Uniformity and the Taylor expansion of ordinary lambda-terms

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Abstract

We define the complete Taylor expansion of an ordinary lambda-term as an infinite linear combination — with rational coefficients — of terms of a resource calculus similar to Boudol's resource lambda-calculus. In this calculus, all applications are (multi-)linear in the algebraic sense, that is commute with linear combination of the function or the argument. We study the beta-reduction of the linear combination of resource terms associated to a lambda-term by Taylor expansion, using a uniformity property that they enjoy.

Introduction

In [ER03], we introduced an extension of the lambda-calculus where terms can be differentiated with respect to their arguments. Typically (in a simply typed version of this *differential lambda-calculus*), if M is a term of type $A \to B$ and if N is a term of type A, we introduce[1] the term $\mathrm{D}M \cdot N$ of type $A \to B$, to be understood as the derivative of the function M with respect to its argument, linearly applied to the value N[2].

Intuitively, in the term $\mathrm{D}M \cdot N$, the term M is provided with exactly one copy N of its argument, and this explains why A is still present as an argument type of $\mathrm{D}M \cdot N$, for the other copies that M might need in computing a result. We argued indeed in the introduction of [ER03] that the mathematical notion of linearity, which is the key concept of differentiation (computing the best possible *linear* approximation of a function), and the logical notion of linearity (a function is *linear* if it uses its argument exactly once) are deeply related, as already strongly suggested by the notations, terminology and denotational semantics of linear logic.

At the end of that paper, we proved a result relating the Taylor expansion of one application of a lambda-term to another one in a special case: given two ordinary lambda-terms M and N such that $(M)\,N$ is β-equivalent to a variable $*$, we studied the Taylor expansion of that application, namely

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\mathrm{D}^n M \cdot N^n\right) 0$$

where we use $\mathrm{D}^n M \cdot N^n$ for the n-th derivative of M with respect to its first parameter (it corresponds to an n-linear function) linearly applied n times to N, that is: $\mathrm{D}(\cdots \mathrm{D}M \cdot N \cdots) \cdot N$. We showed that,

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[1] Actually, the syntax of [ER03] is more complicated since we introduced an explicit notation $\mathrm{D}_i M \cdot N$ for the derivative of M with respect to its ith argument. This has been shown useless by Lionel Vaux in his study of the differential lambda-mu calculus [Vau05].

[2] In standard mathematical notations, the derivative of M is a function M' associating to $x \in A$ a linear map $M'(x)$ from A to B, the differential of M at point x; thus M' has type $A \to A \multimap B$ (where $A \multimap B$ is the type of linear maps from A to B). With these notations, our $\mathrm{D}M \cdot N$ corresponds to $\lambda x^A\, M'(x)(N)$ so that $\mathrm{D}M$ may be considered as having type $A \multimap A \to B$.

with our reduction rules for the differential lambda-calculus, in that sum, there is exactly one term which does not reduce to 0, and that the order n of that term corresponds to the number of times N arrives in head position during the *linear head reduction*[3] of $(M)\,N$ to $*$.

Our aim here is, in some sense, to start generalizing that result in two directions:

- instead of Taylor expanding only one application, we want to Taylor expand all the applications of an ordinary lambda-term;
- instead of considering terms which reduce to a variable, we want to consider all possible situations.

For that reason, we use here a "target language" which is much simpler than the full differential lambda-calculus of [ER03]. Indeed, the general application of lambda-calculus will not be needed anymore, we shall only need iterated "differential applications" followed by an application to 0, corresponding to differential lambda-terms like $(\mathrm{D}^n M \cdot (N_1, \dots, N_n))\, 0$. Keeping in mind that such a differential application is "symmetric" in the sense that its value does not change when we permute the N_i's, in our object language, we replace ordinary application by a multi-set-based notion of application: given a term s and a finite multi-set $T = t_1 \dots t_n$ of terms[4], we allow the formation of a term $\langle s \rangle\, T$ to be understood as corresponding to the differential lambda-term $(\mathrm{D}^n s \cdot (t_1, \dots, t_n))\, 0$.

Interestingly, the calculus we arrive to by these considerations is very similar to Boudol's resource lambda-calculus [Bou93, BCL99] and Kfoury's linearized lambda-calculus [Kfo00], but we insist on its standard algebraic aspects, supported by the fact that it admits the already mentioned quite natural vector space model of [Ehr04] (finiteness spaces).

This calculus has a notion of reduction, which corresponds to the differential beta-reduction of [ER03] : standard substitution is replaced by a linear version of substitution which can be seen as a partial derivative. For this reduction, the calculus enjoys confluence as well as strong normalization, even in the untyped case (from the viewpoint of linear logic, this is due to the fact that the promotion rule is absent from this calculus, see also [ER04]).

In this resource calculus, we are now able to define inductively the Taylor expansion M^* of an ordinary lambda-term M: it will be an infinite formal linear combination of simple[5] resource terms (with coefficients in a field), and should obey, in the case of an application:

$$((M)\,N)^* = \sum_{n=0}^{\infty} \frac{1}{n!} \langle M^* \rangle\, N^{*n},$$

in accordance with the intended meaning, and with the denotational semantics, of application in this resource calculus. Of course we have to give meaning to the operations involved in that sum, and especially to the expression N^{*n}, where N^* will itself be an infinite linear combination of simple terms. This can be done using the usual multinomial equation, and one obtains in that way a direct expression of the Taylor expansion of M:

$$M^* = \sum_{t \in \mathcal{T}(M)} \frac{1}{\mathsf{m}(t)} t$$

where $\mathcal{T}(M)$ is the set of all simple resource terms which have "the same shape" as M, and $\mathsf{m}(t)$ is a positive integer called multiplicity coefficient (because it is larger when t has more repeated patterns) associated to each resource term t. Up to some minor variations, the resource terms which are in some $\mathcal{T}(M)$ are those called *well formed* in [Kfo00]. We characterize these terms as those which are coherent with themselves for a coherence relation on simple resource terms, and call them *uniform* (not "well formed", because we are very much interested by the other terms as well, and also because this usage of

[3]A modified beta-reduction considered explicitly for the first time by De Bruijn and called by him *mini-reduction* [DB87]; it is the reduction implemented by Krivine's abstract machine [Kri85, Kri05] and it has been extensively studied by Danos and Regnier, see for instance [DR99].

[4]Written as a product, for reasons which should be clear if one has in mind the semantics outlined in the final section of [ER03] and thoroughly presented in [Ehr04], where we insist on the fact that the space $!X$ has not only a standard co-algebraic structure which accounts for the structural rules of logic, but also an *algebraic* structure, accounting for this multi-set construction.

[5]We call *simple* a resource term which is not a linear combination of resource terms. Since all the operations of the resource lambda-calculus are linear, any term obtained by combining terms along the syntax of the resource lambda-calculus can be written in an unique way as a linear combination of simple terms, exactly as for polynomials in algebra: simple terms play the role of monomials.

the word "uniform" is reminiscent of a corresponding notion in denotational semantics, see the discussions in [BE01]).

The main purpose of the paper is then to study the behaviour of this Taylor expansion when one reduces its simple summands, which are all strongly normalizing, even if M is not. Thanks to the uniformity and coherence of these resource terms, the situation is quite simple:

- For two distinct simple terms t and t' in $\mathcal{T}(M)$, there is no normal simple term having a non-zero coefficient in the normal forms of both t and t'.
- For that reason, it makes sense to add the normal forms of all the elements t of $\mathcal{T}(M)$, getting a generally infinite sum s of simple terms with rational coefficients.
- Moreover, if t_0 is a normal simple term which has a non-zero coefficient in the normal form of some $t \in \mathcal{T}(M)$, this coefficient is just $\mathsf{m}(t)/\mathsf{m}(t_0)$, and hence the coefficient of a normal simple term t_0 occurring in s is just $1/\mathsf{m}(t_0)$.
- Last, all these normal simple terms are coherent with each other (and in particular, uniform).

So this (generally) infinite sum s of normal simple terms looks like the Taylor expansion of an ordinary lambda-term, and actually it is the Taylor expansion of the Böhm tree of M; this will be explained in a forthcoming paper, using a decorated version of Krivine's machine.

1 Syntax

1.1 Notation and terminology

If X is a finite set, we use $|X|$ for its cardinality. For us the word integer means non-negative integer.

In this paper we deal with some kind of power series. This notion involves two kinds of numbers: coefficients and exponents. Power series have a natural vector space (or more generally module) structure, which requires an addition and a multiplication on coefficients, more precisely, a semi-ring structure. Exponents have to be natural numbers.

I-indexed families. Let R and I be sets; we use R^I for the set of I-indexed families of elements of R, or equivalently the set of applications from I to R. An I-indexed family is denoted as $(x_u)_{u\in I}$ or as a map $x : I \mapsto R$, depending on the context.

