Paths in the lambda-calculus[1]

Three years of communications without understanding

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Abstract

Since the rebirth of λ-calculus in the late sixties, three major theoretical investigations of β-reduction were undertaken: 1) Lévy's analysis of families of redexes (and the associated concept of labeled reductions); 2) Lamping's graph-reduction algorithm; 3) Girard's geometry of interaction.

All these three studies happened to make crucial (if not always explicit) use of a notion of path. Namely and respectively: legal, consistent and regular paths. We prove they are equivalent.

1 Introduction

Let us first survey the three different notions of paths which we want to prove are equivalent.

Lévy took hold in [13] of the difficult notion of two redexes being created in the "same" way during a reduction (in which case they were said to belong to the same "family"). Then he labeled terms and made β-reductions act on labels so that two redexes were in the same family iff they had the same labels.

Labels then slept fifteen years before the awakening in [2] where they were identified with *legal paths*. More precisely: 1) labels of redexes in any reduct N of M denote paths in M; 2) those paths are legal; 3) conversely, any legal path in M denotes a label of a redex to appear somewhere in the set of reducts of M. Legality is a simple and effective condition that intuitively asks for enough symmetry in the path so that the reduction may unfold it into a redex.

In the meantime, people were seeking for a shared reduction faithfully implementing the notion of families, i.e., a reduction where families could be said to be reduced in one step. Such a reduction was discovered by Lamping in [12], and Kathail in [11] (subsequent simplifications were given in [10] and [1]). The invariants used to prove the correctness of Lamping's implementation were *consistent paths*.

Finally Girard unveiled in [9] an interpretation of the cut-elimination procedure for linear logic. Again the alternative computation could be defined as the computation of a particular set of paths on proofs, namely *regular paths* which were defined through an algebraic and computational device the *dynamic algebra* (see [4, 15] where this is extended to λ-calculus).

A fact to marvel at, is that none of the three conditions above seems to bear any relation with β-reduction. There is yet another equivalent definition which may be the most natural one, but which is also the most uneffective one: *persistent paths*. Call a path persistent if its residuals through any reductions are still connected. In [6] this fourth condition is shown to be equivalent to regularity hence to all the three with which we deal in this paper.

Apart from the satisfaction gained in knowing that there is essentially one notion of paths in λ-calculus, we also expect these studies to yield new insights about the (implementation of) β-reduction based on the unification of the different perspectives.

This paper could also have been written in the framework of proof-nets: a syntax for linear logic where paths are easily defined and manipulated (see [8, 4, 15, 6]). This point is important since other calculi may be embedded in linear logic (see [7]), for instance the $\lambda\mu$-calculus of Parigot [14].

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2 Graphical representation of λ-terms

We display here a graph representation of λ-terms which unifies in a single shot the usual representation linking the bound variables to their λ's, the dynamic graph [4, 15, 5] and the sharing graph representation as defined in [1]. Our graphs are unoriented but have a natural orientation defined on the figures below. Each node $\mathtt{m}$ has a *depth* (an *index* in sharing graphs terminology) which is a positive integer marking $\mathtt{m}$. Each node is defined together with an associated set of entering edges (w.r.t. the natural orientation) called its *premises*. Each edge is premise of at most one node and is labeled by a *weight* which is an element of the dynamic algebra LS (see section 4). There are three kind of nodes:

Communication nodes: the *context (*$\mathtt{c}$*)* and the *variable (*$\mathtt{v}$*)* nodes are zero-ary, i.e., have no associated premises;

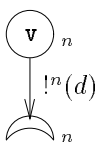
Multiplicative nodes: the *application* and the *lambda* nodes are binary, i.e., have two associated premises;

Exponential nodes: in sharing graph terminology the *control* nodes; the *croissant* is unary, i.e., has one premise; the *fan* is binary; the *bracket* is unary; the *weakening* is zero-ary.

For each integer n and each term M we define by induction on M a graph $\mathcal{G}_n(M)$ together with its *boundary* consisting in a single *root node* (represented on the figure as the topmost node) and a set of control nodes, the *free nodes* of $\mathcal{G}_n(M)$ in one-to-one correspondence with the free variables of M (represented as the bottomost nodes). The translation of a lambda-term M is defined to be $\mathcal{G}(M) = \mathcal{G}_0(M)$.

Variable x**.**

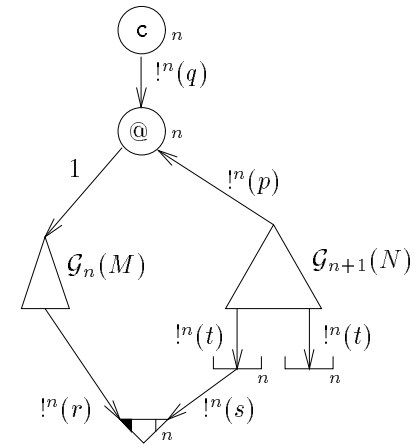
$\mathcal{G}_n(x) =$



The premise of the croissant node is the edge labeled by $!^n(d)$.

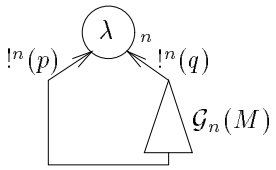
Application (MN)**.** Only free nodes corresponding to free variables both in M and N are linked together by means of a fan:

$\mathcal{G}_n(MN) =$

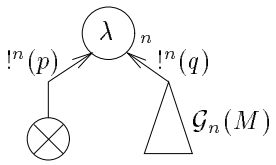


The premises of the application are respectively called the *context edge* (labeled by $!^n(q)$) and the *argument edge* (labeled by $!^n(p)$); the exiting edge is the *function edge* of the application. The premises of the fan node are the edges labeled by $!^n(r)$ and $!^n(s)$, the premise of the bracket node is the edge labeled by $!^n(t)$.

Abstraction $\lambda x.M$**.** There are two cases whether the variable does or does not appear in M. In the former we link the corresponding node to the lambda node; in the latter we link a weakening node to the lambda node:



$\mathcal{G}_n(\lambda xM) =$



The premises of the lambda are respectively called the *body edge* (labeled by $!^n(q)$) and the *variable edge* (labeled by $!^n(p)$).

If $\mathtt{n}$ is a context, variable or lambda node at depth n in $\mathcal{G}(M)$ then the set of nodes below $\mathtt{n}$ at depth greater than n must be of the form $\mathcal{G}_n(N)$ for some subterm N of M. We shall say that $\mathtt{n}$ is the root of N in M. If N is not equal to M than there is an edge on top of $\mathtt{n}$ which will be called the root edge of N in M or the root edge of $\mathtt{n}$.

A *path* is a sequence of consecutive nodes or equivalently a sequence of oriented edges and reversed oriented edges w.r.t. the natural orientation. We shall use both conventions and even mix them without mention. The reverse of a path φ is denoted φ^r. A path is *straight* if it contains no subpath of the form $\varphi\varphi^r$ nor uv^r where u and v are the two premises of some binary node.

