model gives for the first time a mathematically simple and intuitively digestible interpretation of LL. As the reader will experience, the first part of this paper explaining the three games constructions is real easy to understand and very natural.

Exploitation of the model (continued). Turning to potential long term applications, one might expect progresses in giving an interpretation of LL as a logic of processes, a long-standing LL prospect, since now it really is communication which is modeled, as said above.

One could even dream of a programming language with communication primitives cut out from this model. Since time is explicit in the model, agents would have access to the *run* as they may observe it and learn from that.

Another possible use to which one could put this model would be in making games semantics touch more traditional semantical ground. The intuition being that: games semantics becomes usual semantics when one forgets about time. Theorems corroborating this idea already exist in the intuitionistic case relating, for instance, sequential algorithms (which can also be presented as strategies as was shown in [9]) to hypercoherences (in [15]).

So doing, one could also build an understanding of Girard's coherence semantics which in the classical case is a perfect but still only formal apparatus. Beyond this local concern, progress towards a grand unification between the operational and the extensional is the loot, as said at the beginning.

An even farther hope would be to get applications in economics. Think of it, modern markets are getting more abstract: stock exchange, option markets. Sometimes they seem like a dialog between programs: on the one hand there is the market itself governed by rules where prices are computed using sophisticated mathematical modeling. To the point that the model is less modeling the practice than giving rules so that the practice can develop. A situation that a computer scientist might want to model by saying that a market is a kind of a communication protocol. On the other hand, there are programs analyzing signals, taking decisions and acting on the market. Could it be that games give a way to understand that spontaneous self-programming of the markets and help organizing it ? Federation with [20]'s communication and money games interpreting *additive* LL might help.

Unsolved problems. Back to technical matters: there is an obvious but not quite functorial collapse of games into a relational model (which was already studied by Malacaria in the original AJM model). This not-functor doesn't behave nicely with respect to games constructions *but* was the main source of guidance in building our games model. Badly behaved as it was, it gave evidence that saturated strategies are still able to communicate consistently.

Many questions follow. How is coherence to be understood in games concepts, how does it relate with determinism, and how closer is games semantics to catch definability than coherence semantics is.

But to install these questions in a livable framework needs further efforts. Progresses are reported in [6].

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2 Games

2.1 Basic definitions

We will consider games between a *proponent* P and an *opponent* O. Players will play alternatively, finitely many moves, thus generating some alternating sequences of moves, termed *plays* or *matches*. The empty match will be denoted by ϵ.

A *game* A here will consist in the following data:

- a set of moves M_A;
- a map λ_A from M_A to {O,P} telling which player O or P may play a move;
- a prefix-closed subset P_A of finite alternating sequences of moves;
- a length-preserving equivalence relation on matches $\equiv_A$.

2.2 Constructions on games

Let us review the three basic AJM constructions one needs to build MELL games.

Orthogonal. Given A, $A^\perp$ is defined as:

- M_A;
- $\bar{\lambda}_A$, with $\bar{\lambda}(m) = P$ if $\lambda(m) = O$ and $\bar{\lambda}(m) = O$ if $\lambda(m) = P$;
- P_A;
- $\equiv_A$.

Linear negation is here seen as the exchange of players. This operation is clearly involutive.

Times. Given A and B, $A \otimes B$ is defined as:

- $M_A + M_B$;
- $\lambda_A + \lambda_B$;
- $s \in P_{A\otimes B}$ if $s_A \in P_A$, $s_B \in P_B$ and O only switches components in s;
- $s \equiv s'$ if for all n, s and s' n-th moves are in the same component, and both $s_A \equiv s'_A$ and $s_B \equiv s'_B$.

Above s_A stands for the projection of s on A, that is the subsequence of s where only moves in A are kept. Same for s_B. Subgames A and B are often referred to as components of $A \otimes B$.

Note that even if players are both free to switch from one subgame to the other, by alternation and projection conditions, in a given match at most one of them actually does it. Indeed if the match begins in A, the first player to enter B will leave a pending move in s_A, so, by alternation, he only can reenter A. And so on.

The condition that projections on components be valid will be referred to as the projection condition.