Free modules. Suppose R is a commutative semi-ring: R has a commutative addition with a zero, and a commutative multiplication that is distributive over addition. Given an I-indexed family x, we use $\mathsf{supp}(x)$ for the *support* of x, that is, the set $\{u \in I, x_u \neq 0\}$.

We use $R\langle I\rangle$ for the subset of R^I consisting of families with a finite support, that is the free R-module on the set I. Concretely we view $R\langle I\rangle$ as the set of finite linear combinations of elements of I with coefficients in R. We therefore denote the family (x_u) in $R\langle I\rangle$ as the sum $\sum_{u\in I} x_u u$ which has only finitely many nonzero terms.

Multi-sets. In the particular case where $R = \mathbb{N}$, we may alternatively view $R\langle I\rangle$ as the free commutative monoid over I. When this is the case we use $\mathcal{M}_{\mathrm{fin}}(I)$ for $R\langle I\rangle$ and call its elements the *finite multi-sets* over I. Finite multisets are ranged over by the letters $S, T\ldots$

Let S be a finite multi-set over I. We call *multiplicity* of u in S the number $S(u)$. The *cardinality* of S is the number $|S| = \sum_{u\in I} S(u)$ and its *underlying set* is $\mathsf{set}(S) = \{u \in I \mid S(u) \neq 0\}$ ($\mathsf{set}(S)$ is just another notation for $\mathsf{supp}(S)$, dedicated to multi-sets; we use sometimes the notation $u \in S$ instead of $u \in \mathsf{set}(S)$). Let T be another finite multi-set. The *multi-set union* of S and T is the multi-set U defined by $U(u) = S(u) + T(u)$. This is of course the monoid operation on $\mathcal{M}_{\mathrm{fin}}(I)$ and its neutral is the empty multi-set. Depending on the context we use one of two notations for this operation.

Multi-sets as monomials. We mostly use multi-sets for denoting some kind of coefficient free monomials. Suppose I is a set of formal indeterminates and pick for example two indeterminates u and v in I; then we will write $u^p v^q$ for the multi-set where u has multiplicity p, v has multiplicity q, all the multiplicities of the other indeterminates in I being 0. If S and T are two multi-sets/monomials on I, then it is natural to use the product notation ST for the multi-set union of S and T and to use 1 for the empty multi-set.

Multi-sets as multi-exponents. Let now x be a function from I to any commutative monoid R. Then we denote by x^S the value $\prod_{u\in I} x(u)^{S(u)} \in R$ of the monomial S under the valuation x. In this context we consider S as a multi-exponent. If T is another monomial on I then we have $x^S x^T = x^U$ where U is, again, the multi-set union of S and T so we are driven to use an additive notation in order to get the usual equation $x^S x^T = x^{S+T}$.

We also extend to finite multi-sets some notations which are standard for integers. We define the *factorial* of S as $S! = \prod_{u\in I} S(u)!$ (this product having only finitely many factors different from 1). We define the *multinomial coefficient*

$$[S] = \frac{|S|!}{S!} = \frac{\left(\sum_{u\in I} S(u)\right)!}{\prod_{u\in I} S(u)!} \in \mathbb{N}$$

which is the number of distinct enumerations of the elements of S (taking repetitions into account). For instance, if u and v are two distinct elements of I, then $[u^{n-p}v^p] = \binom{n}{p}$. More generally, if $u_1, \ldots, u_k$ are pairwise distinct elements of I and $n_1, \ldots, n_k \in \mathbb{N}$ with $n_1 + \cdots + n_k = n$, then $[u_1^{n_1} \ldots u_k^{n_k}] = \frac{n!}{n_1! \ldots n_k!} = \binom{n}{n_1, \ldots, n_k}$ is the coefficient of the monomial $u_1^{n_1} \ldots u_k^{n_k}$ in the expansion of $(u_1 + \cdots + u_k)^n$ in the algebra of polynomials with indeterminates $u_1, \ldots, u_k$, over any field of characteristic 0.

All these notations are compatible with standard mathematical practice, for instance, given two valuations x and y from I to some commutative semi-ring, the binomial equation generalizes to

$$(x+y)^S = \sum_{T\leq S} \binom{S}{T} x^T y^{S-T}$$

where $T \leq S$ and $S - T$ are defined in the obvious pointwise way, and $\binom{S}{T} = \frac{S!}{T!(S-T)!} = \prod_{u\in I} \binom{S(u)}{T(u)}$.

1.2 Syntax of the resource calculus

Let $\mathcal{V}$ be a countable set of variables.

Simple terms and simple poly-terms. They are defined by mutual induction, as follows.

Variable: if x is a variable, then x is a simple term.

Linear application: if s is a simple term and T is a simple poly-term, then $\langle s\rangle\, T$ is a simple term, the application of s to T.

Abstraction: if x is a variable and t is a simple term, then $\lambda x\, t$ is a simple term in which, as usual, the variable x is bound.

Poly-terms: any finite multi-set of simple terms is a simple poly-term viewed as a monomial of simple terms. The intuition is that each of the elements of such a monomial must be used multi-linearly, that is, exactly as many times as its multiplicity in the monomial.

Let Δ be the set of all simple terms; they will be ranged over by the letters $s, t, \ldots$. Let $\Delta^! = \mathcal{M}_{\mathrm{fin}}(\Delta)$ be the collection of all simple poly-terms, which will be ranged over by the letters $S, T, \ldots$. We use $\Delta^{(!)}$ for Δ or $\Delta^!$ when we do not want to be specific and then we use the letters $\sigma, \tau \ldots$ to range over individuals.

As in lambda-calculus, we have bound and free variables in simple (poly-)terms. Standard lambda-calculus technics may be applied to this system to define α-equivalence and substitution of a term to a variable into a term.

A (poly-)term σ can contain various subterms which are equivalent up to α-equivalence, but nevertheless syntactically distinct. We say that σ is *α-canonical* if this is not the case. Clearly, any (poly-)term admits an α-equivalent α-canonical (poly-)term. We assume all the (poly-)terms we deal with to be in α-canonical form. For instance, an α-canonical form of the simple poly-term $(\lambda x\, x)(\lambda y\, y)$ is $(\lambda x\, x)^2$.

If σ is a simple (poly-)term, we use $\mathcal{V}(\sigma)$ for the set of all free variables of σ.

We define the *size* of a simple (poly-)term by the following induction:

- $\mathsf{S}(x) = 1$;
- $\mathsf{S}(\lambda x\, t) = 1 + \mathsf{S}(t)$;
- $\mathsf{S}(\langle t\rangle\, T) = 1 + \mathsf{S}(t) + \mathsf{S}(T)$;
- $\mathsf{S}(t_1 \dots t_n) = n + \sum_{i=1}^{n} \mathsf{S}(t_i)$.

In particular, observe that in the last clause if $n = 0$ then we get $\mathsf{S}(1) = 0$, where 1 is the empty poly-term.

Finite terms and poly-terms. Let R be a semiring with multiplicative unit[6] 1 and let I be a set. Recall that we use $R\langle I\rangle$ for the free R-module generated by I, the set of finite linear combinations with coefficients in R of elements of I. If f is a function from I to some R-module E, we use $\widetilde{f}$ for the function $R\langle I\rangle \to E$ which is defined in the obvious way, extending f by linearity.

We call *finite terms* and *finite poly-terms* the elements of $R\langle\Delta\rangle$ and $R\langle\Delta^!\rangle$ respectively. We extend by multi-linearity all the constructions of the syntax above to finite terms and poly-terms. For instance, if $U = \sum_{S\in\Delta^!} a_S S$ and $V = \sum_{T\in\Delta^!} b_T T$ are elements of $R\langle\Delta^!\rangle$, the product $UV \in R\langle\Delta^!\rangle$ is defined as $UV = \sum_{S,T\in\Delta^!} a_S b_T ST = \sum_{W\in\Delta^!} c_W W$ where $c_W = \sum_{ST=W} a_S b_T \in R$ vanishes for almost all values of W.

Similarly $\lambda x\, t$ is linear in t and $\langle t\rangle\, T$ is linear in t and in T. This last property justifies the terminology "linear application" for this construction. Standard lambda-calculus application is definitely not linear in the argument (see the introduction of [ER03]).