To each variable node (occurrence of variable) $\mathtt{v}$ we associate its *discriminant* $\gamma_{\mathtt{v}}$: it is the maximal path starting at $\mathtt{v}$ and moving downwards. Note that it is uniquely determined by $\mathtt{v}$ and that it may contain only control nodes. The last node of $\gamma_{\mathtt{v}}$ will be called its *discriminant node*. The discriminant node $\mathtt{m}$ of $\gamma_{\mathtt{v}}$ may have an exiting edge. If $\mathtt{m}$ has an exiting edge it is the variable edge of some λ-node λ and we say that $\mathtt{v}$ and $\gamma_{\mathtt{v}}$ are bound by λ; if not $\mathtt{m}$ is a free node of $\mathcal{G}(M)$.

Let γ be a discriminant; then among all the nodes visited by γ, its variable node has a maximal depth and its discriminant node $\mathtt{m}$ has a minimal depth. The difference between the two will be called the *lift* of γ. Note that if γ is bound by some λ-node λ then the depth of λ is the same than the depth of $\mathtt{m}$.

We will say that a path is of type t_1-t_2 if it starts from a node of type t_1 and ends at a node of type t_2. In particular an edge of type @-λ will be called a redex edge. There is a one to one correspondence between redexes in M and redex edges in $\mathcal{G}(M)$. We say that a path φ crosses a redex if it contains the corresponding redex edge.

Let M be a term and M' be obtained by some β-contraction of M. We define the notion of *residual* of the edges of $\mathcal{G}(M)$ in $\mathcal{G}(M')$ in the obvious way and extend it to the notion of residual of a path. Note that some paths of $\mathcal{G}(M)$ may have no or more than one residual in $\mathcal{G}(M')$.

3 Legal paths

In order to express the legality condition, we begin with a definition on paths. *Well balanced paths* (shortly wbp) are inductively defined by the following clauses:

- the function edge of any application node is a wbp.
- let φ_1 be a wbp ending at a context node $\mathtt{c}$ of an application @, u be the context edge of @, φ_2 be a wbp connecting @ to a lambda node λ and v be the body edge of λ. Then the path

$$\varphi_1\, u\, \varphi_2\, v^r$$

is a wbp (see the left part of the figure 1).

- let φ_1 be a wbp ending at some variable node $\mathtt{v}$, γ be the discriminant starting at $\mathtt{v}$, u be the variable edge starting from the discriminant node of γ and ending in some lambda node λ, φ_2 be a wbp connecting an application node @ to λ and v be the argument edge of @. Then the path

$$\varphi_1\, \gamma\, u\varphi_2^r\, v^r$$

is a wbp (see the right part of the figure 1).

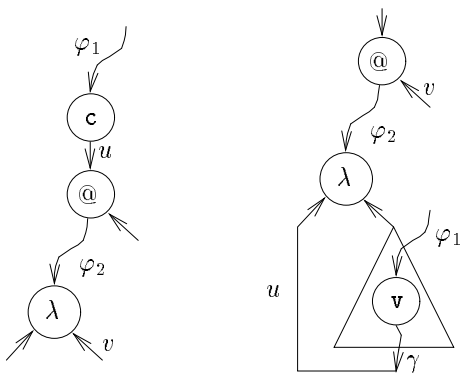


Figure 1: Well balanced paths

Lemma 3.1 (Straight.) *If φ is a wbp then φ is straight, starts from an application node by its function edge and ends downwards at a lambda, context or variable node.*

Lemma 3.2 (Prefix.) *Let φ_1 and φ_2 be two wbp's connecting the same application node @ to respectively the lambda nodes λ_1 and λ_2. If φ_2 is a prefix of φ_1 then $\lambda_2 = \lambda_1$ and $\varphi_2 = \varphi_1$.*

Lemma 3.3 (Sub-wbp.) *Let φ be a wbp of type @-λ and @ (resp. λ) be an application node (resp. a λ-node) visited by φ. Then φ has a unique subpath ϕ which is a wbp of type @-λ and starts at @ (resp. ends at λ).*

These are more or less immediate consequences of the definition of wbp's.

Actually the "interesting" wbp's are those of type @-λ; they intend to be isomorphical to redexes family. However it is not enough to ask only for well balancing on paths of type @-λ to get this result. The stronger condition of legality to come hereafter is needed. But first let us define by crossed induction two other kinds of paths in a term M: *@-cycles* and *v-cycles*.

(@-cycle) Let @ be an application node in $\mathcal{G}(M)$, u its argument edge and denote by N the subterm argument of @. An *@-cycle at @* is a path of the form:

$$@\, u^r\, \psi_1\, \xi_1\, \psi_2\, \cdots\, \psi_n\, \xi_n\, \psi_{n+1}\, u\, @$$

where the ψ_i's are entirely contained in N and the ξ_i's are v-cycles over some free occurrences of variables of N in M. A particular case of @-cycle, which is the very base case of the crossed induction, is $n = 0$; then the @-cycle is just $@u^r\psi u@$ where ψ is a cycle contained in N starting and ending at the root node of N. Note that a path entirely contained in N crosses only nodes at depth strictly greater than the depth of @.

(v-cycles) Let $\mathtt{v}$ be a variable node corresponding to a bound occurrence of variable in M, γ be its discriminant and u be the variable edge of its binder λ. A *v-cycle over* $\mathtt{v}$ is a path of the form

$$\mathtt{v}\, \gamma\, u\, \lambda\, \varphi^r\, @\, \psi\, @\, \varphi\, \lambda\, u^r\, \gamma^r\, \mathtt{v}$$

where @ is some application node, φ is a wbp linking @ to λ, and ψ is an @-cycle at @.

We now state a consequence of lemma 3.3 relating wbp and @-cycles (proved in [2]):

Proposition 3.4 *Let φ be a wbp and $@\,\psi\,@$ be an @-cycle at @ contained in φ. Then φ can be uniquely decomposed into:*

$$\zeta_1\, \gamma_1\, u_1\, \lambda_1\, {\varphi_1}^r\, @\, \psi\, @\, \varphi_2\, \lambda_2\, {u_2}^r\, {\gamma_2}^r \zeta_2$$

where both φ_i's are wbp's linking @ to some lambda nodes λ_1 and λ_2, u_i is the variable edge of λ_i and γ_1, γ_2 are the discriminants of some occurrences of variables bound respectively by λ_1 and λ_2.

In the situation of Proposition 3.4, we will say that φ_1 and φ_2 are the *call* and *return* paths of the @-cycle ψ and that γ_1 and γ_2 are the discriminants of respectively the call and return paths.