In $A \otimes B$, O is stronger which is natural if one follows the intuitive correspondence between provability and the existence of a winning strategy. He also can make a better deal.

Consider the bargaining game B where matches are $p_0^+, p_1, \ldots, p_{2n}^+, p_{2n+1}, ok$, with (p_i) an increasing sequence and P playing p-moves. If O plays in $B \otimes \ldots \otimes B$, he can suspend his final answer, enter another subgame to check for a better offer, and come back whenever it pleases him to ask for an even better one.

Bang. Given A, $!A$ is defined as:

- $\omega \times M_A$;
- $\lambda_{!A}(im) = \lambda_A(m)$;
- $s \in P_{!A}$ if $s_i \in P_A$ and O only may switch components;
- $s \equiv s'$ if there is a permutation θ on ω such that for all n, s n-th move is in the $\theta(i)$ component iff s' n-th move is in the i component, and for all i, $s_{\theta(i)} \equiv_A s'_i$.

Above s_i stands for the projection of s on the i-th copy of A. Note the similarity between $!A$ and $\otimes^\omega A$. The only difference lies in the equivalence relation which can permute subgames in $!A$.

Again O is stronger in $!A$. It has a trivial winning strategy in $!A$ if it has one at all in A, since he can always backtrack.

2.3 Notations

Given a MELL formula $F[X_1, \ldots, X_n]$ and games $A_1, \ldots, A_n$ one can build a game $F[A_1, \ldots, A_n]$ following the constructions above and:

$$\begin{aligned} A \wp B &= (A^\perp \otimes B^\perp)^\perp, \\ ?A &= (!A^\perp)^\perp. \end{aligned}$$

Note that only switching conditions make a difference between $\otimes$ and $\wp$ on the one hand, and ! and ? on the other hand. They play the same rôle in games the coherence relation plays in coherence semantics.

From now on we will take for granted that some games have been chosen to interpret atomic types; nothing that follows depends on this choice.

If Γ is a MELL sequent $A_1, \ldots, A_n$, then the associated game is by convention $A_1 \wp \ldots \wp A_n$. If Δ is a subsequent of Γ and $s \in P_\Gamma$, then s_Δ or s/Δ, the *projection* of s on Δ, means the subsequence of s with moves in M_Δ. Thereafter we simply write $s \in \Gamma$ in place of $s \in P_\Gamma$.

3 Strategies

3.1 Further definitions

A *strategy* in A will simply be a non-empty prefix-closed set of positions in P_A.

No condition of determinism or whatsoever is asked. Whether the strategy belongs to P or O is a matter of who is using it. Yet, by convention, we ask a P-strategy to be closed by O-moves, and dually, an O-strategy to be closed by P-moves.

A P-strategy will simply be termed a strategy in the sequel.

By σ plays s we simply mean $s \in \sigma$.

In the game $A \otimes A^{\perp}$, an O-strategy of particular interest is the identity, denoted by a or id, which is the prefix-closure of the set:

$$\{s \in A \otimes A^{\perp} : s_A = s_{A^{\perp}}\}.$$

Note that the first move of $s \in a$ is a P-move m. Indeed, the first move in the other subgame, which by the switching condition is an O-move, must be equal to m; since subgames are dual, m is a P-move.

Operationnaly, a is generated by the injection:

$$\begin{array}{rcl} i(m_P : A) & = & m_O : A^{\perp}, \\ i(m_P : A^{\perp}) & = & m_O : A. \end{array}$$

It is a winning strategy for O in the minimal sense that O can always extend a play in this game.

It is also a winning strategy for P in the dual game $A^{\perp}, A$ where it interprets the logical axiom $\vdash A^{\perp}, A$, whence the name a as 'axiom'.

3.2 Composition of strategies

How does one play in Γ, Δ, given strategies in Γ, A and $A^{\perp}, \Delta$? Well, the intuitive answer is: play as your strategies and when they answer in A or $A^{\perp}$, make them play against each other. Intuitive as it may be, this operational definition is quite cumbersome. We give below an extensional version of it.