Partial derivatives. We define now formally the finite (poly-)term $\frac{\partial\sigma}{\partial x}\cdot t$ where σ is a finite (poly-)term, x is a variable and t is a finite term. This will be called the *partial derivative* of σ with respect to x in the direction t. We first give the definition for σ simple:

$$
\begin{aligned}
\frac{\partial y}{\partial x}\cdot t &= \begin{cases} t & \text{if } y = x \\ 0 & \text{otherwise} \end{cases} \\
\frac{\partial \lambda y\, s}{\partial x}\cdot t &= \lambda y\, \frac{\partial s}{\partial x}\cdot t \quad \text{with the usual proviso that } x \neq y \text{ and } y \text{ is not free in } t \\
\frac{\partial \langle s\rangle\, T}{\partial x}\cdot t &= \Big\langle \frac{\partial s}{\partial x}\cdot t\Big\rangle T + \langle s\rangle \left(\frac{\partial T}{\partial x}\cdot t\right) \\
\frac{\partial s_1 \dots s_n}{\partial x}\cdot t &= \sum_{i=1}^{n} s_1 \dots s_{i-1}\Big(\frac{\partial s_i}{\partial x}\cdot t\Big) s_{i+1}\dots s_n\,.
\end{aligned}
$$

Therefore we have the following properties:

$$
\begin{aligned}
\frac{\partial 1}{\partial x}\cdot t &= 0 \\
\frac{\partial ST}{\partial x}\cdot t &= \Big(\frac{\partial S}{\partial x}\cdot t\Big) T + S\Big(\frac{\partial T}{\partial x}\cdot t\Big)\,.
\end{aligned}
$$

The definition is extended to the case where σ is a finite (poly-)term by linearity. Partial derivation should be understood as a linear substitution operation. Indeed one shows easily that it is linear in t. Moreover, it is clear that $\frac{\partial\sigma}{\partial x}\cdot t = 0$ as soon as x does not occur free in σ.

One can also define a substitution operation of a term t for a variable x in a simple (poly)-term σ, written $\sigma\,[t/x]$, to be extended by linearity on σ to arbitrary (poly-)terms σ. However, just as ordinary lambda-calculus application is not linear in the argument, this notion of substitution is not linear in t,

[6] At some point, we shall require that each element of the shape $n\cdot 1$ (with $n \in \mathbb{N}^+$) has an inverse, as for instance in the semiring of positive rational numbers.

in sharp contrast with the partial derivative operation defined above. This operation will be only used when $t = 0$, in which case it is a simple *occur-check* of x in σ: $\sigma\,[0/x]$ is equal to 0 if x occurs free in σ and to σ otherwise.

The following lemma is easily proved by induction on the (poly-)term σ and actually boils down to the commutativity of poly-term multiplication.

Lemma 1 *Let σ be a finite (poly-)term and let s and t be finite terms. Let x and y be variables such that x does not occur free in t. Then we have*

$$\frac{\partial}{\partial y}\Big(\frac{\partial \sigma}{\partial x}\cdot s\Big)\cdot t = \frac{\partial}{\partial x}\Big(\frac{\partial \sigma}{\partial y}\cdot t\Big)\cdot s + \frac{\partial \sigma}{\partial x}\cdot\Big(\frac{\partial s}{\partial y}\cdot t\Big)$$

and in particular, when y does not occur free in s,

$$\frac{\partial}{\partial y}\Big(\frac{\partial \sigma}{\partial x}\cdot s\Big)\cdot t = \frac{\partial}{\partial x}\Big(\frac{\partial \sigma}{\partial y}\cdot t\Big)\cdot s\,.$$

So we introduce the standard notation

$$\frac{\partial^n \sigma}{\partial x_1\cdots\partial x_n}\cdot(t_1,\ldots,t_n) = \frac{\partial}{\partial x_n}\Big(\cdots\frac{\partial \sigma}{\partial x_1}\cdot t_1\cdots\Big)\cdot t_n$$

when no x_i occurs free in any of the simple terms t_j. For any permutation f of $\{1,\ldots,n\}$, we have

$$\frac{\partial^n \sigma}{\partial x_1\cdots\partial x_n}\cdot(t_1,\ldots,t_n) = \frac{\partial^n \sigma}{\partial x_{f(1)}\cdots\partial x_{f(n)}}\cdot\big(t_{f(1)},\ldots,t_{f(n)}\big) \tag{1}$$

Degree in a variable. If σ is a simple (poly-)term and x is a variable, the *degree* of x in σ is the number of free occurrences of x in σ, taking multiplicities into account. This number is denoted by $\mathsf{d}_x(\sigma)$. For instance, in the simple term $\langle x\rangle\,(\langle x\rangle\, y^2)^3$, the degree of x is 4 and the degree of y is 6. Due to the fact that all the syntactic constructions of this calculus are linear, this notion of degree coincides with the standard algebraic one. Typically, if σ is a simple (poly-)term and if $a \in R$, we have $\sigma\,[ax/x] = a^{\mathsf{d}_x(\sigma)}\sigma$. Also, $\mathsf{d}_x(ST) = \mathsf{d}_x(S) + \mathsf{d}_x(T)$ when S and T are simple poly-terms.

Big step differentiation. Given a simple term σ, a variable x and a simple poly-term $T = t_1\ldots t_n$ where the variable x does not appear free, we define

$$\partial_x(\sigma,T) = \Big(\frac{\partial^n \sigma}{\partial x^n}\cdot(t_1,\ldots,t_n)\Big)\,[0/x] \in R\langle\Delta^{(!)}\rangle$$

which does not depend on the enumeration $t_1,\ldots,t_n$ of T thanks to Equation (1). Observe that this expression is non zero iff $n = \mathsf{d}_x(\sigma)$. Also note that if x does not occur free in the t_i's, then x does not occur free in $\frac{\partial^n \sigma}{\partial x^n}\cdot(t_1,\ldots,t_n)$.

We use $\partial_{x_1,\ldots,x_n}(\sigma,T_1,\ldots,T_n)$ for the iterated big step differentiation

$$\partial_{x_n}(\cdots\partial_{x_1}(\sigma,T_1)\cdots,T_n)\,.$$

The value of this expression does not depend on the order we put on the pairwise distinct variables $x_1,\ldots,x_n$.

Partial derivation *vs.* substitution. Partial derivation can be understood as a linear substitution. Let σ be a simple (poly-)term and let x be a variable. Let $n = \mathsf{d}_x(\sigma)$ and let $x_1,\ldots,x_n$ be pairwise distinct variables which do not occur free in σ or in t. Let σ' be a simple (poly-)term obtained by replacing the n occurrences of x in σ by the pairwise distinct variables $x_1,\ldots,x_n$. Such a σ' will be called an *x-linearization* of σ in $x_1,\ldots,x_n$. For any simple t, we have

$$\frac{\partial \sigma}{\partial x}\cdot t = \sum_{i=1}^{n}\sigma'\,[t/x_i]\,[x/x_1,\ldots,x_n]\,. \tag{2}$$

Let $T = t_1\ldots t_n$ be a simple poly-term of cardinality n. Iterating the formula above, we get

$$\partial_x(\sigma,T) = \sum_{f\in\mathfrak{S}_n}\sigma'\,\big[t_{f(1)}/x_1,\ldots,t_{f(n)}/x_n\big] \tag{3}$$

where $\mathfrak{S}_n$ is the group of all permutations of $\{1,\ldots,n\}$.

Leibniz law and partial derivation. Let σ be a simple (poly-)term and let t be a simple term. Let x, x_1 and x_2 be variables, with $x_1 \neq x_2$ and x not free in σ. The Leibniz law concerns the interaction between differentiation and contraction, and can be written as follows:

$$\frac{\partial \sigma\,[x/x_1, x_2]}{\partial x} \cdot t = \Big(\frac{\partial \sigma}{\partial x_1} \cdot t\Big)\,[x/x_1, x_2] + \Big(\frac{\partial \sigma}{\partial x_2} \cdot t\Big)\,[x/x_1, x_2]\ .$$

The proof is a simple induction on σ. Iterating, we obtain the following formula.

