Proof-nets and the Hilbert space

V. Danos & L. Regnier

Abstract

Girard's execution formula (given in [Gir88a]) is a decomposition of usual β-reduction (or cut-elimination) in reversible, local and asynchronous elementary moves. It can easily be presented, when applied to a λ-term or a net, as the sum of maximal paths on the λ-term/net that are not cancelled by the algebra L^* (as was done in [Dan90, Reg92]).

It is then natural to ask for a characterization of those paths, that would be only of geometric nature. We prove here that they are exactly those paths that have residuals in any reduct of the λ-term/net. Remarkably, the proof puts to use for the first time the interpretation of λ-terms/nets as operators on the Hilbert space.

1 Presentation

λ-Calculus is simple but not completely convincing as a real machine-language. Real machine instructions have a fixed run-time; a β-reduction step does not. Some implementations do map β-reductions into sequences of real elementary steps (as in environment machines for example) but they use a global time to achieve this. The "geometry of interaction" (GOI) is an attempt to find a low-level combinatorial code within which β-reduction could be implemented and such that:

- elementary reduction steps are local;
- parallelism shows up and global time disappears;
- some mathematics dealing with syntax is uncovered.

— Goal and organization of this paper.
A *persistent* path is a path on a λ-term which survives the action of any reduction (defined in [Reg92]). A *regular* path is a path which is not cancelled by Girard's algebraic device L^* (defined in [Gir88a]). The paper mainly aims at proving the equivalence of these two definitions and proceeds in the following order: first the geometric viewpoint on reduction is given, yielding a

definition of persistence; second the algebraic viewpoint, yielding a definition of regularity; and last the announced equivalence is proved.

Recently, Girard extended the definition of regularity to accomodate additives ([Gir95]). It would be interesting to investigate whether this matching between persistence and regularity still holds in the full case.

— Context of this paper.
The main result proved here was already known (see [Reg92]), but the method is new and leads to a simpler proof which is the first use of the linear structure of the Hilbert space in GOI. In another paper [ADLR94], together with A. Asperti and C. Laneve, we prove that persistence can be defined in two other ways: *consistency* ([AGL92]) and *legality* ([AL93]). Persistence is a most natural concept and these different ways to define it make it inescapable.

— Paths and execution.
It is fairly obvious that persistent paths describe the reduction. Intuitively they just are all the (anticipations of) cuts/redexes that ever happen during the reductions, and that's why they survive the action of any reduction. Indeed, Asperti and Laneve showed in [AL93] that each redex family (see [Lév78] for a definition) generated along the reduction of a given λ-term M corresponds to a unique persistent path in M. The converse also holds modulo a small technical detail (a balance condition). A similar result relating persistent paths to families of cut in nets was given earlier in [Reg92].

— Nets vs λ-calculus.
We present GOI with nets and not λ-terms. First because there are reduction-preserving embeddings of intuitionistic logic and classical logic into linear logic (LL). The intuitionistic embeddings, e.g., mapping $A \to B$ to $!A \multimap B$ or $!A \multimap !B$, allow to study λ-calculus directly inside nets. In the same way, the classical embeddings, e.g., mapping $A \to B$ to $!?A \multimap ?B$ or $!A \multimap ?!B$ (see [DJS94]), allow to study "classical λ-calculi" (for example, the $\lambda\mu$-calculus of Parigot [Par92]).

A second reason for choosing nets is that they provide a syntax that allows easy description of geometric entities like paths.

Everything here works for LL proof-nets as well as for any known kind of (pure) net(s) obtained by translations of λ-calculus, $\lambda\mu$-calculus. This is why we simply speak of nets.

2 A geometric point of view

In this first section we define nets and their reduction, the action of reduction on paths, say what persistent paths are, then prove there is a finite number of them in case the proof net is strongly normalizing (SN).

2.1 Nets

Nets are oriented graph structures whose nodes are called *links* and whose edges are labeled by formulae of linear logic. When drawing a net we represent edges oriented up-down so that we may speak of moving *downwardly* or *upwardly* in the graph. Possibly, e.g., when encoding pure λ-calculus in linear logic, formulae are quotiented by some duality-preserving equivalence relation.[1] Links are defined together with an arity and a coarity, i.e., a given number of incident edges called the *premises* of the link and a given number of emergent edges called the *conclusions* of the link.

- An **axiom** link has no premise and two conclusions labeled by dual formulae;
- a **cut** link has two premises labeled by dual formulae and no conclusion;
- *multiplicative links* **par** and **times** have two premises and one conclusion labeled by the **par** or the **times** of the premises;
- an **of course** link has one premise and one conclusion labeled by the **of course** of the premise;
- a **why not** link has n premises, for some integer n, all labeled by the same formula and one conclusion labeled by the **why not** of the premises.

Each edge is the conclusion of a unique link and the premise of at most one link. Edges which are not premise of a link are the *conclusions* of the net. Nets are required to fulfill two additional conditions:

Box condition: to each **of course** link `n` in the net R is associated a subnet B of R, called a *box* (represented by a rectangular frame), such that one conclusion of B, called the *principal door (pal)* of B, is the premise of `n`. All the other conclusions, called the *auxiliary doors (pax)* of B are premise of **why not** links. Two boxes are either disjoint or included one in the other. In the latter case, they may share some pax's but have distinct pal's.

Sequentialization condition: a net must be definable inductively in the following way: **axiom** is the basic case, **times** or **cut** of two nets, **par**, **why not** or boxing of one net. Purely geometrical conditions known as *correctness conditions* exist that are equivalent to the inductive one. We shall not discuss that here. The original definition of proof-nets is in [Gir87a]. Also a discussion on various correctness conditions may be found in [Dan90, DR89].

[1] I.e., quotiented in such a way that unsound cuts, e.g., between a **times** and an **of course**, never appear during reductions.

Remark. We use one general **why not** link with any number of premises. Possibly none, in which case we speak of a **weakening** link. There is a one-to-one correspondence between **of course** links and boxes. A premise of a **why not** link may be pax of several nested boxes.

The *height* of a link is the number of boxes in which it is contained and the height $h(\alpha)$ of an edge α is the height of its final link. An *exponential branch* is a premise of a **why not** link and an *exponential tree* is the set of premises of a **why not** link. The *lift* $l(\alpha)$ of an exponential branch α is the difference between the height of its initial link and the height of its final link. Thus it is always positive or null.

2.2 Paths in a net

A path is a sequence of edges or reverted edges, i.e., a path may take an edge from its goal to its source.[2] Let $\mathcal{P}(R)$ be the free category over R, that is, $\mathcal{P}(R)$ is the category of paths on R. We invert the usual convention by taking $\phi_1\phi_2$ to mean "ϕ_2 then ϕ_1", because in the sequel we will weight paths by functions. Thus $\phi_1\phi_2$ is a path if ϕ_2 ends where ϕ_1 begins. The operation of reversing paths (denoted ϕ^*) is a contravariant involution making $\mathcal{P}(R)$ an *involutive category.*

Edges will be denoted by α, β, γ, δ, σ whereas we use ϕ and ψ for general paths. The notation $\alpha^{(*)}$ means α and/or α^* depending on the context.