Definition 3.5 (Legal paths.) *A wbp φ is a* legal path *iff for any @-cycle ψ contained in φ, the call and return paths of ψ and their discriminants are pairwise equal.*

As proved in [2] legal paths are related to Lévy's families [13]:

Theorem 3.6 *Given a λ-term M, there is a one-to-one correspondence between legal paths of type @-λ in M and all the possible redex families obtained by reduction of M.*

4 $\mathcal{LS}$ and regular paths

We give a presentation of the dynamic algebra $\mathcal{LS}$ as an equational theory. Terms of $\mathcal{LS}$ will be called *monomials*. We define at the same time the language and the (equational) axioms of $\mathcal{LS}$. Items are:

- an associative operation with a 0 and a 1;
- an involution U^* satisfying $0^* = 0$, $1^* = 1$ and $(UV)^* = V^*U^*$ for any monomials U and V in $\mathcal{LS}$;
- a morphism ! for composition, 0, 1 and $*$;
- two *multiplicative* coefficients p and q satisfying the *zap equations*:

$$\begin{aligned} p^*p = q^*q &= 1, \\ p^*q = q^*p &= 0. \end{aligned}$$

- Four *exponential* constants r, s, t, d satisfying the zap equations:

$$\begin{aligned} r^*r = s^*s = d^*d = t^*t &= 1, \\ s^*r &= 0; \end{aligned}$$

and the *swap equations*:

$$\begin{aligned} !(U)d &= dU; \\ !(U)x &= x!(U) \qquad \text{when } x = r, s; \\ !(U)t &= t!^2(U); \end{aligned}$$

for any monomial U.

The weight of oriented edges was defined in section 2. The weight $w(\psi)$ of a path ψ is inductively given by: if ψ is a null path (a path starting and ending at the same node, crossing no edge) then its weight is 1, if ψ is $u\psi'$ (resp. $u^r\psi'$) where u is an oriented edge then $w(\psi) = w(\psi')w(u)$ (resp. $w(\psi')w(u)^*$). Note that weights are composed antimorphically w.r.t. paths and that we have $w(\psi^r) = w(\psi)^*$.

Important convention. We simply write $U = V$ (resp. $U \neq V$) for $\mathcal{LS} \vdash U = V$ (resp. $\mathcal{LS} \nvdash U = V$).

Definition 4.1 (Regular paths.) *A path ψ is* regular *iff $w(\psi) \neq 0$.*

At first sight this definition looks ineffective since it is usually hard to show that something is not provable in an axiomatic theory. However the theorem AB^* below (proved in [15]) shows that it is actually easy to compute whether a weight of a path is (provably) null or not in $\mathcal{LS}$.

We say that a monomial in which the symbol ! (resp. $*$) doesn't occur is *flat* (resp. *positive*). Note that positive monomials satisfy $A^*A = 1$. We define

inductively the *stable forms* of LS to be the monomials of the form AB^* or $A!(M)B^*$ when A and B are flat and positive and M is in turn a stable form. Note that if M is a stable form of LS, then $M = AB^*$ for some positive (but not necessarily flat) A and B.

Proposition 4.2 (Confluence of $\mathsf{LS}\bigcirc$) *The rewriting system $\mathsf{LS}\bigcirc$ on monomials defined by orienting all the equations from left to right is noetherian. Stable forms are normal w.r.t. $\mathsf{LS}\bigcirc$.*

Let U be a monomial:

- *if $U = V$ for some stable form V of LS then U rewrites to V w.r.t. $\mathsf{LS}\bigcirc$;*
- *if $U = 0$ then U rewrites to 0 w.r.t. $\mathsf{LS}\bigcirc$.*

REMARK. This entails that $\mathsf{LS}\bigcirc$ is confluent on monomials which are either null or equal to some stable form.

Theorem 4.3 (AB^*) *Let M be a term and ψ be a path in $\mathcal{G}(M)$. Then $w(\psi)$ rewrites, w.r.t. $\mathsf{LS}\bigcirc$, either to 0, or to a stable form.*

Suppose $AB^* = 0$ for some positive A and B. Since $A^*A = B^*B = 1$ we immediately get $0 = 1$ in LS. But this is contradicted by the fact that LS has some non trivial models (see the next section). Thus rewriting a weight into AB^* form indeed shows that it is provably not null in LS.

The foregoing argument has an interesting corollary concerning models of LS:

Corollary 4.4 (Semantical AB^*.) *Let $\mathcal{M}$ be a non trivial model of LS and M be a term. Then, for every path ψ in $\mathcal{G}(M)$ we have:*

$$\mathsf{LS} \vdash w(\psi) = 0 \quad \textit{iff} \quad \mathcal{M} \models w(\psi) = 0.$$

REMARK. This property is specific to weights of paths. For instance one may find some non trivial models of LS in which $t^*d = 0$ (see next section) which is not provable in LS (there are some other models in which $t^*d \neq 0$). However it is a strong result since it states that any non trivial model of LS is as good as the theory for checking regularity.

5 A model of LS

In order to help with some of the weight computations to come, we give a first interpretation of the equational theory LS (the second one is the context semantics in the last section). The verification that the interpretation satisfies the equations of LS is left to the reader.

Let $\mathcal{T}$ be the set of partial transformations of $\mathbb{N}$; that is, an element f of $\mathcal{T}$ is a one-to-one mapping of a subset of $\mathbb{N}$ (the *domain* of f) onto a subset of $\mathbb{N}$ (the *codomain* or *range* of f). Composition of partial transformations is defined in the obvious way (the domain of fg is the set of n such that n is in the domain of g and $g(n)$ is in the domain of f). We interpret each monomial U of LS by a partial transformation $|U|_{\mathcal{T}}$ of $\mathcal{T}$.

We denote by $0_{\mathcal{T}}$ the nowhere defined transformation, by $1_{\mathcal{T}}$ the identity on $\mathbb{N}$ and set $|0|_{\mathcal{T}} = 0_{\mathcal{T}}$ and $|1|_{\mathcal{T}} = 1_{\mathcal{T}}$. If f is a partial transformation then f^* is the inverse transformation from the codomain to the domain of f so that for any monomial U we define $|U^*|_{\mathcal{T}} = |U|_{\mathcal{T}}{}^*$.

Let $|p|_{\mathcal{T}}$ and $|q|_{\mathcal{T}}$ be two transformations of $\mathcal{T}$ with full domain and disjoint codomains; for instance take $|p|_{\mathcal{T}}(n) = 2n$ and $|q|_{\mathcal{T}}(n) = 2n + 1$.

Let $(m, n) \mapsto \lceil m, n \rceil$ be a one-to-one mapping of $\mathbb{N}^2$ onto $\mathbb{N}$. We shall denote by $\lceil n_1, \ldots, n_k \rceil$ the integer $\lceil n_1, \lceil n_2, \ldots, \lceil n_{k-1}, n_k \rceil \cdots \rceil \rceil$. For any transformation f we define $!_{\mathcal{T}}(f)$ by:

$$!_{\mathcal{T}}(f)\lceil m, n \rceil = \lceil m, f(n) \rceil$$

whenever n is in the domain of f. For any monomial U we set $|!(U)|_{\mathcal{T}} = !_{\mathcal{T}}(|U|_{\mathcal{T}})$. Note that if f has a full domain, since any integer may be written $\lceil m, n \rceil$ for some m and n, then $!_{\mathcal{T}}(f)$ has full domain. In particular $!_{\mathcal{T}}(1_{\mathcal{T}}) = 1_{\mathcal{T}}$. Also $!_{\mathcal{T}}(0_{\mathcal{T}}) = 0_{\mathcal{T}}$.