Lemma 1 (Zipping) *Let s_1 and s_2 respectively be in Γ, A and $A^{\perp}, \Delta$ and be such that $s_{1A} = s_{2A^{\perp}} \neq \epsilon$. There is a* unique *$s \in \Gamma, A \otimes A^{\perp}, \Delta$ such that:*

$$\begin{array}{rcl} s_{\Gamma, A} & = & s_1, \\ s_{A^{\perp}, \Delta} & = & s_2, \end{array}$$

and $s_{A \otimes A^{\perp}}$ is played by id on the O-side. This s we denote by $s_1; s_2$.

Proof: Note that $s_{1A} = s_{2A^{\perp}} \neq \epsilon$ implies one of s_1 and s_2, has a last P-move, say m_P, in A, $A^{\perp}$. Say it is s_1. This last move m_P in A is also s_1's last move *at all*, since playing again in Γ afterwards is P's prerogative, by the switching condition, and to do this P needs an O-move in A just before, by the alternation condition, an O-move which there is not. So $s_1 = s'_1 m_P$, and since that same move m_P is played as m_O in $A^{\perp}$ in s_2 we can also decompose $s_2 = s'_2 m_O s''_2$.

If both s'_1 and s'_2 have non empty projections on A and $A^{\perp}$ respectively, then one can inductively apply the lemma which gives a unique s' and then put $s = s' m_P m_O s''_2$.

First let us see that s is correct: to check this we only have to bother about possible wrong switches since the projections in all immediate subgames are by assumption correct. It is correct for P to play in A after s' because Γ, A is a P-switching subgame, then m_O is correct since subgame $A \otimes A^{\perp}$ is O-switching and then one plays as in s_2 so s is correct.

Second we have to show that s is the unique solution. Since, as said, m_P is s_1's last move, and any solution has to play m_O after m_P, because the identity is played on O-side in the subgame $A \otimes A^{\perp}$, and thereafter s''_2, any solution ends with $m_P m_O s''_2$. So the proposed s is the only solution.

If either s'_1 or s'_2 have empty projections on A, $A^{\perp}$ then both have and s'_1 is wholly in Γ and $s'_2 = \epsilon$ again by the same argument, so put $s = s'_1 m_P m_O s''_2 = s_1 s_2$. Again this is correct and the only solution. □

If one lets O play freely in the subgame $A \otimes A^{\perp}$, then there is still a solution, of course, but usually a lots of (except if repetition of moves is forbidden).

Let s_1 and s_2 be in Γ, A and $A^{\perp}, \Delta$ respectively, we extend the notation introduced in the lemma, to the case where $s_{1A} = s_{2A^{\perp}} = \epsilon$, by $s_1; \epsilon = s_1$, $\epsilon; s_2 = s_2$ and $s_1; s_2 = \epsilon$ if both $s_1 \neq \epsilon$ and $s_2 \neq \epsilon$.

Definition 1 *If σ is a strategy in Γ, A and τ a strategy in $A^{\perp}, \Delta$ we may define:*

$$\begin{array}{rl} \sigma; \tau = \{s_1; s_2 / \Gamma, \Delta, & s_1 \in \sigma, s_2 \in \tau, \\ & s_{1A} = s_{2A^{\perp}}\} \end{array}$$

It is easy to check that $s_1; s_2 / \Gamma, \Delta$ is correct when s_1 and s_2 are. Thus one can compose strategies. We still have to show that such a formula is associative.

Note that a play in $\sigma; \tau$ might either be an *active* composition, when $s_{1A} = s_{2A^{\perp}} \neq \epsilon$, or a *passive* one, when $s_{1A} = s_{2A^{\perp}} = \epsilon$, in this case, composition succeeds in producing more than ϵ iff one of s_1 and s_2 is empty.