Let ϕ be a path. We say that ϕ is *non-bouncing* (nB) if it doesn't contain any $\alpha^*\alpha$; that ϕ is *non-twisting* (nT) if it doesn't contain any $\alpha_i^*\alpha_j$, with α_i and α_j distinct premises of a same link; that ϕ is *straight* if it is neither bouncing nor twisting. Note that neither nB nor nT is preserved by composition.

A straight path is a path changing direction only in axioms and cuts, bouncing up and down between the ceiling and the floor; so there is a finite number of them in a normal form, because there is no floor.

Let $\mathcal{P}^+(R)$ be the set of finite multisets of paths in $\mathcal{P}(R)$. We denote by $\sum \phi_i$ the multiset whose elements are the ϕ_i's. In particular we denote by ϕ the multiset whose single element is the path ϕ and respectively by 0 and 1 the empty multiset and the multiset whose elements are the empty paths on all the nodes.[3] If ψ and ϕ are two uncomposable paths in $\mathcal{P}(R)$, we write $\phi\psi = 0$ in $\mathcal{P}^+(R)$ and define composition in $\mathcal{P}^+(R)$ by:

$$\left(\sum \phi_i\right)\left(\sum \psi_j\right) = \sum \phi_i\psi_j.$$

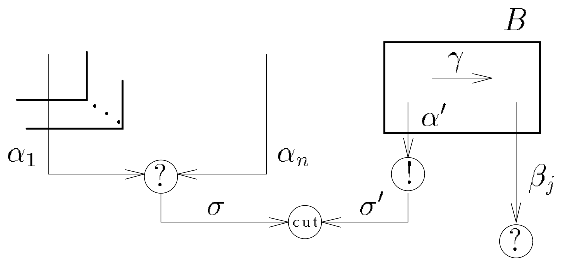
Note that 0 is neutral for $+$, absorbing for composition and 1 is neutral for composition in $\mathcal{P}^+(R)$. If $\Phi = \sum \phi_i$ is an element of $\mathcal{P}^+(R)$ then Φ^* is $\sum \phi_i^*$. Thus $\mathcal{P}^+(R)$ is an *involutive semiring.*

[2] Think of paths as walks (not drives) in one-way streets.

[3] That is, the sum of the identities of $\mathcal{P}(R)$.

2.3 Reduction of a net

Reductions are sequences of *elementary reduction steps* (*ers*). Apart from the axiom step (a), there is exactly one such ers for each couple of dual links: the exponential one (e), the multiplicative one (m). We will forget the weakening ers here, assuming that all the reductions are strict. This assumption doesn't make much sense in λ-calculus (since restriction to strict reductions may in some cases turn a non SN term into a SN one) but it is innocuous when working with nets. Also, since they are much simpler to handle, we will discard the cases of the axiom and multiplicative steps, only dealing with the exponential step.



reduces in:

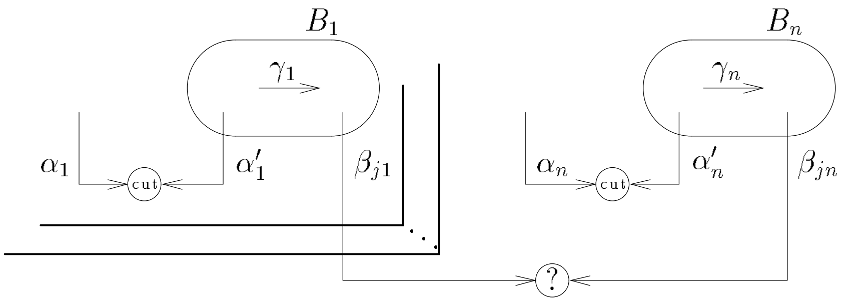


Figure 1: exponential reduction

What happens during the exponential reduction step (see fig. 1 for notations) is that the box B dispatches n copies $B_1, \ldots, B_n$ of itself glued:

- by the (once pal) α' to the top of $\alpha_1, \ldots, \alpha_n$ (thus creating n new cuts);
- and by the (pax's) $\beta_1, \ldots, \beta_m$ to their anchor ?-vertices.

For each i, the ith copy is moved inside the $l(\alpha_i)$ boxes of which α_i is a pax.

Let's classify the edges of R into three groups:

- *consumed* edges: σ and σ';
- *duplicated* edges: γ's inside the box become $\gamma_1, \ldots, \gamma_n$'s, and edges crossing the box, namely α' and β_j's become respectively $\alpha'_1, \ldots, \alpha'_n$ and $\beta_{j1}, \ldots, \beta_{jn}$'s;
- *intact* edges: α_i's and δ's lying elsewhere.

2.4 Reduction on paths

2.4.1 The lifting functor

Let ρ be a reduction on a net R leading to a net $\rho(R)$. The *lifting functor* $\overline{\rho} : \mathcal{P}(\rho(R)) \longrightarrow \mathcal{P}(R)$ is the functor defined by:

- if ρ is a sequence of ers, then $\overline{\rho}$ is the composition of the lifting functors associated to each ers in ρ;
- if ρ is an exponential ers then with the notations of the previous section: $\overline{\rho}(\alpha_i) = \sigma\alpha_i$, $\overline{\rho}(\alpha'_i) = \sigma'\alpha'$, $\overline{\rho}(\beta_{ji}) = \beta_j$, $\overline{\rho}(\gamma_i) = \gamma$ and $\overline{\rho}(\delta) = \delta$ for any intact edge δ;
- in the cases of the a and m ers, the definition is similar.

Because of the exponential ers, $\overline{\rho}$ is not injective (faithful): for instance the α'_i's have the same image. It is not surjective (full) either: for example α_i is not in the image of $\overline{\rho}$, only $\sigma\alpha_i$ is. Some paths even have no extensions in the image of $\overline{\rho}$: typically $\alpha_1^*\alpha_2$ and $\alpha_1^*(\sigma^*\sigma'\alpha')\psi(\sigma^*\sigma'\alpha')^*\alpha_2$ where ψ is any path entirely contained in the box B.

2.4.2 The deformation pseudo-functor

Let ρ be an ers reducing a cut c. A path ψ is *long enough* for ρ if its first and last nodes are neither the cut link c, nor one of the links connected by c. Note that a conclusion-to-conclusion path is long enough for any reduction. Given a nonempty path ϕ in $\mathcal{P}(R)$, we say that ϕ' in $\mathcal{P}(\rho(R))$ is a *residual* of ϕ along ρ if ϕ' is minimal such that ϕ is a subpath of $\overline{\rho}(\phi')$.