Lemma 2 *Let σ be a simple (poly-)term and let T be a simple poly-term. Let x, x_1 and x_2 be variables, with $x_1 \neq x_2$ and x not free in σ. Then*

$$\partial_x(\sigma\,[x/x_1, x_2]\,, T) = \sum_{UV=T} \binom{T}{U} \partial_{x_1,x_2}(\sigma, U, V)\,.$$

Proof. The proof is by induction on $n = \mathsf{d}_x(\sigma\,[x/x_1, x_2]) = \mathsf{d}_{x_1}(\sigma) + \mathsf{d}_{x_2}(\sigma)$, assuming that $|T| = n$, since otherwise both sides of the equation vanish. The case $n = 0$ is trivial, so assume $n > 0$, we can write $T = tS$ for some simple term t and we have

$$\begin{aligned}
\partial_x(\sigma\,[x/x_1, x_2]\,, tS) &= \partial_x\Big(\frac{\partial \sigma\,[x/x_1, x_2]}{\partial x} \cdot t, S\Big) \\
&= \partial_x\Big(\Big(\frac{\partial \sigma}{\partial x_1} \cdot t\Big)\,[x/x_1, x_2]\,, S\Big) + \partial_x\Big(\Big(\frac{\partial \sigma}{\partial x_2} \cdot t\Big)\,[x/x_1, x_2]\,, S\Big) \\
&= \sum_{UV=S} \binom{S}{U} \Big(\partial_{x_1,x_2}\Big(\frac{\partial \sigma}{\partial x_1} \cdot t, U, V\Big) + \partial_{x_1,x_2}\Big(\frac{\partial \sigma}{\partial x_2} \cdot t, U, V\Big)\Big) \\
&= \sum_{UV=S} \binom{S}{U} (\partial_{x_1,x_2}(\sigma, tU, V) + \partial_{x_1,x_2}(\sigma, U, tV)) \\
&= \sum_{\substack{U'V'=T \\ t\in U',\, t\in V'}} \Big(\binom{T-t}{U'-t} + \binom{T-t}{U'}\Big)\,\partial_{x_1,x_2}(\sigma, U', V') \\
&\quad + \sum_{\substack{U'V'=T \\ t\in U',\, t\notin V'}} \binom{T-t}{U'-t} \partial_{x_1,x_2}(\sigma, U', V') + \sum_{\substack{U'V'=T \\ t\notin U',\, t\in V'}} \binom{T-t}{U'} \partial_{x_1,x_2}(\sigma, U', V')
\end{aligned}$$

and we conclude, applying Pascal's binomial identity for the first of these three sums, and observing that, in the two last sums, the binomial coefficients are equal to $\binom{T}{U'}$. □

1.3 Reduction and normal forms

Linear relations. If E and F are two R-modules, we say that a relation $\rho \subseteq E \times F$ is *linear* if it is a linear subspace of the direct product $E \times F$ (in other words, if $u\ \rho\ u'$ and $v\ \rho\ v'$ then $au + bv\ \rho\ au' + bv'$ for any $a, b \in R$).

Let I be a set. Given a relation $\rho \subseteq I \times R\langle I\rangle$, we define a linear relation $R\langle\rho\rangle \subseteq R\langle I\rangle \times R\langle I\rangle$ as the linear span of ρ in this product space and call $R\langle\rho\rangle$ the *linear extension* of ρ. Spelling out this definition, we have $u\ R\langle\rho\rangle\ v$ iff we can find $u_1, \dots, u_n \in I$, $a_1, \dots, a_n \in R$ and $v_1, \dots, v_n \in R\langle I\rangle$ such that $u = \sum_{i=1}^n a_i u_i$, $v = \sum_{i=1}^n a_i v_i$ and $u_i\ \rho\ v_i$ for each i.

Small step reduction. A *redex* is a simple term of the shape $\langle \lambda x\, s\rangle\, S$. The reduction of such a redex is defined by cases, according to whether S is empty or not. The second case is non-deterministic as it consists in choosing an element u in S and then in computing a partial derivative of s in the direction u. The result of such a reduction is a linear combination of simple terms, with integer coefficients.

$$\begin{array}{ccc}
\langle \lambda x\, s\rangle\, 1 & \beta^1_\Delta & s\,[0/x] \\
\langle \lambda x\, s\rangle\, uT & \beta^1_\Delta & \Big\langle \lambda x\, \dfrac{\partial s}{\partial x} \cdot u\Big\rangle T
\end{array}$$

By extending this reduction to all contexts, we define the *one step reduction relation* $\bar{\beta}^1_\Delta \subseteq \Delta \times R\langle\Delta\rangle$, together with a corresponding auxiliary relation on (poly-)terms for which we use the same symbol. More precisely, we say that $\sigma\ \bar{\beta}^1_\Delta\ \sigma'$ in one of the following situations:

(Redex) $\sigma\ \beta^1_\Delta\ \sigma'$;

(Abs) $\sigma = \lambda x\, t$ and $\sigma' = \lambda x\, t'$ with $t\ \bar{\beta}^1_\Delta\ t'$;

(App) $\sigma = \langle t\rangle\, S$ and

- $\sigma' = \langle t'\rangle\, S$ with $t\ \bar{\beta}^1_\Delta\ t'$ or
- $\sigma' = \langle t\rangle\, S'$ with $S\ \bar{\beta}^1_\Delta\ S'$;

(Prod) σ is the poly-term uS and $\sigma' = u'S$ with $u\ \bar{\beta}^1_\Delta\ u'$.

We use β_Δ for the reflexive and transitive closure of $R\langle\bar{\beta}^1_\Delta\rangle$, a relation from terms to terms which is contextual (in the natural sense) by construction. We use the same notation for the corresponding relation on poly-terms.

Theorem 3 *The relation β_Δ is Church-Rosser. Moreover, if $R = \mathbb{N}$ then β_Δ is strongly normalizing*[7]*.*

Proof. The confluence property is proved as in [ER03] (and is simpler in the present context). For the normalization property, observe by inspection of the reduction rules that if σ is simple and $\sigma\ \bar{\beta}^1_\Delta\ \sigma'$, then the size of each element of $\mathsf{supp}(\sigma')$ is strictly smaller than the size of σ. □

Remark: This untyped calculus is strongly normalizing, and so cannot represent general recursive computations as lambda-calculus does. Later we shall introduce infinite sums which will allow us to encode ordinary lambda-terms, making explicit the potential infinite of lambda-calculus.

If $\sigma \in \Delta^{(!)}$, we use $\mathsf{NF}(\sigma)$ for the unique normal form of σ, which is an element of $\mathbb{N}\langle\Delta^{(!)}\rangle$ (and so can be considered as an element of any $R\langle\Delta^{(!)}\rangle$).

Big step reduction. We define now a big step reduction relation $\bar{\beta}^{\mathrm{b}}_\Delta$ which is more convenient for dealing with the problems at hand. The definition is the same as the definition of $\bar{\beta}^1_\Delta$, replacing the small step redex reduction β^1_Δ by the following one:

$$\langle\lambda x\, s\rangle\, T \quad \beta^{\mathrm{b}}_\Delta \quad \partial_x(s, T)\,.$$

This reduction is very similar to the β-reduction of ordinary λ-calculus — $(\lambda x\, M)\, N\ \beta\ M\,[N/x]$ — and for that reason, it is the good notion of reduction on simple terms for studying the Taylor expansion of ordinary lambda-terms. Observe that this reduction is deterministic.

The relation $\bar{\beta}^{\mathrm{b}}_\Delta \subseteq \Delta \times R\langle\Delta\rangle$ is included in the transitive closure of $\bar{\beta}^1_\Delta$, and has the same normal forms. Therefore, we can compute $\mathsf{NF}(\sigma)$ by iteratively applying the reduction $\bar{\beta}^{\mathrm{b}}_\Delta$ to σ.

An explicit formula for normal forms. As in the ordinary lambda-calculus, any simple term s can be written (in a unique way) as follows:

$$s = \lambda x_1 \dots \lambda x_n\, \langle t\rangle\, T_1 \dots T_k$$

where t is a simple term which is either a variable possibly equal to one of the x_i's, and in that case we say that s is in *head normal form*, or a redex, and in that case we say that t (or rather, this particular occurrence of t) is the *head redex* of s.

Lemma 4 *Let σ be a simple (poly-)term.*

[7]This very strong hypothesis can be weakened a little bit as explained in [ER03], but not really significantly.

- *If $\sigma = \lambda x_1 \dots \lambda x_n \, \langle t \rangle \, T_1 \dots T_k$ with $t = \langle \lambda y \, s \rangle \, S$, then*

$$
\begin{aligned}
\mathsf{NF}(\sigma) &= \widetilde{\mathsf{NF}}(\lambda x_1 \dots \lambda x_n \, \langle \partial_y(s,S) \rangle \, T_1 \dots T_k) \\
&= \sum_{u \in \Delta} \partial_y(s,S)_u (\lambda x_1 \dots \lambda x_n \, \mathsf{NF}(\langle u \rangle \, T_1 \dots T_k))
\end{aligned} \tag{4}
$$

  (Remember that we use $\widetilde{\mathsf{NF}}$ for the linear extension of NF to arbitrary finite (poly-)terms and that $\partial_y(s,S)_u$, the coefficient of u in the linear combination of simple terms $\partial_y(s,S)$, is an integer.)

- *If $\sigma = \lambda x_1 \dots \lambda x_n \, \langle t \rangle \, T_1 \dots T_k$ with $t = y \in \mathcal{V}$, then $\mathsf{NF}(\sigma) = \lambda x_1 \dots \lambda x_n \, \langle y \rangle \, \mathsf{NF}(T_1) \dots \mathsf{NF}(T_k)$.*

- *If $\sigma = t_1 \dots t_n$ then $\mathsf{NF}(\sigma) = \prod_{i=1}^n \mathsf{NF}(t_i)$.*

The proof is straightforward, once observed that for each $u \in \mathsf{supp}(\partial_y(s,S))$, one has $\mathsf{S}(u) < \mathsf{S}(\langle \lambda y \, s \rangle \, S)$. For that reason, and by the confluence property, the lemma above can be considered as an inductive definition of NF and will be used as such.