Let k be an integer, ρ, σ and τ be three transformations with full domains. Assume that the codomains of ρ and σ are disjoint. We define $|d|_{\mathcal{T}}$, $|r|_{\mathcal{T}}$, $|s|_{\mathcal{T}}$ and $|t|_{\mathcal{T}}$ by:

$$\begin{aligned} |d|_{\mathcal{T}}(n) &= \lceil k, n \rceil, \\ |r|_{\mathcal{T}}\lceil m, n \rceil &= \lceil \rho(m), n \rceil, \\ |s|_{\mathcal{T}}\lceil n, m \rceil &= \lceil \sigma(n), m \rceil, \\ |t|_{\mathcal{T}}\lceil l, \lceil m, n \rceil \rceil &= \lceil \tau(\lceil l, m \rceil), n \rceil \end{aligned}$$

This interpretation is subjected to a lot of arbitrary choices, beginning with the choice of the one-to-one mapping from $\mathbb{N}^2$ onto $\mathbb{N}$. The effect of this is that we can easily build some interpretation satisfying some additional equations. For example $|t^*d|_{\mathcal{T}} = 0_{\mathcal{T}}$ iff k is not in the range of τ.

The following property is a kind of converse of the semantical version of AB^* (corollary 4.4).

Proposition 5.1 *A monomial U rewrites w.r.t. $\mathsf{LS}\bigcirc$ into an AB^* form iff it is non null in each non trivial model of LS.*

PROOF. The only if part is obvious. For the if part, let U_0 be some normal form of U w.r.t. LSO (which exists since, by the confluence proposition 4.2, LSO is noetherian) and suppose that U_0 is not an AB^* form. Then there must be a configuration $x^*!^k(y)$ or $!^k(x^*)y$ in U_0 where x and y are two coefficients of LS, k is a positive or null integer and such that no oriented equation can be applied. By symmetry we may consider only the first case. We can show that whatever k, x and y are, there is some interpretation of the constants of LS in $\mathcal{T}$ such that $|x^*!^k(y)|_{\mathcal{T}}$ is the null transformation. For example if k is strictly positive and x and y are both p then take:

$$|p|_{\mathcal{T}}\lceil n_0, \ldots, , n_k\rceil = \lceil 0, n_0, \ldots, n_{k-1}, 1, n_k\rceil.$$

We let the reader check that $|p^*!^k(p)|_{\mathcal{T}}$ is the null transformation. Since $x^*!^k(y)$ is occurring in U this entails that $|U|_{\mathcal{T}} = 0_{\mathcal{T}}$. ■

6 More about regular paths

If γ is a discriminant of lift c in $\mathcal{G}(M)$, then its weight is $!^k(G)$ for a monomial G of the form:

$$G = \omega_1 t!(\omega_2 t!(\ldots \omega_c t!(\omega_{c+1} d)\ldots))$$

where the ω_i's are products of r's and s's only. Such a G will be called a *w-discriminant* of lift c. A w-discriminant G of lift c satisfies the *generalized swap equation*:

$$!(U)G = G!^c(U)$$

for any monomial U. If γ and γ' are two discriminants bound by the same lambda node λ, then their weights are respectively $!^k(G)$ and $!^k(G')$ where k is the depth of λ and G and G' are some w-discriminants. Furthermore G and G' satisfy the *generalized zap equation*:

$$\begin{aligned} G^*G' &= 1 \qquad \text{if } \gamma = \gamma' \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Proposition 6.1 (Rendez-vous property) *Let φ be a wbp linking two nodes* $\mathtt{m}$ *and* $\mathtt{n}$ *of respective depths m and n. Then $w(\varphi)$ satisfies the* rendez-vous equation*:*

$$!^n(X)w(\varphi) = w(\varphi)!^m(X).$$

PROOF. By induction on the definition of wbp's. We take the notations of the definition (section 3).

• (φ is an edge). φ is the function edge of some application node so $w(\varphi) = 1$ and $n = m$.

• ($\varphi = \varphi_1\, u\, \varphi_2\, v^r$). Let U_1 and U_2 be the respective weights of φ_1 and φ_2 and k be the depth of @. Notice that $\mathtt{c}$ being the context node of @ is also at depth k and that λ has the same depth n than $\mathtt{n}$. Let $U = w(\varphi)$; with these notations we have:

$$U = !^n(q^*)U_2!^k(q)U_1$$

By induction hypothesis on φ_2 (which begins at depth k) we get $!^n(q^*)U_2!^k(q) = !^n(q^*)!^n(q)U_2$, which is equal to U_2 by the zap equation of q so that finally $U = U_2U_1$ Now again by induction on φ_1 and φ_2 we have:

$$\begin{aligned} !^n(X)U &= !^n(X)U_2U_1 \\ &= U_2!^k(X)U_1 \\ &= U_2U_1!^m(X) \\ &= U!^m(X) \end{aligned}$$

which shows that φ in turn satisfies the rendez-vous equation.

• ($\varphi = \varphi_1\, \gamma\, u\, \varphi_2^r\, v^r$). We denote by k the depth of λ and by c the lift of γ. By definition $\mathtt{v}$ is therefore at depth $k + c$. Since φ ends at the argument node $\mathtt{n}$ of @ (i.e., the initial node of v) which is at depth n it must be that n is strictly positive and that the depth of @ is $n-1$. Let U_i be the weight of φ_i and G be the w-discriminant such that $w(\gamma) = !^k(G)$. With these notations we have:

$$U = !^{n-1}(p^*)U_2^*!^k(p)!^k(G)U_1$$

Now since φ_1 starts at depth m and ends at depth $k+c$ and φ_2 starts at depth k and ends at depth $n-1$, the induction hypothesis tells us that $!^{k+c}(X)U_1 = U_1!^m(X)$ and $!^k(X)U_2 = U_2!^{n-1}(X)$. Thus we have $!^{n-1}(p^*)U_2^*!^k(p) = !^{n-1}(p^*)!^{n-1}(p)U_2^*$ so that by the zap equation of p this is equal to U_2^* and therefore $U = U_2^*!^k(G)U_1$. Again the rendez-vous equation applied to U_2 gives $U = !^{n-1}(G)U_2^*U_1$. Finally by the generalized swap equation of G we may write:

$$\begin{aligned} !^n(X)U &= !^n(X)!^{n-1}(G)U_2^*U_1 \\ &= !^{n-1}(!(X)G)U_2^*U_1 \\ &= !^{n-1}(G!^c(X))U_2^*U_1 \\ &= !^{n-1}(G)!^{n-1+c}(X)U_2^*U_1 \\ &= !^{n-1}(G)U_2^*!^{k+c}(X)U_1 \\ &= !^{n-1}(G)U_2^*U_1!^m(X) \end{aligned}$$

so that φ indeed satisfies the rendez-vous property. ■

REMARK. The proof uses the equations of LS in an unoriented fashion. In other words the rendez-vous equation cannot be obtained only by using the rewriting system LSO.