In case ϕ is long enough for ρ, if ϕ' is a residual of ϕ then $\overline{\rho}(\phi') = \phi$ so that the set of residuals of ϕ is exactly $\overline{\rho}^{-1}(\phi)$. If ϕ is not long enough for ρ then ϕ is a proper subpath of $\overline{\rho}(\phi')$.

If ϕ is nonempty and has a residual ϕ' along ρ (which is not always the case as the example above show) then ϕ' is nonempty.

We extend by induction on the length of a reduction ρ the definitions of residuals along ρ and paths long enough for ρ. For each reduction ρ, we define the *deformation* action $\rho : \mathcal{P}(R) \longrightarrow \mathcal{P}^+(\rho(R))$ by:

$$\rho(\phi) = \sum \text{residuals}(\phi).$$

The deformation ρ is not a functor. However, if we restrict to long enough paths we have:

$$\rho(\phi) = \sum\{\phi' \in \mathcal{P}(\rho(R)), \quad \overline{\rho}(\phi') = \phi\},$$

and ρ is functorial from long enough paths of $\mathcal{P}(R)$ into $\mathcal{P}^+(\rho(R))$.

Note that because $\overline{\rho}$ is not injective, in general $\rho(\phi)$ is a sum of paths (of equal lengths if ρ is one-step). Also, because $\overline{\rho}$ is not surjective, there are some ϕ such that $\rho(\phi) = 0$. The kernel of ρ, i.e., the set of paths with no residuals, measures the connectivity loss of the net under the deformation ρ.

2.4.3 Computing the deformation of a long enough straight path

Let ρ be an exponential ers, ϕ be a straight path long enough for ρ and assume the notations of fig. 1. Note that ϕ may cross the box B in three different ways (see fig. 2), and since the remaining parts of ϕ are left untouched by ρ, it clearly suffices to compute the action of ρ on B-*crossings* of ϕ, i.e., subpaths of ϕ that cross B. There are three kinds of B-crossings.

palpal (represented by a continuous line in fig. 2): paths starting from an α_i, crossing the cut, entering B by its pal, moving into B, exiting B by its pal, crossing the cut (the other way round) and ending in some α_j^*:

$$\alpha_j^*(\sigma^*\sigma'\alpha')\,\psi\,(\sigma^*\sigma'\alpha')^*\alpha_i$$

where ψ is a path inside B;

palpax (and its dual **paxpal**, represented by a dashed line in fig. 2): paths starting like the preceding ones but this time exiting B by some pax β_j:

$$\beta_j\psi\,(\sigma^*\sigma'\alpha')^*\alpha_i;$$

paxpax (represented by a small dashed line in fig. 2): paths entering and exiting B by some pax's β_i and β_j:

$$\beta_j\psi\beta_i^*$$

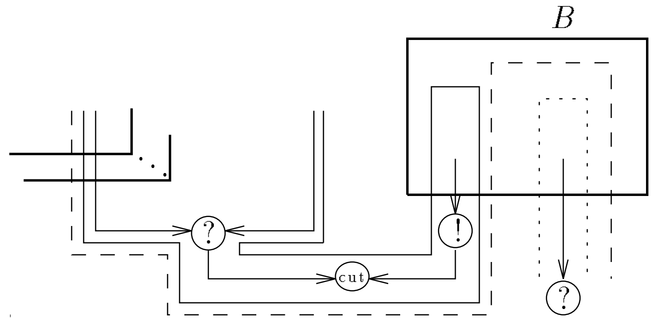


Figure 2: Crossings of a box

It is an easy computation following the definition of ρ to see that our three different kinds of crossings are deformed by ρ in the following way:

$$\begin{array}{rcll} \alpha_j^*(\sigma^*\sigma'\alpha')\,\psi\,(\sigma^*\sigma'\alpha')^*\alpha_i & \xrightarrow{\rho} & \alpha_i^*\alpha_i'\,\psi_i\,\alpha_i'^*\alpha_i & \text{if } j=i,\ 0 \text{ otherwise} \\ \beta_j\psi\,(\sigma^*\sigma'\alpha')^*\alpha_i & \xrightarrow{\rho} & \beta_{ji}\psi_i\alpha_i'^*\alpha_i & \\ \beta_j\psi\beta_k^* & \xrightarrow{\rho} & \sum_l \beta_{jl}\psi_k\beta_{kl}^* & \end{array}$$

In the case of the first type of crossing (palpal) with $i \neq j$ the deformation simply cancels the path. In this case we say that ϕ *exchanges the premises* of the **why not** link. Conversely, it is only if ϕ contains such a crossing that it is cancelled by the deformation. For example, if ρ is a reduction of R such that no residual of ϕ ever crosses a cut reduced by ρ, then $\rho(\phi) \neq 0$.

2.4.4 Persistent paths

Definition 1 *A* persistent *path is any straight path ϕ such that for all reduction ρ, $\rho(\phi) \neq 0$ in $\mathcal{P}^+(\rho(R))$.*

Persistent paths are those that survive all possible deformations. An equivalent definition would be: ϕ in $\mathcal{P}(R)$ is persistent iff for any reduction ρ, there is a ϕ' in $\mathcal{P}(\rho(R))$ such that ϕ is a subpath of $\overline{\rho}(\phi')$.

Note that being straight is preserved by the deformation ρ. And also that any nonempty subpath of a persistent path is still persistent. Thanks to that remark, in the future we will often restrict to long enough paths.

As said before, a typical example of non-persistent path is a palpal crossing exchanging the premises of the **why not** link; and persistence amounts to saying that such a crossing never happen along any reduction.

2.5 Preservation of finiteness of computations

Lemma 2 (Contraction of paths) *Let ρ be an ers, ϕ be a path long enough for ρ and $|\rho(\phi)|$ denote the common length of all terms in the sum $\rho(\phi)$:*

$$|\phi|/2 - 2 \leq |\rho(\phi)| \leq |\phi|$$

under the condition for the left inequation that ϕ is nB and $\rho(\phi) \neq 0$. Furthermore the right inequation is strict iff ϕ crosses the cut reduced by ρ.

Theorem 3 *R is SN iff there is a finite number of persistent paths in $\mathcal{P}(R)$.*

Proof. The left contraction inequation says that reducing a persistent path costs about the logarithm of its length in the fastest case, hence: $|\phi| \leq O(2^r)$ where r is the length of a longest reduction of R.[4]

The right contraction inequation says an ers (through the cut of which the path goes) contracts the path. Hence, following an infinite branch in the reduction space of R, one will get arbitrarily long paths by lifting back cuts to R. But will these lifted cuts be persistent ? It needs the adequacy theorem (regular equivalent to persistent), proved in the last section, to conclude that they are, since they are certainly regular. □

3 An algebraic point of view

In this section we borrow from Girard's original paper ([Gir88a]) the algebraic tool to characterize regular paths, namely L^*. We then build models of this algebra, most notably within $B(H)$ the set of bounded operators on the Hilbert space, and show that any model is as good as L^* in catching regular paths. This property will be at the heart of the proof of the main result in the next and last section.