3.3 Associativity of composition

Theorem 1 *Composition is associative; that is, if σ_1 is a strategy in Γ, A, σ_2 a strategy in $A^\perp, \Delta, B$ and σ_3 a strategy in $B^\perp, \Sigma$, then:*

$$(\sigma_1; \sigma_2); \sigma_3 = \sigma_1; (\sigma_2; \sigma_3).$$

To prove this we need a ternary version of the zipping lemma:

Lemma 2 (Ternary zipping) *Let s_1, s_2 and s_3 respectively be in Γ, A, $A^\perp, \Delta, B$ and $B^\perp, \Sigma$ and be such that: $s_{1A} = s_{2A^\perp} \neq \epsilon$ and $s_{2B} = s_{3B^\perp} \neq \epsilon$. There is a* unique *$s \in \Gamma, A \otimes A^\perp, \Delta, B \otimes B^\perp, \Sigma$ such that:*

$$\begin{aligned} s_{\Gamma,A} &= s_1, \\ s_{A^\perp,\Delta,B} &= s_2, \\ s_{B^\perp,\Sigma} &= s_3, \end{aligned}$$

and both $s_{A\otimes A^\perp}$ and $s_{B\otimes B^\perp}$ are played by id on the O-side. This s we denote by $s_1; s_2; s_3$.

Proof: That such an s exists follows from the binary lemma, just take $s = (s_1; s_2); s_3$ or $s = s_1; (s_2; s_3)$. So we only have to show there is a unique solution. We distinguish four cases.

1. s_1's last move in A is a P-move m_P and s_2's last move in B is a P-move m'_P.

By the switching conditions, as in the binary case, we know m_P is s_1's last move and m'_P is s_2's last move. Since any solution after m_P in A plays m_O in $A^\perp$ and after m'_P in B plays m'_O in $B^\perp$ (because identity plays in these subgames on O-side) and plays m'_P after m_O since m'_P is already after m_O in s_2 (because it is s_2's last move), we deduce that any solution ends with $m'_P m'_O s''_3$ with s''_3 wholly in Σ. Define then $s'_1 = s_1$, $s_2 = s'_2 m'_P$ and $s_3 = s'_3 m'_O s''_3$.

If both s'_2 and s'_3 have non empty projections on B and $B^\perp$ respectively, then one can inductively apply the ternary lemma to s'_1, s'_2 and s'_3 which gives a unique s' ending where s'_2 does, and then put $s = s' m'_P m'_O s''_3$ which will be correct and unique since as said any solution ends with $m'_O s''_3$.

If either s'_2 or s'_3 have empty projections on B, $B^\perp$ then both have and s'_2 is wholly in $A^\perp, \Delta$ and $s'_3 = \epsilon$ because then m'_O is s_3's first move again by the same argument, so put $s = (s_1; s'_2) m'_P m'_O s''_3 = (s_1; s_2) s_3$ the unique solution by the binary lemma applied to s_1 and s'_2.

2. s_1's last move in A is a P-move m_P and s_3's last move in $B^\perp$ is a P-move m'_P.

2.1. Suppose s_2's last move m_O in $A^\perp$ comes before s_2's last move m'_O in B in s_2. Then, arguing in the usual way, any solution ends with $m'_P m'_O s''_2$ with s''_2 wholly in Δ. Define $s'_1 = s_1$, $s_2 = s'_2 m'_O s''_2$ and $s_3 = s'_3 m'_P$.

Again if both s'_2 and s'_3 have non empty projections on B and $B^\perp$ respectively, then one applies the ternary lemma to s'_1, s'_2 and s'_3 which gives a unique s' ending where s'_3 does, and then put $s = s' m'_P m'_O s''_2$ which is unique since, as said, any solution ends with $m'_O s''_2$.

If either s'_2 or s'_3 have empty projections on B, $B^\perp$ by the, by now, usual argument $s'_2 = \epsilon$ which contradicts the fact that $m_O \in s'_2$.

2.2. Suppose s_2's last move m_O in $A^\perp$ comes after s_2's last move m'_O in B in s_2. That case is argued about as 2.1., exchanging s_1 and s_3.

3. s_2's last move in $A^\perp$ is a P-move and s_2's last move in B is a P-move. This case is impossible since there would be two moves for s_2 to end with, which is one too many.

4. The fourth case is argued about as the first one, exchanging s_1 and s_3. □

Now we go for the proof of associativity of composition.