Lemma 6.2 (Subweight) *Let ϕ be a regular path such that $w(\phi) = U\pi V$ for some monomials U, π and V. There are some positive monomials A and B such that π rewrites into AB^* w.r.t. LSO.*

If π is idempotent in LS then $A = B$ so that $\pi = AA^*$. If $\pi = \pi_1\pi_2$ where π_1 and π_2 are two idempotents of LS, then π is an idempotent of LS.

PROOF. Since ϕ is regular, $w(\phi)$ and thus π are not null in any model of LS. Hence by the proposition 5.1, π has an AB^* form.

Now suppose that $\pi^2 = \pi$, thus we have $AB^*AB^* = AB^*$. Since A and B are positive, we have $A^*A = B^*B = 1$ so that $B^*A = 1$. Now it is easy to show by induction on the length of A and B that this is possible only if $A = B$ so that we finally get $\pi = AA^*$.

If $\pi = \pi_1\pi_2$ with π_1 and π_2 idempotent, then we have just shown that there are some positive A_1 and A_2 in LS such that $\pi_i = A_iA_i^*$. Therefore $A_1^*A_2$ is a subweight of $w(\phi)$ thus by applying the first part of the lemma we get two positive A_i' so that $A_1^*A_2 = A_2'A_1'^*$. Hence $\pi = A_1A_2'A_1'^*A_2^*$ and we have:

$$\begin{aligned}\pi^2 &= A_1A_2'A_1'^*A_2^*A_1A_2'A_1'^*A_2^* \\ &= A_1A_2'A_1'^*A_1'A_2'^*A_2'A_1'^*A_2^* \\ &= A_1A_2'A_1'^*A_2^* \\ &= \pi\end{aligned}$$

showing that π is an idempotent. ■

7 Regular well balanced paths are legal

Proposition 7.1 (@-cycle property) *Let @ be an application node in $\mathcal{G}(M)$ at depth k, u be its application edge and $u^r\psi u$ be an @-cycle at @. There are two monomials U and π such that:*

$$w(\psi) = !^{k+1}(U)\pi,$$

*and π is a k-*swapping *idempotent, i.e., π satisfies:*

$$\begin{aligned}\pi^2 &= \pi, \\ !^k(X)\pi &= \pi!^k(X),\end{aligned}$$

for any monomial X.

REMARK. This proposition expresses that the weight of any @-cycle is essentially similar to the weight of an elementary @-cycle. The difference only lies in some idempotent which is interpreted in the model $\mathcal{T}$ by a partial identity; furthermore the k-swapping property says that π is in some sense "invisible", at least it doesn't interact with U.

PROOF. If the weight of ψ is null then the proposition is true by taking $U = \pi = 0$. So we may suppose that $w(\psi)$ is non null in LS. Let N be the subterm argument of the application @. By definition of @-cycles, ψ has the form:

$$\psi_0\,\mathtt{v}_1\,\gamma_1\,v_1\,\lambda_1\,\varphi_1^r\,@_1\,u_1^r\,\phi_1\,u_1\,@_1\,\varphi_1\,\lambda_1\,v_1^r\,\gamma_1^r\,\mathtt{v}_1\,\psi_1\ldots\psi_n$$

where for each i, ψ_i is a path entirely contained in N, γ_i is the discriminant of some free occurrence of variable in N corresponding to the variable node $\mathtt{v}_i$, λ_i is the lambda node binding $\mathtt{v}_i$ and v_i is its variable edge, φ_i is a wbp linking an application node $@_i$ to the lambda node λ_i, u_i is the argument edge of $@_i$ and $u_i^r\,\phi_i\,u_i$ is an @-cycle at the application node $@_i$.

We shall prove by induction on n then on ψ that any path with the shape of ψ (defined in the foregoing paragraph) has the @-cycle property. Note that this is a bit more general than the statement of the proposition since for the sake of induction loading, we don't suppose that the starting node of ψ_0 and the ending node of ψ_n is the argument node of @.

The base case is $n = 0$. Then $\psi = \psi_0$ is entirely contained in N. But N being the argument of @ which is at depth k, the weight of ψ must be of the form:

$$w(\psi) = w(\psi_0) = !^{k+1}(W_0)$$

for some monomial W_0. Thus the proposition is proved with $\pi = 1$.

If $n > 0$ let l_1 be the depth of λ_1 so that the weight of γ_1 is of the form $!^{l_1}(G_1)$ for some w-discriminant G_1 of lift c_1. Furthermore let U_1 be the weight of φ_1 and k_1 the depth of $@_1$. Since $u_1^r\,\phi_1\,u_1$ is an @-cycle, ϕ_1 has the good shape so we may suppose by induction on ϕ_1 that its weight is of the form

$$w(\phi_1) = !^{k_1+1}(V_1)\pi_1$$

for some monomial V_1 and some k_1-swapping idempotent π_1. Furthermore the suffix $\psi_1\ldots\psi_n$ of ψ again has the right shape so that its weight is by induction on n

$$w(\psi_2\ldots\psi_n) = !^{k+1}(V)\pi$$

for some V and some k-swapping idempotent π of LS. Thus the weight of ψ is equal to:

$$\begin{aligned}w(\psi) = {} & !^{k+1}(V)\pi!^{l_1}(G_1^*) \\ & !^{l_1}(p^*)U_1\,!^{k_1}(p)!^{k_1+1}(V_1)\pi_1!^{k_1}(p^*)U_1^*!^{l_1}(p) \\ & !^{l_1}(G_1)!^{k+1}(W_0)\end{aligned}$$

Now since φ_1 is a wbp linking two nodes whose respective depths are k_1 and l_1 the rendez-vous property of φ_1 states that for any monomial X,

$$!^{l_1}(X)U_1 = U_1!^{k_1}(X).$$

Thus we have $!^{l_1}(p^*)U_1!^{k_1}(p) = !^{l_1}(p^*)!^{l_1}(p)U_1$ so that by the morphism equations of ! and the zap equation of p this is equal to U_1. The weight of ψ is therefore equal to:

$$!^{k+1}(V)\pi!^{l_1}(G_1^*)U_1!^{k_1+1}(V_1)\pi_1U_1^*!^{l_1}(G_1)!^{k+1}(W_0)$$