3.1 The weight

Let R be a net and $T_1, \ldots, T_n$ be an enumeration of the exponential trees of R. We define the equational theory L^*_R attached to R and then the so-called *weight* functor w.

The theory L^*_R is defined by the following generators and relations: L^*_R is a free involutive monoid, i.e., there is a mapping $u \longrightarrow u^*$ in L^*_R such that:

$$(u^*)^* = u \qquad \text{and} \qquad (uv)^* = v^*u^*,$$

[4] [DR93] was our reaction against this frightening bound which is reached by the family $(\cdots(I\,I)\cdots I)$ where I denotes the term $\lambda x.x$.

for any u and v. Note that this entails in particular that $1^* = 1$.

There are a 0, a morphism !:

$$!(0) = 0, \quad !(1) = 1, \quad !(u^*) = !(u)^* \quad \text{and} \quad !(u)!(v) = !(uv);$$

two *multiplicative generators* p and q satisfying:

$$p^*p = q^*q = 1, \tag{1}$$
$$p^*q = 0; \tag{2}$$

and for each exponential tree $T_i = (\beta_{ij})_j$ in R, a family $(b_{ij})_j$ of *exponential generators* satisfying the zapping (or annihilation, or orthogonality) relations:

$$b^*_{ij}\, b_{ik} = \delta_{jk}; \tag{3}$$

and the swapping (or commutation) relations:

$$!(u)\, b_{ij} = b_{ij}\, !^{e_{ij}}(u) \tag{4}$$

where u is any element and $e_{ij} = l(\beta_{ij})$ is the lift of β_{ij}. We say that the generator b_{ij} has lift $l(b_{ij}) = e_{ij}$.

In the sequel, unless explicitly stated, we shall write $u = v$ for "u and v are provably equal in the equational theory L^*_R".

The *weight* functor w from $\mathcal{P}(R)$ to L^*_R is defined on edges by: if α is the left (resp. right) premise of a multiplicative link then $w(\alpha) = !^{h(\alpha)}(p)$ (resp. $!^{h(\alpha)}(q)$); if β_{ij} is an exponential branch then $w(\beta_{ij}) = !^{h(\beta_{ij})}(b_{ij})$; if α is any other edge then $w(\alpha) = 1$.

Remarks.

- The equations for p and q are the same as the zapping equations of a binary exponential tree;
- commutations are linear equations (a remark of use only if one finds a model of L^*_R that allows linear combinations);
- annihilations are self-dual; by dualizing the commutations equations one finds: $b^*_{ij}\, !(u) = !^{e_{ij}}(u)\, b^*_{ij}$;
- the equations have a natural orientation: from lhs to rhs. W.r.t. that orientation, starred generators are inclined to travel eastwards and unstarred ones westwards;
- w is a local valuation on each exponential tree of R, i.e., there are no relations between exponential generators of two different exponential trees. On the other hand, the same p and q are used to weight all multiplicative links.

Definition 4 (regular paths) *A path ϕ in $\mathcal{P}(R)$ is* regular *iff $w(\phi) \neq 0$.*

Note that twisting paths are not regular; neither are the palpal crossings with i distinct from j as the reader can check.

We denote by S_R the quotient category $\mathcal{P}(R)/w$ of *weighted paths*. By analogy with $\mathcal{P}^+(R)$, we denote S_R^+ the set of finite multisets of weighted paths in which all the 0-paths are identified to the empty multiset, and composition is defined by distributivity over sums (multiset unions). Therefore S_R^+ is in turn an involutive semiring.

3.2 Execution formula

To a net R, we associate a self-dual element of S_R^+ by the so-called *execution formula*:

$$\mathsf{EX}(R) = \sum_{\phi \in \mathcal{P}_c(R)} w(\phi),$$

where $\mathcal{P}_c(R)$ denotes the set of conclusion-to-conclusion straight paths of $\mathcal{P}(R)$. We'll see at the end of the next section (second part of the adequacy theorem), that it contains the same information as the set of persistent conclusion-to-conclusion paths, and hence may be thought of as a computation of R.

This is another presentation of the formula given in [Gir87b, Gir88a].

3.3 Small models of L_R^*

We will now present the *small models* of the equational theory L_R^*. This construction proves by the way that $0 \neq 1$ in L_R^*.

The basic structure. Let $\mathbb{N}$ be the set of natural numbers and $\mathrm{PI}(\mathbb{N})$ be the inverse monoid of partial injective mappings from $\mathbb{N}$ to $\mathbb{N}$. Product is (partial) composition, * is the inversion, 0 is the nowhere defined mapping and 1 is the full identity on $\mathbb{N}$.

The shift morphism. Fix a bijection $\lceil .,. \rceil : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ and, given u and v in $\mathrm{PI}(\mathbb{N})$, define $u \otimes v$ in $\mathrm{PI}(\mathbb{N})$ by:

$$(u \otimes v)(\lceil x, y \rceil) = \lceil u(x), v(y) \rceil.$$

Then $!(u) = 1 \otimes u$ is a morphism on $\mathrm{PI}(\mathbb{N})$, which we call the *shift morphism*. Define $\lceil x_1, \ldots, x_n \rceil$ as an abbreviation for $\lceil x_1, \ldots \lceil x_{n-1}, x_n \rceil \ldots \rceil$.

Modeling the swapping relations. Let $\tau : \mathbb{N}^n \longrightarrow \mathbb{N}$ be a partial injection (where in the case $n = 0$, τ is a constant in $\mathbb{N}$); then $u_\tau(\lceil x_1, \ldots, x_{n+1} \rceil) = \lceil \tau(x_1, \ldots, x_n), x_{n+1} \rceil$ satisfies the swapping relation with lift n ($n \geq 0$). Any solution to that relation, $\lceil .,. \rceil$ and $!(.)$ being given, has to be of that form; in particular, no solution with lift zero is to be found here with a full range.

Modeling the zapping relations. If τ has full domain, then $u_\tau^* u_\tau = 1$; if τ and τ' have disjoint ranges in $\mathbb{N}$, then $u_\tau^* u_{\tau'} = 0$. This is enough to interpret the zapping relations.

3.4 How to make them bigger

Take H to be the Hilbert space with a countable basis $X = (e_n)$. The *big model* of L_R^* is the set of bounded operators on H, denoted by $B(H)$. The interest of this new model is that we are going to use our remark about the linearity of the commutation relation by superposing "small" solutions; another interest may be understood by reading [Gir88b] or [MR91].