Proof: Take $s \in (\sigma_1; \sigma_2); \sigma_3$. By definition there are $s_{12} \in \sigma_1; \sigma_2$ and $s_3 \in \sigma_3$ such that $s = s_{12}; s_3 / \Gamma, \Delta, \Sigma$, and by definition again there are $s_1 \in \sigma_1$ and $s_2 \in \sigma_2$ such that $s_{12} = s_1; s_2 / \Gamma, \Delta, B$. We distinguish three cases.

1. Suppose both compositions are *active*, that is $s_{1A} = s_{2A^\perp} \neq \epsilon$ and $s_{2B} = s_{3B^\perp} \neq \epsilon$. Then unicity in the ternary lemma says $(s_1; s_2); s_3 = s_1; (s_2; s_3)$, hence $s = s_1; (s_{23}) / \Gamma, \Delta, \Sigma$ with $s_{23} = s_2; s_3 / A^\perp, \Delta, \Sigma$ and belongs to $\sigma_1; (\sigma_2; \sigma_3)$.

2. Suppose the B composition is *passive*, that is $s_{2B} = s_{3B^\perp} = \epsilon$. Then either $s_3 = \epsilon$ or $s_{12} = \epsilon$.

2.1. If $s_3 = \epsilon$, then $s_2 = s_2; \epsilon = s_2; \epsilon / A^\perp, \Delta$ since $s_{2B} = \epsilon$ and $s = (s_1; s_2) / \Gamma, \Delta$ belongs to $\sigma_1; (\sigma_2; \sigma_3)$.

2.2. If $s_{12} = \epsilon$, then $s = s_3 = s_3/\Sigma$ belongs to $\sigma_1; (\sigma_2; \sigma_3)$ because ϵ belongs to σ_2 so that $s_3 \in \sigma_2; \sigma_3$ and ϵ belongs to σ_1 so that $s_3 \in \sigma_1; (\sigma_2; \sigma_3)$.

3. Suppose finally the B composition is *active*, and the A one is *passive*. So $s_{1A} = s_{2A^\perp} = \epsilon$. Then either $s_1 = \epsilon$ or $s_2 = \epsilon$.

3.1. If $s_1 = \epsilon$, $s = (\epsilon; s_2); s_3/\Delta, \Sigma = \epsilon; (s_2; s_3)/\Delta, \Sigma$ since $s_{2A^\perp} = \epsilon$. Thus s belongs to $\sigma_1; (\sigma_2; \sigma_3)$.

3.2. If $s_2 = \epsilon$ then the B composition is passive and we assume it is not. □

Observe that something happens in case 2.2. Namely that $(s_1; s_2); s_3 = s_3$ while $s_1; (s_2; s_3) = \epsilon$. Associativity nevertheless holds because ϵ belongs to any strategy.

Also note that there is no global orientation of $\sigma_1; \sigma_2; \sigma_3$. Matches may begin anywhere in Γ, Δ and Σ.

3.4 The identity

The *id* strategy in $A^\perp, A$ indeed is the unit for composition, that is, for any strategy σ in Γ, A:

$$\sigma; id = \sigma$$

The proof is in the same vein as the proof of the zipping lemma. Be aware that the equality only holds because of the convention that strategies be closed under O-moves, which is why we need it.

4 Saturated strategies

By now we have a category of games. Games are the objects and morphisms are strategies in $A^\perp, B$.

Let us turn it into a model of Linear Logic.

Definition 2 *Set:*

$$[\sigma] = \{s : \exists s' \in \sigma, s \equiv s'\}.$$

Say a strategy is saturated *if* $[\sigma] = \sigma$.

Note that a such strategy is not deterministic. Still, since $\equiv$ respects interleaving and only forgets about the names of the threads in the exponential constructs, time exists within $[\sigma]$.

Also note that in the special case of simple types $[\sigma]$ is the set of all digitalisations of a same HO set of matches as was shown in [11]. Since in this case two matches pointify to the same HO match iff they are equivalent. Finally note that in MLL only s is equivalent to s, so saturation is the identity.

First we see that determinism is only relaxed and not lost.