2 The Taylor expansion of ordinary lambda-terms

We show now how to represent ordinary lambda-terms in this calculus by recursively Taylor expanding all ordinary applications. As remarked above, this requires dealing with infinite linear combinations of (poly-)terms.

Infinite terms and poly-terms. If I is a set, we use $R\langle I \rangle_\infty$ for the R-module of all formal linear combinations $x = \sum_{u \in I} x_u u$ where (x_u) is an arbitrary I-indexed family of scalars taken in R (so that $R\langle I \rangle_\infty = R^I$). Let J be a countable set. We say that a family $(x(j))_{j \in J}$ of elements of $R\langle I \rangle_\infty$ is *summable* if, for each $u \in I$, the family $(x(j)_u)_{j \in J}$ vanishes for almost all values of j. We then define its sum $x = \sum_{j \in J} x(j)$ by setting $x_u = \sum_{j \in J} x(j)_u$, a finite sum in R by assumption. This is just usual convergence for the product topology, R being endowed with the discrete topology. If $J = \mathbb{N}$, observe that for this topology, the convergence of a series is equivalent to the convergence to 0 of its general term. Observe also that all the module operations on $R\langle I \rangle_\infty$ are continuous (R being endowed with the discrete topology).

If I has a structure of commutative monoid (with multiplicative notation) with the property that for each $u \in I$ there are only finitely many pairs $(v,w) \in I^2$ such that $u = vw$, then $R\langle I \rangle_\infty$ is an algebra, with multiplication given by

$$
xy = \sum_{u \in I} \Bigg(\sum_{vw = u} x_v y_w \Bigg) u \, .
$$

Moreover, it is easily checked that this multiplication is continuous with respect to the product topology on $R\langle I \rangle_\infty \times R\langle I \rangle_\infty$.

This is the case in particular of $R\langle \Delta^! \rangle_\infty$. Let $(T(j))_{j \in J}$ be a summable family in this algebra and let $n \in \mathbb{N}$. If $\mu \in \mathcal{M}_{\mathrm{fin}}(J)$, then remember that we write $T^\mu = \prod_{j \in J} T(j)^{\mu(j)} \in R$ (this is a finite product since μ is a finite multi-set). We use $\mathcal{M}_n(J)$ for the set of all multi-sets over J whose cardinality is n and if $\mu \in \mathcal{M}_n(J)$, remember also that we have defined a multinomial coefficient as follows: $[\mu] = n! / \prod_{j \in J} \mu(j)! \in \mathbb{N}$. The family $\Big([\mu] \, T^\mu \Big)_{\mu \in \mathcal{M}_n(J)}$ is summable in the algebra $R\langle \Delta^! \rangle_\infty$ and we have the following "multinomial identity":

$$
\Bigg(\sum_{j \in J} T_j \Bigg)^n = \sum_{\mu \in \mathcal{M}_n(J)} [\mu] \, T^\mu \, . \tag{5}
$$

The constructions of the syntax of our resource calculus can now be extended to these infinite linear combinations of simple (poly-)terms in an obvious way, by linearity (and "continuity" since we require the constructs to commute to arbitrary linear combinations, not only to finite ones).

Differentiation of infinite terms. We want first to make sense of the expression $\frac{\partial\sigma}{\partial x} \cdot t$ when $\sigma \in R\langle\Delta^{(!)}\rangle_\infty$, $t \in R\langle\Delta\rangle_\infty$ and x is not free in t.

Lemma 5 *Let $\tau \in \Delta^{(!)}$, let x be a variable and let $t \in \Delta$. There are only finitely many $\sigma \in \Delta^{(!)}$ such that $\tau \in \mathsf{supp}(\frac{\partial\sigma}{\partial x} \cdot t)$.*

Proof. Assume that $\tau \in \cap_{i=1}^n \mathsf{supp}(\frac{\partial\sigma_i}{\partial x} \cdot t)$ for a finite family $(\sigma_i)_{i=1,\dots,n}$ of pairwise distinct simple (poly-)terms. Then, examining Equation (2), observe that there exists a simple (poly-)term σ and pairwise distinct variables $x_1, \dots, x_n$ such that:

- $\mathsf{d}_{x_i}(\sigma) = 1$ for each i, that is σ is linear in x_i ;
- $\tau = \sigma\,[t/x_i]_{i=1,\dots n}$;
- $\sigma_i = \sigma\,[x/x_i]\,[t/x_j]_{j\neq i}$.

From this one clearly sees that n is upper bounded by the size of τ. $\square$

Lemma 6 *Let x be a variable and let $\tau \in \Delta^{(!)}$. There are only finitely many $\sigma \in \Delta^{(!)}$ and $t \in \Delta$ such that $\tau \in \mathsf{supp}(\frac{\partial\sigma}{\partial x} \cdot t)$.*

Proof. If $(\sigma_i, t_i)_{i\in I}$ is a family of pairwise distinct pairs of simple (poly-)terms and terms and if $\tau \in \cap_{i\in I} \mathsf{supp}(\frac{\partial\sigma_i}{\partial x} \cdot t_i)$ then each simple term t_i must appear as a sub-term of τ and therefore there can be only a finite number of distinct t_i's. If I is infinite, this leads to a contradiction with Lemma 5. Therefore I is finite and the lemma is proved. $\square$

For that reason the whole family of finite (poly-)terms $\left(\frac{\partial\sigma}{\partial x} \cdot t\right)_{\sigma\in\Delta^{(!)},t\in\Delta}$ is summable. So, if $\sigma \in R\langle\Delta^{(!)}\rangle_\infty$ and $t \in R\langle\Delta\rangle_\infty$, it always make sense to define

$$\frac{\partial\sigma}{\partial x} \cdot t = \sum_{\varphi\in\Delta^{(!)},\, u\in\Delta} \sigma_\varphi t_u \frac{\partial\varphi}{\partial x} \cdot u \in R\langle\Delta\rangle_\infty\,.$$

We can derive a bit more from Lemma 6.

Lemma 7 *The map $(\sigma, t) \mapsto \frac{\partial\sigma}{\partial x} \cdot t$ from $R\langle\Delta^{(!)}\rangle_\infty \times R\langle\Delta\rangle_\infty$ to $R\langle\Delta^{(!)}\rangle_\infty$ is continuous (these spaces being endowed with the product topology). In particular, if $(\sigma_i)_{i\in I}$ and $(t_j)_{j\in J}$ are summable families in $R\langle\Delta^{(!)}\rangle_\infty$ and $R\langle\Delta\rangle_\infty$ respectively (with respective sums σ and t), then the family $(\frac{\partial\sigma_i}{\partial x} \cdot t_j)_{i\in I,j\in J}$ is summable, with sum equal to $\frac{\partial\sigma}{\partial x} \cdot t$.*

Proof. By linearity, it suffices to prove continuity at the origin $(0, 0)$ of $R\langle\Delta^{(!)}\rangle_\infty \times R\langle\Delta\rangle_\infty$. We take a neighborhood of 0 in $R\langle\Delta^{(!)}\rangle_\infty$: it is induced by a finite subset W of $\Delta^{(!)}$ (the corresponding neighborhood of 0 in $R\langle\Delta^{(!)}\rangle_\infty$ is the collection $\mathrm{V}_W(0)$ of all $\theta \in R\langle\Delta^{(!)}\rangle_\infty$ such that $W \cap \mathsf{supp}(\theta) = \emptyset$). Then by Lemma 6, for each $\varphi \in W$, we can find two finite sets $U_\varphi \subseteq \Delta^{(!)}$ and $V_\varphi \subseteq \Delta$ such that $\varphi \notin \mathsf{supp}(\frac{\partial\sigma}{\partial x} \cdot t)$ for each $(\sigma, t) \notin U_\varphi \times V_\varphi$. Then taking $U = \bigcup_{\varphi\in W} U_\varphi$ and $V = \bigcup_{\varphi\in W} V_\varphi$, we have $\frac{\partial\sigma}{\partial x} \cdot t \in \mathrm{V}_W(0)$ for each $\sigma \in \mathrm{V}_U(0)$ and $t \in \mathrm{V}_V(0)$. $\square$

So $\frac{\partial\sigma}{\partial x} \cdot t \in R\langle\Delta^{(!)}\rangle_\infty$ is well defined for all $\sigma \in R\langle\Delta^{(!)}\rangle_\infty$ and $t \in R\langle\Delta\rangle_\infty$ and has all the required linearity and continuity properties. We can of course iterate this construction and define $\frac{\partial^n\sigma}{\partial x_1\cdots\partial x_n} \cdot (t_1, \dots, t_n)$ for arbitrary $t_1, \dots, t_n$ of $R\langle\Delta\rangle_\infty$. Again, this operation is linear in each of its parameters $\sigma, t_1, \dots, t_n$, and is continuous in these parameters (for the product topology).