The basic structure. $B(H)$ is clearly an involutive monoid (w.r.t. composition and adjunction) wherein there is a zero and a one. It even is a C^*-algebra.

The shift morphism. Take $\mu : H \otimes H \longrightarrow H$ to be the isomorphism defined by $\mu(e_i \otimes e_j) = e_{\lceil i,j \rceil}$, where $H \otimes H$ means the completion of the usual vector space tensor product w.r.t. the inner product defined by $\langle h_1 \otimes h_1', h_2 \otimes h_2' \rangle = \langle h_1, h_2 \rangle . \langle h_1', h_2' \rangle$.

For each U in $B(H)$ put $!(U) = \mu(1 \otimes U)\mu^{-1}$. Then $!(.)$ is a C^*-algebra isomorphism mapping $B(H)$ isometrically into itself.

The only point which is not obvious is $\|!(U)\| = \|U\|$. Take h in H of norm 1, then $\|(1 \otimes U)(h \otimes h)\| = \|U(h)\|$ so $\|!(U)\| = \|1 \otimes U\| \geq \|U\|$.

There is another way to see this. Take D as defined in the second paragraph below with (d_i) such that D is an isometry. We have $!(U)D = DU$, so $\|!(U)\| \geq \|!(U)D\| = \|DU\| = \|U\|$.

Conversely, for any h_1, h_2 in $B(H)$, $\|(1 \otimes U)(h_1 \otimes h_2)\| = \|h_1\| . \|U(h_2)\| \leq \|U\| . \|h_1\| . \|h_2\|$ which yields the converse inequality.

We have:

$$\langle !(U)(e_{\lceil i_1, i_2 \rceil}), e_{\lceil j_1, j_2 \rceil} \rangle = \delta_{i_1 j_1} \langle U(e_{i_2}), e_{j_2} \rangle \tag{5}$$

Hence the shift morphism preserves being a partial permutation of the basis (see next paragraph) and being a partial isometry.

Upgrading partial injections to partial isometries. For any u in $\mathrm{PI}(\mathbb{N})$ there is a partial isometry U in $B(H)$ defined by:

$$U(e_n) = e_{u(n)} \text{ if } u(n) \text{ is defined, } 0 \text{ otherwise.}$$

Such U's are partial permutations of the basis X. One can easily check that zapping and swapping equations are preserved by upgrading.

Modeling the 0-lift swapping equation. The following lemma is proved by calculation.

Lemma 5 *The solutions D in $B(H)$ of the 0-lift swapping equation $!(U)D = DU$ have the form: $D = \sum_i d_i D_i$ where $D_i(e_n) = e_{\lceil i,n \rceil}$ (upgrading of the 0-lift injection $n \longrightarrow \lceil i,n \rceil$) and $\sum_i |d_i|^2 = d^2 < \infty$. Moreover $\|D\| = d$ (so such a D is indeed continuous).*

Note that:
$$D^*D = \sum_{ij} \bar{d}_i d_j D_i^* D_j = d^2,$$

so if $d = 1$, then D^*D is a self-adjoint idempotent (a projector) and D an isometry. Its final projector satisfies:

$$DD^*(e_{\lceil n_1,n_2 \rceil}) = \sum_{ij} d_i \bar{d}_j D_i D_j^*(e_{\lceil n_1,n_2 \rceil}) = \sum_i d_i \bar{d}_{n_1} D_i(e_{n_2}) = \bar{d}_{n_1} \sum_i d_i e_{\lceil i,n_2 \rceil}$$

Hence:
$$\langle DD^*(e_{\lceil n_1,n_2 \rceil}), e_{\lceil n_1,n_2 \rceil} \rangle = \|DD^*(e_{\lceil n_1,n_2 \rceil})\|^2 = |d_{n_1}|^2, \tag{6}$$

therefore, if we assume that $d_n \neq 0$ for all n, then $DD^*(e_n) \neq 0$ for all n, which is a key property of our interpretation.

X-positivity. An operator U on H is said to be *X-positive* if for all i, j, we have $\langle U(e_i), e_j \rangle \geq 0$. In other words U sends H^+ (all h such that $\langle h, e_i \rangle \geq 0$ for all i) into itself. We denote the set of X-positive bounded operators on H by $B(H^+)$.

Lemma 6 (Everybody can be taken X-positive) *The algebra $B(H^+)$ is stable by product, sum, adjunction, shift, multiplication by a positive real and contains all upgraded partial injections as well as D-like operators with $d_i > 0$ for all i.*

The interpretations of L^*_R in $B(H)$ that we will consider in the next section will be in fact in $B(H^+)$.

The non-interception argument. This is a preliminary step for the adequacy proof, see next section.

Define $h_1 \geq h_2$ iff $h_1 - h_2$ is in H^+.

Lemma 7 *V is X-positive iff it is increasing w.r.t. this partial ordering.*

Proof. By linearity of V we have: $\langle V(h), e_i\rangle = \sum_j \langle h, e_j\rangle\langle V(e_j), e_i\rangle$, so if $h_1 \geq h_2$, since V is X-positive, for all i, j we have that $\langle h_1, e_j\rangle\langle V(e_j), e_i\rangle$ is greater than $\langle h_2, e_j\rangle\langle V(e_j), e_i\rangle$. Hence $\langle V(h_1), e_i\rangle$ is greater than $\langle V(h_2), e_i\rangle$. This computation shows that V, if not zero, is even strictly increasing. Conversely, $e_i \geq 0$, hence if V is increasing, for all j, $\langle V(e_i), e_j\rangle \geq 0$. □

Lemma 8 (No-interception) *Take U_1, U_2 and P in $B(H^+)$, P being a projector. Suppose further that for all n, $P(e_n) \neq 0$, then:*

$$U_1 P U_2 = 0 \quad \Rightarrow \quad U_1 U_2 = 0.$$

Proof. Suppose $U_1U_2 \neq 0$. There exist a vector of the basis, say f_2, such that $U_1U_2(f_2) \neq 0$. Now: $U_1U_2(f_2) = \sum_i \langle U_2(f_2), e_i\rangle U_1(e_i)$, so there also exist another vector of the basis, say f_1, such that $\langle U_2(f_2), f_1\rangle \neq 0$ and $U_1(f_1) \neq 0$

Put $\langle U_2(f_2), f_1\rangle = \lambda$ and $\langle P(f_1), f_1\rangle = \lambda'$. Since U_2 and P are X-positive, λ and λ' are non-negative reals; by construction of f_2, $\lambda > 0$; by the further assumption on P, $P(f_1) \neq 0$. But P being a projector, $\langle P(f_1), f_1\rangle = \|P(f_1)\|^2$, so $\lambda' > 0$. Now, f_1 and f_2 being basis vectors, $U_2(f_2) \geq \lambda f_1$; since P is X-positive, by the preceding lemma, P is increasing w.r.t. $\geq$, so $PU_2(f_2) \geq \lambda P(f_1) \geq \lambda\lambda' f_1$. Finally, U_1 being also X-positive (by the preceding lemma again), we get $U_1PU_2(f_2) \geq \lambda\lambda' U_1(f_1) > 0$. □

3.5 The ab^*-like theorems

We consider the rewriting system associated to L^*_R by orienting the equations of section 3.1 from left to right. An ab^*-form (or stable form) is a term of the equational theory L^*_R of the form: $m := a!(m)b^* \mid 1$ where a and b are flat (no !(.)'s), positive (no *'s) and possibly empty words on the generators. Such a form is normal w.r.t. the oriented equations, although other normal forms exist: consider for instance $b^*_{00}\, b_{10}$.