4.1 The category of saturated strategies

Proposition 3 *If σ and τ are saturated then so is $\sigma; \tau$.*

Proof: Suppose $s \in \sigma; \tau$ and consider $t \equiv s$. By definition, there is an $s' \in \sigma \otimes \tau$ such that $s = s'_{\Gamma,\Delta}$, $s'_A = s'_{A^\perp}$ and $s'_{A\otimes A^\perp}$ is played by the identity on the O-side. Define t' as the play which coincides with s' on A, $A^\perp$ and with t on Γ, Δ; in other words $t' = s'[t/s]$. Then $t'_{\Gamma,A} \equiv s'_{\Gamma,A}$, so $t'_{\Gamma,A} \in \sigma$ since σ is saturated; for the same reason $t'_{A^\perp,\Delta} \in \tau$. So $t' \in \sigma \otimes \tau$, by construction $t = t'_{\Gamma,\Delta}$ and $t'_A = s'_A = s'_{A^\perp} = t'_{A^\perp}$. So by definition $t \in \sigma; \tau$. □

So it makes sense to compose saturated strategies. Associativity obviously still holds, but we still have to take care of units.

Lemma 4 (Synchronization) *If σ and τ are composable strategies then:*

$$[\sigma]; [\tau] = [[\sigma]; \tau] = [\sigma; [\tau]]$$

The proof follows the same argument as the proposition above. Note that is is not true that $[\sigma]; [\tau] = [\sigma; \tau]$. In fact only the right to left inclusion holds, because in the left hand-side strategies are allowed to communicate up to equivalence.

Proposition 5 *The saturation $[id]$ of the former unit id is the prefix-closure of the set:*

$$\{s \in A^{\perp}, A : s_{A^{\perp}} \equiv s_A\}.$$

It is the unit in the category of saturated strategies.

Proof: The first statement is obvious.

For the second one, take a saturated σ, so that $\sigma = [\sigma]$. Then $\sigma; [id] = [\sigma]; [id]$, which is equal to $[[\sigma]; id]$ by the synchronization lemma, which is $[[\sigma]] = \sigma$ because id was the unit. □

The category of saturated strategies is not exactly a subcategory of the raw category, because it does not have the same units.

Looking at the new unit $[id]$, one sees what composition is in the new category; it consists in the old composition 'scattered' by the equivalence on matches.

Lemma 6 *Set $\sigma \leq \tau$ if for all $s \in \sigma$ there is $s' \in \tau$ such that $s \equiv s'$. If $\sigma \leq \tau$, then $[\sigma] \subseteq [\tau]$. And conversely.*

We will use this convenient way to test equality of saturated strategies in the proof of invariance by reduction below. Don't be impressed by the notation since σ might be squarely bigger than τ, for instance $[\sigma] \leq \sigma$.

5 The model

The model as expected is that of games and saturated strategies.

5.1 Interpretation of multiplicatives

The interpretation of multiplicative rules should be pretty obvious. Let us just define the $\otimes$-functor.

Given σ_i in Γ_i, A_i, $i = 1, 2$, one defines:

$$\begin{aligned}\sigma_1 \otimes \sigma_2 = \{s \in \Gamma_1, \Gamma_2, A_1 \otimes A_2 : {} & s/\Gamma_i, \\ & A_i \in \sigma_i, \\ & \text{O switches between } s/\Gamma_i, A_i\text{'s}\}.\end{aligned}$$

Note that, in general, given a play $s \in \Gamma_1, \Gamma_2, A_1 \otimes A_2$, the projections $s/\Gamma_i, A_i$ need not be valid matches. It is only in the case above that they are, because $\sigma_1 \otimes \sigma_2$ can be equivalently defined, from the operational viewpoint, as the strategy that plays like σ_i in the subgame Γ_i, A_i.