For that reason, for each given $n \in \mathbb{N}$, we can extend the construction $\partial_x(\sigma, T)$ to $\sigma \in R\langle\Delta^{(!)}\rangle_\infty$ and $T \in R\langle\mathcal{M}_n(\Delta)\rangle_\infty$, and this operation is bilinear and continuous in σ and T. Observing that, for $\sigma \in \Delta^{(!)}$ and $T \in \mathcal{M}_n(\Delta)$, the size of any element of the support of $\partial_x(\sigma, T)$ must be greater than n, we see that, for any $\sigma \in R\langle\Delta^{(!)}\rangle_\infty$ and any $T \in R\langle\Delta^!\rangle_\infty$, the sequence $(\partial_x(\sigma, T^{(n)}))_{n\in\mathbb{N}}$ converges to 0 in $R\langle\Delta^{(!)}\rangle_\infty$ (where we use $T^{(n)}$ for the restriction of T to $\mathcal{M}_n(\Delta)$, that is $T^{(n)} = \sum_{S\in\mathcal{M}_n(\Delta)} T_S\, S$). So the series $\sum_{n=0}^\infty \partial_x(\sigma, T^{(n)})$ converges. Its sum is denoted by $\partial_x(\sigma, T)$; this operation is bilinear and continuous in (σ, T).

So all the differentiation operations we have considered for finite (poly-)terms make sense in the infinite case as well, without any restriction on the infinite linear combinations we consider. This observation will be used at the end of the present paper, when we shall give a "substitution-oriented" version of Taylor's formula in Theorem 19 and shows that the infinitary resource lambda-calculus is a sound extension of the finitary one.

The exponential and the promotion. Any $t \in R\langle\Delta\rangle_\infty$ can canonically be seen as an element of $R\langle\Delta^!\rangle_\infty$ (identifying $t \in \Delta$ with $t \in \Delta^!$, the multi-set whose only element is t, with multiplicity 1). It is clear that $t^n \to 0$ when $n \to \infty$ so that the following sum converges:

$$\exp t = \sum_{n=0}^{\infty} \frac{1}{n!} t^n \in R\langle\Delta^!\rangle_\infty$$

where the exponents correspond to multiplication in the algebra $R\langle\Delta^!\rangle_\infty$. Using formula (5), one can check that actually

$$\exp t = \sum_{T \in \Delta^!} \frac{t^T}{T!} T$$

(remember that, with our notations, $T! = \prod_{u\in\Delta} T(u)! \in \mathbb{N}^+$ and that $t^T = \prod_{u\in\Delta} t_u^{T(u)} \in R$).

Without surprises, we have $\exp 0 = 1$ and $\exp(s + t) = \exp s \ \exp t$. This operation $t \mapsto \exp t$ corresponds to promotion in linear logic.

Remark: this exponential operation could be defined not only for $t \in R\langle\Delta\rangle_\infty$ but for arbitrary $S \in R\langle\Delta^!\rangle_\infty$, as soon as $S_1 = 0$. When $S_1 \neq 0$, computing $\exp S$ involves an infinite sum of scalars, or maybe the use of an "exponential map" e_R on the semi-ring R, setting $\exp S = e_R(S_1)\exp(S - S_1 \cdot 1)$.

2.1 Complete Taylor expansion of an ordinary lambda-term

Multiplicity coefficients. Given a simple term t, we define a positive integer $\mathsf{m}(t)$, the *multiplicity coefficient* of t by the following inductive definition.

$$\begin{aligned}
\mathsf{m}(x) &= 1\\
\mathsf{m}(\lambda x\, s) &= \mathsf{m}(s)\\
\mathsf{m}(\langle s\rangle\, T) &= \mathsf{m}(s) \prod_{t\in\Delta} T(t)!\, \mathsf{m}(t)^{T(t)} = \mathsf{m}(s)\ T!\ \mathsf{m}^T
\end{aligned}$$

with our concise notations for arithmetic operations on multi-sets. For a poly-term T, we define accordingly $\mathsf{m}(T) = T!\ \mathsf{m}^T$, so that $\mathsf{m}(\langle s\rangle\, T) = \mathsf{m}(s)\mathsf{m}(T)$. In Section 4, we shall give a combinatorial interpretation of these coefficients.

The expansion. Given an ordinary lambda-term M, we define a subset $\mathcal{T}(M)$ of Δ which is the collection of all simple terms having *the same shape as* M. This set is defined as follows, by induction on M.

$$\begin{aligned}
\mathcal{T}(x) &= \{x\}\\
\mathcal{T}(\lambda x\, M) &= \{\lambda x\, t \mid t \in \mathcal{T}(M)\}\\
\mathcal{T}((M)\, N) &= \{\langle t\rangle\, T \mid t \in \mathcal{T}(M) \text{ and } T \in \mathcal{M}_{\mathrm{fin}}(\mathcal{T}(N))\}\,.
\end{aligned}$$

We define the *complete Taylor expansion* of an ordinary lambda-term M:

$$M^* = \sum_{t\in\mathcal{T}(M)} \mathsf{m}(t)^{-1}\, t \ \in R\langle\Delta\rangle_\infty\,.$$

This expansion satisfies the following lemma, whose last statement means that M^* can be obtained by recursiveley Taylor expanding all applications in M. This motivates our terminology for this operation.

Lemma 8 *If x is a variable and if M and N are terms of the standard lambda-calculus, one has*

- $x^* = x$,
- $(\lambda x\, M)^* = \lambda x\, M^*$ *and*
- $((M)\,N)^* = \langle M^*\rangle \exp N^* = \sum_{n=0}^{\infty} \frac{1}{n!}\langle M^*\rangle\, N^{*n}$.

Proof. The only interesting case is the last one. We have

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{1}{n!}\langle M^*\rangle\, N^{*n} &= \sum_{n=0}^{\infty}\frac{1}{n!}\Big\langle \sum_{s\in\mathcal{T}(M)}\frac{1}{\mathsf{m}(s)}s\Big\rangle\Big(\sum_{t\in\mathcal{T}(N)}\frac{1}{\mathsf{m}(t)}t\Big)^n\\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\Big\langle \sum_{s\in\mathcal{T}(M)}\frac{1}{\mathsf{m}(s)}s\Big\rangle\Big(\sum_{T\in\mathcal{M}_n(\mathcal{T}(N))}[T]\,\frac{1}{\mathsf{m}^T}T\Big)\\
&= \sum_{\substack{s\in\mathcal{T}(M)\\ T\in\mathcal{M}_{\mathrm{fin}}(\mathcal{T}(N))}}\frac{1}{|T|!}[T]\,\frac{1}{\mathsf{m}(s)\mathsf{m}^T}\langle s\rangle\, T\\
&= \sum_{\substack{s\in\mathcal{T}(M)\\ T\in\mathcal{M}_{\mathrm{fin}}(\mathcal{T}(N))}}\frac{1}{T!\mathsf{m}(s)\mathsf{m}^T}\langle s\rangle\, T \quad \text{since } [T]=\frac{|T|!}{T!}\\
&= ((M)\,N)^*\,.
\end{aligned}$$

□

The question. Our main goal is to understand the behaviour of this Taylor expansion with respect to beta-reduction. The first thing to observe is that the resource terms occurring in the Taylor expansion of an ordinary lambda-term are coherent with each other and with themselves (we shall say "uniform"), in a sense we define below. Then we shall see that the normal form operator is *stable* (in the sense of [Ber78] and [Gir86]) with respect to this coherence relation. This is a *qualitative* property whose main consequence will be a "non-interference" effect: the supports of the normal forms of two distinct terms of the Taylor expansion are disjoint. Then we shall see that the multiplicity coefficients of uniform terms evolve very simply during big step differential reduction —a *quantitative* property—.

These two main observations will lead to our final Theorem 21.

3 Qualitative properties: the coherence relation on simple terms and poly-terms

We define a binary *coherence* relation $\frown\!\!\!\!\smile$ on simple terms and on simple poly-terms, which is easily seen to be symmetric (but neither reflexive nor anti-reflexive). We use the notation $\frown$ for the largest anti-reflexive sub-relation of $\frown\!\!\!\!\smile$. The definition is by induction on simple terms.