Theorem 9 (ab^* **version 1)** *For all straight ϕ, either there is a unique form ab^* into which $w(\phi)$ rewrites, or $w(\phi)$ rewrites to zero by means of the oriented equations.*

Proof. (Sketchy, see [Reg92] for a complete proof) By the right "contraction inequation", we can sort out an inductive proof on the length of ϕ using *special* exponential steps ρ w.r.t. ϕ. Special means that ϕ contains no cut edge below the pax's of the door associated to ρ and crosses the cut associated to ρ.

Given ϕ, such a ρ always exists; this follows from the correctness condition on nets. Now in the case ρ is special relative to ϕ, paxpax crossings can't occur, so $\rho(\phi)$ is a path (not a sum of paths) the weight of which rewrites to ab^* iff the weight of ϕ does as an easy calculation shows. □

In fact the theorem is true for any path, not necessarily straight.

A first easy consequence of ab^* is the endogamy of exponential trees: during the rewriting of $w(\phi)$ no two generators of different exponential trees are ever in zapping position. This is why we don't need any relation between b_{ij} and $b_{i'j'}$ when $i \neq i'$ in the definition of L^*_R. An immediate but interesting corollary is that any straight path that never uses more than one branch per exponential tree is regular.

A second consequence of ab^*, stating a kind of no-backtrack property, is:

Lemma 10 (Stretching) *Any regular straight path that has straight extensions, has at least one regular proper straight extension.*

Another more significant consequence, which needs the extension of ab^* to bouncing paths to be proved is:

Theorem 11 (ab^* version 2) *The category of weighted paths S_R is an involutive category which is:*
(i) inversive : it satisfies $(x^*)^* = x$, $xx^*x = x$ and $xx^*yy^* = yy^*xx^*$ for all weighted paths x and y;
(ii) decidable in linear time.

Finally, the computation of regularity doesn't depend on the model of L^*_R. This lemma will be at the heart of the proof of the adequacy theorem in the next section.

Lemma 12 *Let $\mathcal{L}$ be a non trivial model of L^*_R and ϕ be a path in $\mathcal{P}(R)$. Then $w(\phi) = 0$ iff $\mathcal{L} \models w(\phi) = 0$.*

Proof. The only if part is just the definition of model. For the if part, suppose $w(\phi) \neq 0$ in L^*_R. By ab^* version 1, there exist some positive words a and b such that $w(\phi) = ab^*$. Thus $a^*w(\phi)b = 1$ in L^*_R, therefore also in $\mathcal{L}$. Since $\mathcal{L}$ is non trivial we deduce that $\mathcal{L} \models w(\phi) \neq 0$. □

Conversely, lemma 12 implies ab^* version 1. Take a counterexample to ab^*, it must be of the form x^*x' where x and x' are either generators belonging to different exponential trees, either one of them is an exponential generator whereas the other is multiplicative, or one of them is a multiplicative whereas the other has the form $!(y)$; these are the only cases in which no oriented equation of L^*_R applies. A look at the small models shows that one can choose x and x' orthogonal (i.e., such that $x^*x' = 0$), hence by the lemma they are already zero in L^*_R.

This last version of ab^* explains why the "context semantics" of Lamping and [AGL92] can be made to compute persistent paths as well as the two original models of Girard in [Gir88a].

4 Where they meet

Now we are all set to prove what we were aiming at. We first state the fundamental lemma.

Lemma 13 (Preservation of kernels) *Let ϕ be in $\mathcal{P}(R)$, and ρ be a reduction of R:*

$$w(\phi) = 0 \qquad \textit{iff for all } \phi' \in \rho(\phi), \quad w(\phi') = 0.$$

The if part is called "hardlemma", and the converse "easylemma".

Proof. First a word on tactics. By the semantic formulation of the ab^* property (lemma 12), it is enough to prove the preservation in some non-trivial model $\mathcal{L}$ of both L^*_R and $\mathsf{L}^*_{\rho(R)}$. Ideally we would like to build $\mathcal{L}$ together with an interpretation of the weight functors such that $w(\phi) = w(\rho(\phi))$ is satisfied in $\mathcal{L}$. That's almost what we are going to achieve (that is except in one limit case, where ρ is such that all α_i have lift zero, which calls for a weaker result).

Before continuing, let us fix a small gap in what we have just said. We want to compare $w(\phi)$ and $w(\rho(\phi))$, in some model to be built. However, $\rho(\phi)$ being, in general, a multiset of paths, the expression $w(\rho(\phi))$ has no sense. For this reason, during the proof we shall add a formal sum in the usual way to the definition of L^*_R, and define $w(\sum_i \psi_i)$ to be $\sum_i w(\psi_i)$ for any family of paths ψ_i. We interpret this sum in the big model $B(H)$ by the sum of operators. In the small models we define a partial sum by: $f+g$ is defined only if $fg^* = f^*g = 0$ (i.e., f and g have disjoint domains and disjoint codomains), in which case $f+g(x)$ is $f(x)$ if x is in the domain of f, $g(x)$ is x is in the domain of g. Then $f+g$ is obviously a partial injection and composition is distributive over partial sum. Note that the upgrading from small to big models maps this partial sum on the sum of operators. It is enough to define only a partial sum since as we will see, we only consider sums of weights $\sum_i u_i$ such that $u_i u_j^* = u_i^* u_j = 0$ in L^*_R.

After these preliminaries let us go in the proof. First note that we obviously may restrict to a one-step reduction. We will consider only the case of an exponential ers ρ. By the stretching property, we may without loss of generality assume that ϕ is long enough w.r.t. ρ. Remark also that we may as well assume that ϕ neither twists nor contains palpal crossings with i distinct from j, for if it does the lemma boils down to $0 = 0$ which holds indeed.

We use the notations of the definition of the exponential ers in section 2.3. If δ is an intact edge of R we assume it has the same weight in R and $\rho(R)$.