If we let X be $!^{l_1}(G_1^*)U_1!^{k_1+1}(V_1)\pi_1U_1^*!^{l_1}(G_1)$ then we have:

$$\begin{aligned}
X &= U_1!^{k_1}(G_1^*)!^{k_1+1}(V_1)\pi_1U_1^*!^{l_1}(G_1)\\
&= U_1!^{k_1}(G_1^*!(V_1))\pi_1U_1^*!^{l_1}(G_1)\\
&= U_1!^{k_1}(!^{c_1}(V_1)G_1^*)\pi_1U_1^*!^{l_1}(G_1)\\
&= U_1!^{k_1+c_1}(V_1)!^{k_1}(G_1^*)\pi_1U_1^*!^{l_1}(G_1)\\
&= U_1!^{k_1+c_1}(V_1)\pi_1!^{k_1}(G_1^*)U_1^*!^{l_1}(G_1)\\
&= U_1!^{k_1+c_1}(V_1)\pi_1U_1^*!^{l_1}(G_1^*)!^{l_1}(G_1)\\
&= U_1!^{k_1+c_1}(V_1)\pi_1U_1^*\\
&= !^{l_1+c_1}(V_1)U_1\pi_1U_1^*
\end{aligned}$$

by using the rendez-vous equation of U_1, the generalized swap equation of G_1, the k_1-swapping property of π_1, the rendez-vous equation of U_1, the generalized zap equation of G_1 and finally the rendez-vous equation of U_1. Thus we get:

$$w(\psi) = !^{k+1}(V)\pi!^{l_1+c_1}(V_1)U_1\pi_1U_1^*!^{k+1}(W_0).$$

Now note that l_1 being the depth of λ_1, it is also the depth of the discriminant node of γ_1 thus we have $l_1 \leq k$. In other words there is a positive or null d_1 such that $k = l_1 + d_1$. Let X be any monomial. By using (again) the rendez-vous property of φ_1 and the k_1-swapping property of π_1 we have that:

$$\begin{aligned}
U_1\pi_1U_1^*!^k(X) &= U_1\pi_1U_1^*!^{l_1+d_1}(X)\\
&= U_1\pi_1!^{k_1+d_1}(X)U_1^*\\
&= U_1!^{k_1+d_1}(X)\pi_1U_1^*\\
&= !^{l_1+d_1}(X)U_1\pi_1U_1^*\\
&= !^k(X)U_1\pi_1U_1^*
\end{aligned}$$

so that $U_1\pi_1U_1^*$ is k-swapping. We have to show that it is an idempotent. But by the subweight lemma (6.2), π_1 being an idempotent there is a positive monomial A_1 such that $\pi_1 = A_1A_1^*$. Furthermore the same lemma states that there are some positive B and C such that $U_1A_1 = BC^*$. Thus we have

$$\begin{aligned}
U_1\pi_1U_1^* &= BC^*CB^*\\
&= BB^*
\end{aligned}$$

which by positivity of B is clearly idempotent.

So we are in position to write

$$w(\psi) = !^{k+1}(V)\pi!^{l_1+c_1}(V_1)!^{k+1}(W_0)\pi_1'$$

where $\pi_1' = U_1\pi_1U_1^*$ is a k-swapping idempotent of LS. But c_1 is the lift of the discriminant γ_1 which starts at depth strictly greater than k and ends at depth l_1. Thus $l_1 + c_1$ is strictly greater than k; in other words there is a positive or null d_1' such that $l_1+c_1 = k+1+d_1'$ so that by the k-swapping property of π we may write:

$$\begin{aligned}
w(\psi) &= !^{k+1}(V)\pi!^{k+1+d_1'}(V_1)!^{k+1}(W_0)\pi_1'\\
&= !^{k+1}(V)\pi!^{k+1}(!^{d_1'}(V_1)W_0)\pi_1'\\
&= !^{k+1}(V)!^{k+1}(!^{d_1'}(V_1)W_0)\pi\pi_1'\\
&= !^{k+1}(V!^{d_1'}(V_1)W_0)\pi\pi_1'.
\end{aligned}$$

Since π and π_1' are k-swapping their product also is and since both are idempotent, by the subweight lemma, their product also is. ■

Theorem 7.2 *Every regular wbp is legal.*

Proof. Let φ be a regular wbp. We have to check that any @-cycle in φ satisfies the legality condition; so let $u^r\psi u$ be an @-cycle at some application node @ (whose argument edge is u) and for $i = 1$ or 2, let φ_i be a wbp linking @ to a lambda node λ_i, u_i be the variable edge of λ_i, $\mathtt{v}_i$ be a variable node bound by λ_i, γ_i be the discriminant of $\mathtt{v}_i$ and suppose that the path

$$\phi = \mathtt{v}_1\,\gamma_1\,u_1\,\lambda_1\,\varphi_1{}^r\,@\,u^r\,\psi\,u\,@\,\varphi_2\,\lambda_2\,u_2{}^r\,\gamma_2{}^r\,\mathtt{v}_2$$

is entirely contained in φ. We are to show that $\psi_1 = \mathtt{v}_1\,\gamma_1\,u_1\,\lambda_1\,\varphi_1{}^r\,@\,u^r$ and $\psi_2 = \mathtt{v}_2\,\gamma_2\,u_2\,\lambda_2\,\varphi_2{}^r\,@\,u^r$ are equal.

Let k and k_i be the respective depths of @ and λ_i; the weights of φ_i and γ_i are respectively of the form U_i and $!^{k_i}(G_i)$ where G_i is some w-discriminant of lift c_i. Then the weight of ϕ is:

$$!^{k_2}(G_2^*)!^{k_2}(p^*)U_2!^k(p)w(\psi)!^k(p^*)U_1^*!^{k_1}(p)!^{k_1}(G_1)$$

By the rendez-vous property of φ_1 and φ_2, and the zap equations of p this is equal to:

$$w(\phi) = !^{k_2}(G_2^*)U_2w(\psi)U_1^*!^{k_1}(G_1)$$

But the @-cycle property tells us that $w(\psi) = !^{k+1}(U)\pi$ for some U and some idempotent π. Furthermore G_i is a w-discriminant with lift c_i. So by

the rendez-vous property again, the morphism equation of ! and the generalized swap equation of G_i we have:

$$\begin{aligned} w(\phi) &= !^{k_2}(G_2^*)U_2\,!^{k+1}(U)\,\pi U_1^*!^{k_1}(G_1) \\ &= !^{k_2}(G_2^*)\,!^{k_2+1}(U)\,U_2\pi U_1^*!^{k_1}(G_1) \\ &= !^{k_2}(G_2^*\,!(U))\,U_2\pi U_1^*!^{k_1}(G_1) \\ &= !^{k_2}(!^{c_2}(U)\,G_2^*)\,U_2\pi U_1^*!^{k_1}(G_1) \\ &= !^{k_2+c_2}(U)\,!^{k_2}(G_2^*)U_2\pi U_1^*!^{k_1}(G_1) \end{aligned}$$

Let $W_i = !^k(p^*)U_i^*!^{k_i}(p)!^{k_i}(G_i)$ be the weight of ψ_i. By the rendez-vous property and the zap equations of p we have $W_i = U_i^*!^{k_i}(G_i)$. Thus $w(\phi) = !^{k_2+c_2}(U)\,W_2^*\pi W_1$.