5.2 Interpretation of exponentials

We give the GoI interpretation of dereliction, contraction, weakening and promotion, as partial injections on moves, as adapted to the typed case by AJM. And take then the saturation of the history-free strategy generated by these partial injections to zero in our intended strategy.

dereliction: take k an integer, say 0 to simplify notation, then $[d]$ is the saturation of the strategy d in $?A^{\perp}, A$ generated by the following partial injection:
$d(m) = 0m$ for $m \in M_A^O$, and inversely
$d(0m) = m$ for $m \in M_A^P$.

contraction: take $l+r$ a bijection from $\omega+\omega$ to ω, then $[c]$ is the saturation of the strategy in $?A^{\perp}, !A \otimes !A$ generated by the following partial injection:
$c(im) = l(i)m$ for $m \in M_A^O$ and im played in the left component of $!A \otimes !A$,
$c(im) = r(i)m$ for $m \in M_A^O$ and im played in the right component of $!A \otimes !A$, and inversely
$c(im) = l^*(i)m$ played in the left component of $!A \otimes !A$ when $l^*(i)$ is defined and $m \in M_A^P$,
$c(im) = r^*(i)m$ played in the right component of $!A \otimes !A$ when $r^*(i)$ is defined and $m \in M_A^P$.

weakening: $[w]$ only contains the empty match in $?A^{\perp}$.

promotion: take '$\lceil , \rceil$' a bijection from $\omega \times \omega$ to ω, and let $s \in ?\Gamma, !A$, we extend our notation pertaining to plays in a !/? game by defining s_i to be obtained by, first, selecting the subsequence of s consisting of moves im in $M_{!A}$ and moves $\lceil i, j \rceil m$ in $M_{?\Gamma}$ for any $j \in \omega$, and second, replacing im's by m's and $\lceil i, j \rceil m$'s by jm's.

Given a strategy σ in $?\Gamma, A$, we define $\sigma^!$ by explaining how it extends the empty match or matches ending with an O-move (which is enough by O-closure).

Intuitively $\sigma^!$ records the current s_i where the last O move was taking place, if any, and plays in it as σ would have, it is only O that changes from one copy, that is from one s_i, to another.

For any i, $\sigma^!$ plays the one move match im in $!A$ if σ plays m in A. Similarly, $\sigma^!$ plays the one move match $\lceil i, j \rceil m$ in $?\Gamma$ if σ plays jm in $?\Gamma$.
Suppose $s \in \sigma^!$ ends with a P-move in $!A$:

$$\frac{s_i, m, m' \in \sigma \quad m, m' \in M_A}{s, im, im' \in \sigma^!}$$

$$\frac{s_i, m, jm' \in \sigma \quad m \in M_A, m' \in M_\Gamma}{s, im, \lceil i, j \rceil m' \in \sigma^!}$$

Suppose $s \in \sigma^!$ ends with a P-move in $?\Gamma$:

$$\frac{s_i, jm, j'm' \in \sigma \quad m, m' \in M_\Gamma}{s, \lceil i, j \rceil m, \lceil i, j \rceil' m' \in \sigma^!}$$

$$\frac{s_i, jm, m' \in \sigma \quad m \in M_\Gamma, m' \in M_A}{s, \lceil i, j \rceil m, im' \in \sigma^!}$$

Note that this definition implies that if s belongs to $\sigma^!$, then only O ever switches from an s_i to an s_j, and conversely, one sees:

$$\sigma^! = \{s : s_i \in \sigma, \text{O switches between } s_i\text{'s}\};$$

Note also that O can only switch in $!A$ because the switching condition binds him to preserve the copy-tag in $?\Gamma$.

This extensional definition of $\sigma^!$ parallels the one given for $\sigma_1 \otimes \sigma_2$, as the operational definition first given parallels the operational definition of $\sigma_1 \otimes \sigma_2$.

We now interpret promotion by $[\sigma^!]$.

Remarkably, all arbitrary choices in the exponential constructions are vanishing in the saturation process. Any other choice of k, l, r, and '$\lceil , \rceil$' would end in the same saturated strategy. It might even be that saturating a strategy only means to pack up in one strategy the union of all possible GoI generated history-free strategies. But this remains to be seen.

5.3 Invariance by reduction

Theorem 2 *Composition of strategies models relevant reduction in MELL proof-nets; that is, if σ, σ' are generated by interconvertible proof-nets (without $[w]$-steps), then $[\sigma] = [\sigma']$.*

Proof: Let σ, τ be saturated strategies respectively on games $?\Gamma, A$ and $?A^\perp, \Delta$. The exponential reduction steps are the following ones:

$$\begin{array}{rcl} [\sigma^!]; [d_A] & = & \sigma \\ [\sigma^!]; [c_A] & = & [c_\Gamma]; ([\sigma^!] \otimes [\sigma^!]) \\ [\sigma^!]; [\tau^!] & = & [([\sigma^!]; \tau)^!] \end{array}$$

where c_Γ is the tensor product of the c_{A_i} over $?A_i$ in $?\Gamma$.