Let h be the height of the box B and $l_i = l(\alpha_i)$ be the lift of α_i. If γ is an edge lying inside B, its weight is $!^{h+1}(u)$ where u is some generator of L^*_R. By definition, γ has n residuals $(\gamma_i)_{1\leq i\leq n}$ in $\rho(R)$ and the weight of γ_i is $!^{h+l_i}(u_i)$ where u_i satisfy the same zapping and swapping equations as u. Therefore we may interpret u and u_i by the same object in $\mathcal{L}$ and that's what we do. We call this property of the interpretation the *box relation*. With the box relation, if ψ is a path contained in the box B and ψ_i is one of its residual in $\rho(R)$, then we have $w(\psi) = !^{h+1}(u)$ and $w(\psi_i) = !^{h+l_i}(u_i)$ where u and u_i are elements of L^*_R and $\mathsf{L}^*_{\rho(R)}$ such that $\mathcal{L} \models u = u_i$.

Let's call the weights of α_i, β_j and the residuals β_{ji} of the latter, respectively x_i, y_j and z_{ji}. Thus x_i is an exponential generator of lift $l(x_i) = l_i$.

Assuming the notations of section 2.4.3 we now inspect each way ϕ crosses the box B.

(a) For the palpal crossing the (weight of the) lhs is:

$$w(\alpha_i^*(\sigma^*\sigma'\alpha')\,\psi\,(\sigma^*\sigma'\alpha')^*\alpha_i) = !^h(x_i^*)!^{h+1}(u)!^h(x_i) = !^{h+l_i}(u)$$

while the rhs is:

$$w(\alpha_i^*\alpha_i'\psi_i\alpha_i'^*\alpha_i) = !^{h+l_i}(u_i)$$

so that the equality holds in $\mathcal{L}$ by the box relation.

(b) For the palpax crossing let h_j denote the height of β_j (which by the way is the same as that of β_{ji}); thus h_j is lower or equal than h. The lhs is:

$$w(\beta_j\psi\,(\sigma^*\sigma'\alpha')^*\alpha_i) = !^{h_j}(y_j)!^{h+1}(u)!^h(x_i) = !^{h_j}(y_j)!^h(x_i)!^{h+l_i}(u),$$

while the rhs is:

$$w(\beta_{ji}\psi_i\alpha_i'^*\alpha_i) = !^{h_j}(z_{ji})!^{h+l_i}(u_i).$$

We ask that $\mathcal{L}$ satisfies the so-called *coagulation relation*:

$$z_{ji} = y_j!^{h-h_j}(x_i). \tag{7}$$

An easy computation shows that this definition of (the interpretation of) z_{ji} is consistent w.r.t. the zapping and swapping relations, because $l(z_{ji}) = l_i + l(y_j) - 1$. With the box and the coagulation relations we get identity of weights of palpax crossings in $\mathcal{L}$.

(c) For the paxpax crossing, the lhs is:

$$w(\beta_j\psi\beta_k^*) = !^{h_j}(y_j)!^{h+1}(u)!^{h_k}(y_k^*).$$

Now by the coagulation relation above, the rhs in $\mathcal{L}$ is:

$$\begin{aligned}\sum_i w(\beta_{ji}\psi_i\beta^*_{ki}) &= \sum_i !^{h_j}(z_{ji})!^{h+l_i}(u_i)!^{h_k}(z^*_{ki})\\ &= \sum_i !^{h_j}(y_j)!^{h}(x_i)!^{h+l_i}(u_i)!^{h}(x^*_i)!^{h_k}(y^*_k)\\ &= !^{h_j}(y_j)\left(\sum_i !^{h}(x_ix^*_i)\right)!^{h+1}(u_i)!^{h_k}(y^*_k).\end{aligned}$$

Let $P = \sum_i !^h(x_ix^*_i)$; since $\mathcal{L}$ satisfies the box and coagulation relations we have:

$$\mathcal{L} \models w(\rho(\phi)) = v_1P \ldots v_pPv_{p+1}$$

where the v_i's are such that:

$$\mathcal{L} \models w(\phi) = v_1 \ldots v_{p+1}.$$

Since the x_i's satisfy the zapping equations together, P is easily seen to be an idempotent. Assume $\mathcal{L}$ is a small model and suppose one of the residuals of ϕ in $\rho(R)$ is regular. Being the sum of the weights of the residuals of ϕ, $w(\rho(\phi))$ is non null, thus by lemma 12, non null in $\mathcal{L}$. But idempotents in $\mathcal{L}$ are identities on subsets of $\mathbb{N}$ so that $w(\phi)$ is in turn non null in $\mathcal{L}$, thus in L^*_R and this ends the proof of easylemma.

To get hardlemma we should ask that $\mathcal{L}$ satisfy the so called *partition relation*:

$$\sum_i !^h(x_ix^*_i) = 1. \tag{8}$$

Now set $\mathcal{L} = B(H)$ and $x_i = U_{\tau_i}$ in $B(H)$. Then the partition relation above is equivalent to: $H = \bigoplus_i \mathrm{Rg}(U_{\tau_i})$. But this in turn only means that the ranges of the τ_i are a partition of $\mathbb{N}$, which is possible except in the degenerate case where all x_i have zero lift.

Therefore, to end the proof of hardlemma, it only remains to consider this degenerate case.

A side-remark, before concluding: up to this point we only had to use small models, yet there is no hope of the proof going through in that context in degenerate cases. This because any 0-lift injection's range has the form $\lceil i_0, X\rceil$ for some i_0 in X. In fact it's even worse, for adding $dd^* = 1$ to the theory for some 0-lifted d yields a proof of $0 = 1$.

Suppose, to begin with, that $h = 0$ and that $n = 1$ (where n is the number of premises of the **why not** link being cut). We interpret x_1 by $D = \sum_i d_iD_i$ with $d_i > 0$ for all i, and z_{j1} according to the coagulation relation. Since everything lies in $B(H^+)$ and DD^* satisfies the further assumption of the no-interception lemma, we may apply the latter to (the interpretations of) $w(\rho(\phi))$ and $w(\phi)$. But this is just hardlemma.

To deal with situations where $h > 0$, observe that by instantiating equation (5) one get:

$$\langle !(U)(e_{\lceil i_1, i_2 \rceil}), e_{\lceil i_1, i_2 \rceil}\rangle = \langle U(e_{i_2}), e_{i_2}\rangle,$$

which when iterated yields:

$$\langle !^h(U)(e_{\lceil i_1,\ldots,i_{h+1} \rceil}), e_{\lceil i_1,\ldots,i_{h+1} \rceil}\rangle = \langle U(e_{i_{h+1}}), e_{i_{h+1}}\rangle,$$

so $!^h(P)$ is going to satisfy the further assumption of the no-interception lemma if P does. To deal with the case where there is more than one x_i, say n, just cut D in n pieces of which the final projectors are orthogonal. □

Now for a vague but intuitive remark: as announced in the tactics, the argument goes through, though the partition relation is not satisfied by D, for $DD^* \neq 1$. This is no accident that no model equates fully $w(\rho(\phi))$ to $w(\phi)$, and is due to the fact that the L^*-way to compute is less duplicating than the usual.