Now by lemma 3.1 we easily get that ψ_1 and ψ_2 are straight paths ending at the same node. Since they are straight, we must have either $\psi_i = \phi_i v_i \phi$ where v_1 and v_2 are two distinct premises of a binary node, or one ψ_i is a suffix of the other.

In the first case let x_i be the weight of v_i, V_i and V be the respective weights of ψ_i and ψ so that $W_i = V x_i V_i$. Since the v_i's are distinct premises of a binary node we have $x_2^* x_1 = 0$ in $\mathcal{LS}$, thus they are interpreted by partial transformations with disjoint codomains in $\mathcal{T}$. But

$$w(\phi) = !^{k_2+c_2}(U)V_2^* x_2^* V^* \pi V x_1 V_1.$$

$V^*\pi V$ being an idempotent of $\mathcal{T}$ we get that $x_2^* V^* \pi V x_1$ and therefore $w(\phi)$ are null in $\mathcal{T}$. By the corollary 4.4 (semantical AB^*) we deduce that $w(\phi)$ is null in $\mathcal{LS}$ which contradicts the hypothesis that ϕ is regular.

Hence we have for example that ψ_2 is a suffix of ψ_1. This entails that φ_2 is a prefix of φ_1. But both are wbp's of type @-λ thus by the lemma 3.2 are equal. Thus we have $\psi_i = \mathtt{v}_i\,\gamma_i\,u_1\,\lambda_1\,\varphi_1{}^r\,@\,u^r$; joined to the fact that ψ_2 is a suffix of ψ_1 we obtain $\psi_2 = \psi_1$. ■

8 Legal paths are regular

We shall prove in this section the coincidence of legal paths and (well-balanced) regular paths. For the regularity of every legal path, we need two lemmas. The former is proved in [2], the latter in [15].

Lemma 8.1 *Let M be a term, φ a legal path in $\mathcal{G}(M)$, ρ the leftmost outermost redex crossed by φ and M' the term obtained by reducing ρ. Then φ has a* unique *residual φ' in M' and φ' is in turn legal.*

Lemma 8.2 (The Lifting Lemma) *Let φ be a straight path in $\mathcal{G}(M)$, ρ be the leftmost outermost redex crossed by φ. Suppose φ has a residual φ' by the reduction of ρ. Then we have:*

$$w(\varphi) = Aw(\varphi')B^*$$

for some positive A and B in $\mathcal{LS}$.

REMARK. The theorem AB^* is a straightforward consequence of this lemma. One has just to show that if φ has no residual then its weight is null.

PROOF (sketch). Let @ and λ be the application and the lamda node defining the redex ρ, u be the function edge of @. We use some key properties of φ w.r.t. its leftmost outermost redex, namely that φ may be decomposed into:

$$\begin{aligned} \varphi = (&\varphi_0\,\gamma_0\,v\,u\,v'^*) \\ &\psi_0\,v'\,u\,v^*\,\gamma_0{}^*\,\varphi_1 \cdots \varphi_n\,\gamma_n\,v\,u\,v'^*\,\psi_n \\ &(v'\,u\,v^*\,\gamma_n{}^*\,\varphi_{n+1}) \end{aligned}$$

where v and v' are respectively the variable edge of λ and the argument edge of @, γ_i's are discriminants of occurrences of the variable bound by λ, φ_i's are subpaths of φ lying completely outside the argument of @ and ψ_i's are subpaths of φ lying completely inside the argument of @. The parenthesized subpaths may possibly be empty. From this a straightforward computation of the weight gives the result. ■

Theorem 8.3 *Every legal path φ is regular.*

PROOF. We prove by induction on the length of φ that $w(\varphi)$ has an AB^* form. If φ is the function edge of an application node then its weight is 1 so it is in AB^* form. Otherwise by definition of wbp, φ must cross some redex. Let ρ be the leftmost outermost redex of φ and φ' be the residual of φ by the contraction of ρ which uniquely exists and is legal by lemma 8.1. Then by induction hypothesis, the weight of φ' is AB^* for some positive elements A and B of $\mathcal{LS}$. But by the lifting lemma, there are some positive C and D such that $w(\varphi) = Cw(\varphi')D^* = CAB^*D^*$. ■

8.1 Another proof.

We shall give here the outline of another proof of the foregoing theorem. We call *cycle* a straight path of the form $\psi \equiv u\psi' u^r$ where u is the variable edge of a lambda node λ. Let $\gamma_1, \ldots \gamma_n$ be the discriminants bound by λ. We say that the cycle ψ is *deterministic* if ψ satisfies:

$$w(\gamma_i \psi \gamma_j{}^r) = 0 \iff \gamma_i \neq \gamma_j\,.$$

REMARK. In other words $\gamma_i\psi$ has a unique regular extension among the n possible ones as the name "deterministic" suggests.

With this definition, it is possible to show the following proposition:

Proposition 8.4 *If ψ is a deterministic cycle at some lambda node λ, then for any discriminant γ bound by λ the path $\gamma\psi\gamma^r$ is a v-cycle.*

REMARK. This proposition mirrors the @-cycle property 7.1. The @-cycle property states that a path having such geometrical property (being an @-cycle) has such weight property; the deterministic cycles property states that a path having such weight property has such geometrical property (it is a v-cycle). Also note that a v-cycle is a path of the form $\gamma u\varphi^r\psi\varphi u^r\gamma^r$ where γ is a discriminant bound by some lambda node λ, u is the variable edge of λ, φ is a wbp linking some application node @ to λ and ψ is an @-cycle at @; now the @-cycle property of ψ and the rendez-vous-property of φ show that $u\varphi^r\psi\varphi u$ is a deterministic cycle. This allows us to see the deterministic cycle property as a kind of converse of the @-cycle property.

Let now φ be a non regular wbp. Then one may show that φ has a subpath of the form $\gamma_1\psi\gamma_2$ where γ_1 and γ_2 are two *distinct* discriminants bound by the same lambda node λ and ψ is a deterministic cycle at λ. Therefore φ cannot be legal since this would entail that $\gamma_1 = \gamma_2$.

9 Consistent paths

Paths provide a basic semantics of sharing graphs [12, 10]. Let us rephrase the notions in [10] in $\mathcal{G}(M)$.