- $x \frown\!\!\!\!\smile t'$ if $t' = x$;
- $\lambda x\, s \frown\!\!\!\!\smile t'$ if $t' = \lambda x\, s'$ with $s \frown\!\!\!\!\smile s'$;
- $\langle s\rangle\, T \frown\!\!\!\!\smile t'$ if $t' = \langle s'\rangle\, T'$ with $s \frown\!\!\!\!\smile s'$ and $T \frown\!\!\!\!\smile T'$.
- $s_1 \dots s_n \frown\!\!\!\!\smile s_{n+1}\dots s_m$ if $s_i \frown\!\!\!\!\smile s_j$ for all i, j.

This is not a (partial) equivalence relation, due to the potential presence of empty poly-terms 1 as arguments. Typically, $\langle x\rangle\, y \frown \langle x\rangle\, 1 \frown \langle x\rangle\, z$, but we have not $\langle x\rangle\, y \frown\!\!\!\!\smile \langle x\rangle\, z$ if y and z are two distinct variables.

We say that a simple (poly-)term σ is *uniform* if $\sigma \frown\!\!\!\!\smile \sigma$. This corresponds to the notion of well-formed term in [Kfo00] (however, in that paper, the relation corresponding to $\frown\!\!\!\!\smile$ is a partial equivalence relation because empty multi-sets are not accepted as arguments).

A clique for this coherence relation is a subset U of $\Delta^{(!)}$ such that $\tau \frown\!\!\!\!\smile \tau'$ whenever $\tau, \tau' \in U$. In particular, each element of a clique must be uniform. Observe by the way that it results from the definition that if $\sigma \frown\!\!\!\!\smile \sigma'$ for two simple terms σ and σ', then automatically σ and σ' are uniform.

Lemma 9 *If M is a lambda-term, then $\mathcal{T}(M)$ is a maximal clique in $(\Delta, \frown)$.*

The proof is straightforward. However, not all maximal cliques of Δ are of the shape $\mathcal{T}(M)$ for some lambda-term M. For instance, a maximal extension of the clique $\{\langle x\rangle\, 1, \langle x\rangle\, \langle x\rangle\, 1, \dots\}$ cannot be of that shape.

Coherence and differentiation. Coherence is not preserved by partial differentiation. For instance, the poly-term x^2 is uniform and y is a uniform term, but $\frac{\partial x^2}{\partial x} \cdot y = 2xy$ is not uniform if x and y are distinct variables.

However, big step differentiation satisfies a "stability" property with respect to the coherence relation we have defined on (poly-)terms, similar to the characterization of the trace of stable linear functions between coherence spaces in [Gir87, GLT89].

Theorem 10 *Let x be a variable. Let $\sigma, \sigma' \in \Delta^{(!)}$ and $S, S' \in \Delta^!$. Let $\varphi \in \mathsf{supp}(\partial_x(\sigma, S))$ and $\varphi' \in \mathsf{supp}(\partial_x(\sigma', S'))$. If $\sigma \frown \sigma'$ and $S \frown S'$, then $\varphi \frown \varphi'$ and if moreover $\varphi = \varphi'$, then $\sigma = \sigma'$ and $S = S'$.*

Proof. We assume that $\sigma \frown \sigma'$ and $S \frown S'$. We proceed by induction on the sum of the sizes of σ and σ', for σ and σ' in $\Delta^{(!)}$.

Assume that σ is a variable y. Then $\sigma' = y$. If $y \neq x$, we must have $S = S' = 1$ since $\varphi \in \mathsf{supp}(\partial_x(\sigma, S))$ and $\varphi' \in \mathsf{supp}(\partial_x(\sigma', S'))$ (otherwise at least one of these sets would be empty). So $\varphi = \varphi' = y$ and we conclude trivially. If $y = x$ then S and S' must be singleton multi-sets (otherwise again at least one of the two sets $\mathsf{supp}(\partial_x(\sigma', S'))$ and $\mathsf{supp}(\partial_x(\sigma, S))$ would be empty). Say $S = t$ and $S' = t'$ (with $t, t' \in \Delta$, $t \frown t'$). Then we have $\varphi = t$ and $\varphi' = t'$ and we conclude straightforwardly.

The case where σ is an abstraction is trivial.

Assume that $\sigma = \langle t\rangle\, T$ (with $t \in \Delta$ and $T \in \Delta^!$). Then by definition of coherence we must have $\sigma' = \langle t'\rangle\, T'$ with $t \frown t'$ and $T \frown T'$. Since $\varphi \in \mathsf{supp}(\partial_x(\sigma, S))$, we must have $\varphi = \langle u\rangle\, U$ and there must exist $S_1, S_2 \in \Delta^!$ such that $S = S_1 S_2$, $u \in \mathsf{supp}(\partial_x(t, S_1))$, $U \in \mathsf{supp}(\partial_x(T, S_2))$. Similarly, $\varphi' = \langle u'\rangle\, U'$ and there exist $S'_1, S'_2 \in \Delta^!$ such that $S' = S'_1 S'_2$, $u' \in \mathsf{supp}(\partial_x(t', S'_1))$, $U' \in \mathsf{supp}(\partial_x(T', S'_2))$. But by definition of coherence we have $S_1 \frown S'_1$ and $S_2 \frown S'_2$ and hence by inductive hypothesis $u \frown u'$ and $U \frown U'$, so $\varphi \frown \varphi'$. If furthermore $\varphi = \varphi'$, then $u = u'$ and $U = U'$ and the inductive hypothesis yields $t = t'$, $S_1 = S'_1$ and $S_2 = S'_2$ and we conclude.

Assume last that σ and σ' are poly-terms. If $\sigma = 1$, we must have $S = 1$ (as otherwise $\mathsf{supp}(\partial_x(\sigma, S))$ would be empty) and there are two sub-cases: the case $\sigma' = 1$ is straightforward. Let us assume that $\sigma' \neq 1$ so that we can write $\sigma' = u'U'$. In that case we have $\varphi = 1$ and $\varphi' = v'V'$ with $v' \in \mathsf{supp}(\partial_x(u', S'_1))$ and $V' \in \mathsf{supp}(\partial_x(U', S'_2))$ for some $S'_1, S'_2 \in \mathcal{M}_{\mathrm{fin}}(\Delta)$ satisfying $S'_1 S'_2 = S'$. We have to show that $1 \frown v'V'$, or equivalently that $\{v'\} \cup \mathsf{set}(V')$ is a clique. That $\mathsf{set}(V')$ is a clique results from the inductive hypothesis. So let $w' \in \mathsf{set}(V')$ and let us show that $v' \frown w'$. Then $w' \in \mathsf{supp}(\partial_x(w'_0, S'_3))$ where $w'_0 \in \mathsf{set}(U')$ and S'_3 is a factor of S'_2. We have $u' \frown w'_0$ and $S'_1 \frown S'_3$, hence the inductive hypothesis yields $v' \frown w'$ as desired. In the present case we know that $\varphi \neq \varphi'$ so there is nothing more to prove.

The last sub-case to consider is the case where σ and σ' are simple poly-terms both distinct from 1. Then we can write $\varphi = vV$ and $\varphi' = v'V'$ where $v \in \mathsf{supp}(\partial_x(t, S_1))$, $V \in \mathsf{supp}(\partial_x(U, S_2))$, $v' \in \mathsf{supp}(\partial_x(t', S'_1))$ and $V' \in \mathsf{supp}(\partial_x(U', S'_2))$ with $tU = \sigma$ and $t'U' = \sigma'$, for some $S_1, S_2, S'_1, S'_2 \in \Delta^{(!)}$ satisfying $S_1 S_2 = S$ and $S'_1 S'_2 = S'$. One shows exactly as above that $\varphi \frown \varphi'$. If moreover $\varphi = \varphi'$, then we can take $v = v'$ and $V = V'$ and again we conclude straightforwardly by inductive hypothesis, since we know that $t \frown t'$ and $S_1 \frown S'_1$ (and hence $t = t'$ and $S_1 = S'_1$) on one hand, and $U \frown U'$ and $S_2 \frown S'_2$ (and hence $U = U'$ and $S_2 = S'_2$) on the other hand. This concludes the proof. □

Stability of the normal form operator. As a consequence of Theorem 10 and Lemma 4, the NF operator satisfies also a stability property with respect to the coherence relation we have defined on (poly-)terms.

Theorem 11 *Let $\sigma, \sigma' \in \Delta^{(!)}$ and assume that $\sigma \frown \sigma'$. Then for all $\varphi \in \mathsf{supp}(\mathsf{NF}(\sigma))$ and $\varphi' \in \mathsf{supp}(\mathsf{NF}(\sigma'))$ we have $\varphi \frown \varphi'$, and if furthermore $\varphi = \varphi'$, then $\sigma = \sigma'$.*

Proof. We proceed by induction on the sum of the sizes of the simple (poly-)terms σ and σ', using Lemma 4.

If $\mathsf{S}(\sigma) + \mathsf{S}(\sigma') = 0$ then σ and σ' are poly-terms and $\sigma = \sigma' = 1$; one concludes straightforwardly.