Let us only consider the dereliction reduction step. A similar analysis works fine for the contraction reduction step and the commutative reduction step.

We want $[\sigma^!]; [d] = [\sigma]$ for all σ in $?\Gamma, !A$.

Let s be in $\sigma^!; d$. By definition there are s_1 and s_2 respectively in $\sigma^!$ and d such that $s = s_1; s_2/?\Gamma, A$.

If the composition is active, then all moves in $s_1; s_2$ happening in $!A \otimes ?A^\perp$ are of the form $0m$ with $m \in M_A$ and those happening in $?\Gamma$ of the form $\lceil 0, j \rceil m$ with $m \in M_A$ and $j \in \omega$. Replacing them respectively by m and jm yields $s'_1; s'_2$ in $\sigma \otimes id$, but $\sigma; id = \sigma$ as we know, so $s' = s'_1; s'_2/?\Gamma, A$ belongs to σ, and also by construction $s' \equiv s$ by the permutation $\theta(j) = \lceil 0, j \rceil$.

If the composition is passive and $s_1 = \epsilon$, then $s = s_2$ has at most one O-move and belongs to σ indeed.

If the composition is passive and $s_2 = \epsilon$, then $s = s_1$ that takes place wholly in Γ and has all its moves of the form $\lceil i, j \rceil m$ with $m \in M_A$ and $j \in \omega$ with i a *fixed* integer. Indeed, consider the first move, say $\lceil i, j \rceil m$, if O is to play, by the switching condition, he must answer $\lceil i, j \rceil m'$, if P is to play, by definition of $\sigma^!$, as said, its answer is of the form $\lceil i, j \rceil' m'$. Put $s' = s_i$, by construction $s' \in \sigma$ and $s' \equiv s$ by the permutation $\theta(j) = \lceil i, j \rceil$. (Believe it or not, but this is the one case where the plain AJM model fails. But we are now working with saturated strategies, and everything works smoothly.)

Thus we have that $\sigma^!; d \leq \sigma$. Conversely, the same analysis shows that when $s \in \sigma$, then s', obtained by applying $\theta(j) = \lceil 0, j \rceil$ on s moves in $?\Gamma$, is in $\sigma^!; d$. So $\sigma^!; d \geq \sigma$. Whence we may conclude that $[\sigma^!; d] = [\sigma]$ as said in the section about saturated strategies.

It remains to show that $[\sigma^!; d] = [\sigma^!]; [d]$. We already have the left to right inclusion.

The argument above works for any constant in place of 0. Thus, if $\cup d$ denotes the union of the dereliction strategies over all choices of constants, we have: $\sigma^!; \cup d \leq \sigma$.

Now $[d] = \cup d; [id]$. Therefore using the synchronization lemma and the last remark we have:

$$\begin{aligned} [\sigma^!]; [d] &= [\sigma^!; [d]] \\ &= [\sigma^!; \cup d; [id]] \\ &\subseteq [\sigma; [id]] \\ &= [\sigma] \\ &= [\sigma^!; d]. \end{aligned}$$

To go from the second line to the third, argue in this way: if $\tau \leq \tau'$ then $[\tau; [\sigma]] \subseteq [\tau'; [\sigma]]$. Indeed, $[\tau] \subseteq [\tau']$ and by synchronization:

$$[\tau; [\sigma]] = [\tau]; [\sigma] \subseteq [\tau']; [\sigma] = [\tau'; [\sigma]] \quad \Box$$

To model the erasure step, that is $[\sigma^!]; [w] = [w]$, seems desperate with games, since an s in $\sigma^!; w$ which is obtained by a passive composition $s_1; \epsilon$ with s_1 wholly in $?\Gamma$ and non-empty, will never have any equivalent in w which contains only the empty play. In this case one only gets $[\sigma^!]; [w] \geq [w]$.

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