The (finite) *contexts* are terms defined by the following grammar:

$$a ::= \Box \mid \circ\cdot a \mid \star\cdot a \mid \sharp\cdot a \mid \natural\cdot a \mid \langle a, b\rangle.$$

We denote by $A_n[a]$ a context of the form:

$$\langle \ldots \langle a, a_n\rangle, \ldots a_1\rangle.$$

The *basic transformations of level n* are partial transformations on contexts defined for each n by:

$$\begin{aligned}
\mathtt{d}^n(A_n[\langle c, a\rangle]) &= A_n[\langle\langle c, a\rangle, \Box\rangle] \\
\mathtt{r}^n(A_n[\langle c, a\rangle]) &= A_n[\langle c, \star\cdot a\rangle] \\
\mathtt{s}^n(A_n[\langle c, a\rangle]) &= A_n[\langle c, \circ\cdot a\rangle] \\
\mathtt{t}^n(A_n[\langle\langle c, a\rangle, b\rangle]) &= A_n[\langle c, \langle a, b\rangle\rangle] \\
\mathtt{p}^n(A_n[\langle c, a\rangle]) &= A_n[\langle c, \natural\cdot a\rangle] \\
\mathtt{q}^n(A_n[\langle c, a\rangle]) &= A_n[\langle c, \sharp\cdot a\rangle]
\end{aligned}$$

To each oriented edge u (w.r.t. the natural orientation) of a graph $\mathcal{G}(M)$ we associate a basic transformation $\mathtt{h}_u$ of contexts in the following way:

- if u is not the premise of a node then $\mathtt{h}_u$ is the identity;
- if u is the premise of a node $\mathtt{n}$ of depth n then $\mathtt{h}_u$ is respectively $\mathtt{d}^n$, $\mathtt{r}^n$, $\mathtt{s}^n$, $\mathtt{t}^n$, $\mathtt{p}^n$ or $\mathtt{q}^n$ when we have respectively that $\mathtt{n}$ is a croissant node, $\mathtt{n}$ is a fan node and u is its left premise, $\mathtt{n}$ is a fan node and u is its right premise, $\mathtt{n}$ is a bracket node, $\mathtt{n}$ is an application or a lambda node and u is its argument or its variable edge, $\mathtt{n}$ is an application or a lambda node and u is its context or its body edge.

Partial transformations on contexts may be composed in the obvious way. For each oriented edge u of $\mathcal{G}(M)$, $\mathtt{h}_u$ is injective; therefore there is a partial transformation $\mathtt{h}_u^*$ defined by $\mathtt{h}_u^*(c) = d$ iff $\mathtt{h}_u(d) = c$. We define $\mathtt{h}_{u^r}$ to be $\mathtt{h}_u^*$. If ψ is a path in $\mathcal{G}(M)$ then $\psi = u_1 \ldots u_n$ where the u_i's are oriented edges or reversed oriented edges and we define:

$$\mathtt{h}_\psi = \mathtt{h}_{u_n} \ldots \mathtt{h}_{u_1}.$$

Definition 9.1 (Consistent paths.) *A* consistent path *is a path ψ such that $\mathtt{h}_\psi$ is defined on some context.*

Note that $\mathtt{h}_u$ is not in general defined on every context. For instance $\mathtt{p}^1$ is not defined on the context $\Box$. This is why we shall now shift to *infinite contexts*, i.e., functions from natural numbers to contexts.

We denote $\mathcal{C}^\infty$ the set of infinite contexts. We write $\langle\langle \cdots C(1)\rangle, C(0)\rangle$ the infinite context C so that any infinite context may be written $\langle C, a\rangle$ where C is in turn an infinite context and a is a context. With this convention the definition of the basic transformations may be easily extended to infinite contexts (replace the context c in the definition by the infinite context C), and since they are clearly injective on infinite contexts, we also may extend the definition of $\mathtt{h}_u^*$ and of $\mathtt{h}_\psi$ to infinite contexts. We denote by $\mathcal{H}$ the set of partial transformations over $\mathcal{C}^\infty$.

Lemma 9.2 *A path ψ is consistent iff its associated partial transformation $\mathtt{h}_\psi$ on* infinite *contexts has a non void domain.*

This is an immediate consequence of the definitions.

Proposition 9.3 *There is an interpretation of monomials in $\mathcal{H}$ such that $\mathcal{H}$ becomes a non trivial model of LS.*

PROOF. We note $\mathtt{0}$ nowhere defined transformation and $\mathtt{1}$ the identity on $\mathcal{C}^\infty$. If $\mathtt{h}$ is a partial transformation then we denote by $!(\mathtt{h})$ the transformation defined by:

$$!(\mathtt{h})\langle C, a\rangle = \langle \mathtt{h}(C), a\rangle .$$

The constants 0, 1 are interpreted by $\mathtt{0}$ and $\mathtt{1}$; p and q by $\mathtt{p}^0$ and $\mathtt{q}^0$; for each i the exponential coefficients d, t, r and s are interpreted respectively by $\mathtt{d}^0$, $\mathtt{t}^0$, $\mathtt{r}^0$, $\mathtt{s}^0$; the involution $*$ by the inversion of partial transformations and the morphism ! (in $\mathcal{LS}$) by ! (in $\mathcal{H}$). When it is not ambiguous, we shall drop the superscript 0 from the interpretations of the constants.

Note that if $\mathtt{b}^n$ is a basic transformation of level n then we have $!(\mathtt{b}^n) = \mathtt{b}^{n+1}$.

Let us check some of the axioms. The facts that the ! of $\mathcal{H}$ is a morphism and that the inversion is an antimorphism for composition are immediate. In particular if $\mathtt{h}$ is a transformation with full domain then $!(\mathtt{h})$ has also a full domain. This wasn't true on finite contexts and is the very reason why we shifted to infinite contexts.

Let $C = \langle C_0, a\rangle$ be an infinite context; then $\mathtt{p}^*\mathtt{p}(C) = \mathtt{p}^*\langle C_0, \sharp \cdot a\rangle = \langle C_0, a\rangle = C$. Furthemore $\mathtt{p}^*\mathtt{q}(C) = \mathtt{p}^*\langle C_0, \natural \cdot a\rangle$; but $\langle C_0, \natural \cdot a\rangle$ is not in the codomain of $\mathtt{p}$ thus not in the domain of $\mathtt{p}^*$, thus $\mathtt{p}^*\mathtt{q}$ is nowhere defined, i.e., equal to $\mathtt{0}$. Now C_0 may be written $\langle C_1, b\rangle$ so that $C = \langle\langle C_1, b\rangle, a\rangle$. Let $\mathtt{h}$ be any partial transformation; then $\big(!(\mathtt{h})\mathtt{t}\big)(C) = !(\mathtt{h})\langle C_1, \langle a, b\rangle\rangle = \langle \mathtt{h}(C_1), \langle a, b\rangle\rangle$ and $\big(\mathtt{t}!^2(\mathtt{h})\big)(C) = \mathtt{t}\langle\langle \mathtt{h}(C_1), b\rangle, a\rangle = \langle \mathtt{h}(C_1), \langle a, b\rangle\rangle$ so they are indeed equal; this shows that $\mathtt{t}$ satisfies the swap axiom of t. The computations for the other axioms are similar. ■

Theorem 9.4 *A path ψ in $\mathcal{G}(M)$ is consistent iff it is regular.*

PROOF. Obviously the interpretation of $w(\psi)$ in $\mathcal{H}$ is $\mathtt{h}_\psi$. By definition, ψ is consistent iff $\mathtt{h}_\psi$ is not the $\mathtt{0}$ transformation. The only if part of the theorem follows from the fact that $\mathcal{H}$ is a model of $\mathcal{LS}$. The if part is a consequence of the corollary 4.4 of the AB^* theorem. ■

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