4.1 The adequacy theorems

Theorem 14 (Adequacy, part 1) *Let ϕ be straight in $\mathcal{P}(R)$. Then ϕ is persistent iff it is regular (that is $w(\phi) \neq 0$ in L^*_R).*

Proof. (a) Suppose ϕ is not persistent. Then by definition there is a ρ such that $\rho(\phi) = 0$ in $\mathcal{P}^+(\rho(R))$. Hence $w(\rho(\phi)) = 0$ and by hardlemma $w(\phi) = 0$.

(b) Suppose conversely that ϕ is persistent. There are two cases.

(b1) Suppose further that ϕ is an *elementary path*, i.e., ϕ crosses no cut. Being straight, ϕ is an upward-then-downward path, for to switch from traveling downward to traveling upward needs either a rebounce, a twist or a cut. Therefore $w(\phi)$ is not zero, since it is already in stable form.

(b2) Suppose on the contrary that ϕ crosses a cut c and let ρ be the one-step reduction associated to c. At least one of the terms of $\rho(\phi)$ is persistent and shorter than ϕ. Hence by induction hypothesis $w(\rho(\phi)) \neq 0$, and by easylemma $w(\phi) \neq 0$. □

The adequacy theorem has an immediate corollary inducing a simplification of persistence.

Corollary 15 *A straight path ϕ is persistent iff there is reduction ρ such that an element of $\rho(\phi)$ is an elementary path.*

Proof. Suppose there is a reduction ρ such that an element of $\rho(\phi)$ is an elementary path. Such a path has a non null weight as proved in case b1 above, therefore by the preservation lemma 13, ϕ has a non null weight which, by adequacy, is equivalent to ϕ is persistent.

Conversely, the argument sketched in the proof of the ab^* theorem (theorem 9) builds a special reduction ρ of the net such that at each step of ρ, ϕ has exactly one residual the size of which strictly decreases along ρ. Therefore at some point $\rho(\phi)$ must be an elementary path and we are done. □

Theorem 16 (Adequacy, part 2) *L^*_R separates persistent paths, i.e., if ϕ_1 and ϕ_2 are persistent paths in R with same source and goal and not subpath one of the other, then $w(\phi_1\phi_2^*) = 0$ in L^*_R.*

Proof. These two paths have a first bifurcation point, i.e., there are paths ϕ, ϕ'_1 and ϕ'_2 and two distinct premises α_1 and α_2 of a link such that $\phi_i = \phi'_i\alpha_i^*\phi$, $i = 1, 2$. Let w_i, x_i and w be respectively the weights of ϕ'_i, α_i and ϕ. By ab^* version 3, we may apply the equations of inversive category to weights of paths so that we get:

$$w(\phi_1)w(\phi_2)^* = w_1x_1^*ww^*x_2w_2^* = w_1x_1^*ww^*x_1x_1^*x_2w_2^* = 0.$$

□

Remark. Note that L^*_R does not separate regular bouncing paths because $w(\phi\phi^*\phi) = w(\phi)$.

This last result says that not only L^*_R catches all persistent paths, but also, in some sense, that there is as much information in $\mathsf{EX}(R)$ as in the set of persistent paths.

References

[ADLR94] Andrea Asperti, Vincent Danos, Cosimo Laneve, and Laurent Regnier. Paths in the lambda-calculus. In *Proceedings of the Ninth Annual Symposium on Logic in Computer Science*, Paris, 1994. IEEE Computer Society Press.

[AGL92] Martin Abadi, Georges Gonthier, and Jean-Jacques Lévy. Linear logic without boxes. In *Proceedings of the Seventh Annual Symposium on Logic in Computer Science*, San Diego, 1992. IEEE Computer Society Press.

[AL93] Andrea Asperti and Cosimo Laneve. Paths, computations and labels in lambda-calculus. In *Proceedings of the fifth International Conference on Rewriting Techniques and Applications*, volume 690 of *Lecture Notes in Computer Science*, pages 152–167. Springer-Verlag, 1993.

[Dan90] Vincent Danos. *Une Application de la Logique Linéaire à l'Étude des Processus de Normalisation (Principalement du λ-Calcul).* Thèse de doctorat, Université Paris 7, 1990.

[DJS94] Vincent Danos, Jean-Baptiste Joinet, and Harold Schellinx. LKQ and LKT: Sequent calculi for second order logic based upon dual linear decompositions of classical implication. In *this volume*, 1994.

[DR89] Vincent Danos and Laurent Regnier. The structure of multiplicatives. *Archives for Mathematical Logic*, 28:181–203, 1989.

[DR93] Vincent Danos and Laurent Regnier. Local and asynchronous beta-reduction. In *Proceedings of the Eight Annual Symposium on Logic in Computer Science*, Montreal, 1993. IEEE Computer Society Press.

[Gir87a] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50(1):1–102, 1987.

[Gir87b] Jean-Yves Girard. Multiplicatives. In Lolli, editor, *Logic and Computer Science : New Trends and Applications*, pages 11–34, Torino, 1987. Università di Torino. Rendiconti del seminario matematico dell'università e politecnico di Torino, special issue 1987.

[Gir88a] Jean-Yves Girard. Geometry of interaction I: interpretation of system F. In Ferro, Bonotto, Valantini, and Zanardo, editors, *Proceedings of the Logic Colloquium 88*, pages 221–260, Padova, Italy, 1988. North Holland.

[Gir88b] Jean-Yves Girard. Geometry of interaction II: Deadlock free algorithm. In Martin-Löf & Mints, editor, *Proceedings of COLOG'88*, volume 417 of *Lecture Notes in Computer Science*, pages 76–93. Springer-Verlag, 1988.

[Gir95] Jean-Yves Girard. Geometry of interaction III: Accomodating the additives. In *this volume*, 1995.

[Lév78] Jean-Jacques Lévy. *Réductions Correctes et Optimales dans le Lambda-Calcul.* Thèse de doctorat, Université Paris 7, 1978.

[MR91] Pasquale Malacaria and Laurent Regnier. Some results on the interpretation of λ-calculus in operator algebra. In *Proceedings of the Sixth Annual Symposium on Logic in Computer Science*, Amsterdam, 1991. IEEE Computer Society Press.

[Par92] Michel Parigot. $\lambda\mu$-calculus: an algorithmic interpretation of classical natural deduction. In *Proceedings LPAR'92*, volume 624 of *Lecture Notes in Computer Science*, pages 190–201. Springer-Verlag, 1992.

[Reg92] Laurent Regnier. *Lambda-Calcul et Réseaux*. Thèse de doctorat, Université Paris 7, 1992.