Otherwise, assume first that σ is a simple term, we consider the following cases.

- If $\sigma = \lambda\bar{x}\,\langle x\rangle\, S_1 \ldots S_n$, then $\sigma' = \lambda\bar{x}\,\langle x\rangle\, S'_1 \ldots S'_n$ with $S_i \subset\!\!\!\supset S'_i$ for $i = 1, \ldots, n$. Since $\varphi \in \mathsf{supp}(\mathsf{NF}(\sigma))$ and $\varphi' \in \mathsf{supp}(\mathsf{NF}(\sigma'))$, these simple terms are of the shape $\varphi = \lambda\bar{x}\,\langle x\rangle\, T_1 \ldots T_n$ and $\varphi' = \lambda\bar{x}\,\langle x\rangle\, T'_1 \ldots T'_n$ with $T_i \in \mathsf{supp}(\mathsf{NF}(S_i))$ and $T'_i \in \mathsf{supp}(\mathsf{NF}(S'_i))$ for each i. Then we apply the inductive hypothesis for each i (since $S_i \subset\!\!\!\supset S'_i$) and we conclude.

- If $\sigma = \lambda\bar{x}\,\langle\langle\lambda x\, t\rangle\, U\rangle\, S_1 \ldots S_n$ then σ' must be of the shape $\sigma' = \lambda\bar{x}\,\langle\langle\lambda x\, t\rangle\, U'\rangle\, S'_1 \ldots S'_n$ with of course $t \subset\!\!\!\supset t'$, $U \subset\!\!\!\supset U'$ and $S_i \subset\!\!\!\supset S'_i$ for each i. There exists $u \in \mathsf{supp}(\partial_x(t, U))$ and $u' \in \mathsf{supp}(\partial_x(t', U'))$ such that $\varphi \in \mathsf{supp}(\mathsf{NF}(\lambda\bar{x}\,\langle u\rangle\, S_1 \ldots S_n))$ and $\varphi' \in \mathsf{supp}(\mathsf{NF}(\lambda\bar{x}\,\langle u'\rangle\, S'_1 \ldots S'_n))$. By Theorem 10 we have $u \subset\!\!\!\supset u'$ and hence, since the size of $\lambda\bar{x}\,\langle u\rangle\, S_1 \ldots S_n$ is strictly smaller than the size of σ (and similarly for $\lambda\bar{x}\,\langle u'\rangle\, S'_1 \ldots S'_n$), we have $\varphi \subset\!\!\!\supset \varphi'$ by inductive hypothesis. If moreover $\varphi = \varphi'$, then the inductive hypothesis implies that $u = u'$ and $S_i = S'_i$ for each i and hence (applying again Theorem 10), we obtain that $\sigma = \sigma'$.

Assume last that $\sigma = S$ and $\sigma' = S'$ are poly-terms. Let $T \in \mathsf{supp}(\mathsf{NF}(S))$ and $T' \in \mathsf{supp}(\mathsf{NF}(S'))$, we must show that $T \subset\!\!\!\supset T'$, so let $t, t' \in \mathsf{set}(T) \cup \mathsf{set}(T')$. We are reduced to showing that $t \subset\!\!\!\supset t'$. There exists $s, s' \in \mathsf{set}(S) \cup \mathsf{set}(S')$ such that $t \in \mathsf{NF}(s)$ and $t' \in \mathsf{NF}(s')$. We know that $s \subset\!\!\!\supset s'$ (by definition of coherence for poly-terms) and moreover, with our definition of the size, we have $\mathsf{S}(s)+\mathsf{S}(s') < \mathsf{S}(S)+\mathsf{S}(S')$. Therefore the inductive hypothesis applies and yields $t \subset\!\!\!\supset t'$ and hence $T \subset\!\!\!\supset T'$. Assume moreover that $T = T' = t_1 \ldots t_k$. Then S and S' must be of the shape $S = s_1 \ldots s_k$ and $S' = s'_1 \ldots s'_k$ with $t_i \in \mathsf{supp}(\mathsf{NF}(s_i)) \cap \mathsf{supp}(\mathsf{NF}(s'_i))$ for each i, and hence $s_i = s'_i$ for each i (by inductive hypothesis again). Hence $S = S'$. □

Let M be an ordinary lambda-term. The theorem above expresses a "non-interference" property: if s and t are two distinct simple terms which appear in the complete Taylor expansion of M, then the normal forms of s and t will have non overlapping supports (and, by the way, these supports will contain only uniform simple terms).

4 Quantitative properties: combinatorial considerations

We want now to understand the behaviour of the mutiplicity coefficients of a simple (poly-)term along its big step reduction. In the present paper, we want to solve this question when the simple (poly-)term under consideration appears in the complete Taylor expansion of an ordinary lambda-term, and hence is uniform. This hypothesis will be extremely useful.

For this purpose, the basic fact to observe is that $\mathsf{m}(\sigma)$ is the number of permutations of the *free or bound* variable *occurrences* in σ which respect the variables associated to these occurrences and leave σ unchanged. These permutations form a subgroup of a symmetric group, the *isotropy* group of σ. This group is generally non trivial because the multi-set construction used in the syntax of poly-terms is commutative. For instance, the term $\lambda x\,\langle\langle z\rangle\, x^3\rangle\, y^2$ has multiplicity coefficient $3! \times 2!$.

Doing that, we have transformed our problem into a group-theoretic one: relate the isotropy groups of a term to the isotropy group of the same term where a big step differential substitution has been performed. This is what we do in this section for proving the *Uniform Plugging Equation*.

4.1 A group equation

Let G be a finite group and let L and R be subgroups of G. Then $LR = \{lr \mid l \in L \text{ and } r \in R\} \subseteq G$ is not a subgroup of G in general. Nevertheless, the cardinality of this set satisfies the following well known equation which is essential in the forthcoming considerations.

Lemma 12 *If L and R are subgroups of a finite group G, then*

$$|LR| = \frac{|L|\,|R|}{|L \cap R|}\,.$$

Proof. The set LR is the union of the left cosets lR (for $l \in L$), and these cosets are either disjoint or equal and have $|R|$ as cardinality. Let $l, l' \in L$, one has $lR = l'R$ exactly when $l^{-1}l' \in L \cap R$ and so LR is the disjoint union of exactly $|L| / |L \cap R|$ disjoint sets of cardinality $|R|$, whence the equation. □

We shall also use the fact that if $h : G \to H$ is a group homomorphism and G is finite, then $|h(G)| = |G| / |\ker h|$.

4.2 The uniform plugging equation

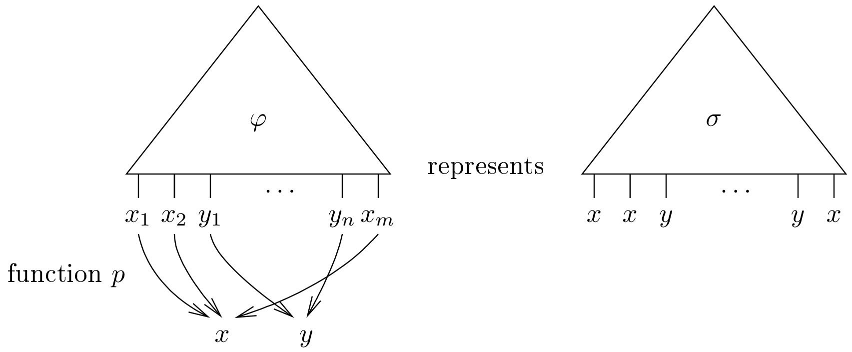
In order to give a precise definition of the group of permutations of variable occurrences in a simple (poly-)term σ which leave σ unchanged, we need to separate the various occurrences of all variable appearing, free or bound, in σ. This is exactly the purpose of the notion of multilinear-free (poly-)term we introduce now. The idea is to separate the occurrences in σ by using pairwise distinct variables, producing a term φ, and then recovering the original names of variables through a "naming function" (we will use letters p, q... for these functions from variables to variables). Such a pair (φ, p) will be called a multilinear-free representation of σ. Because the permutations we consider should act also on the *bound* occurrences of σ, all the variables occurring in φ will be required to be free.

Multilinear-free representation of a (poly-)term. Let us say that a simple (poly-)term φ is *multilinear-free* if each variable occurring in φ occurs exactly once, and occurs free in φ. Let us say that a partial function (substitution) Φ from $\mathcal{V}$ to multilinear-free terms is multilinear-free if $\mathcal{V}(\Phi(x)) \cap \mathcal{V}(\Phi(x')) = \emptyset$ when x and x' are two distinct elements of $\mathsf{Dom}\,\Phi$ (the domain of Φ). We use $\mathcal{V}(\Phi)$ for the disjoint union $\bigcup_{x \in \mathsf{Dom}\,\Phi} \mathcal{V}(\Phi(x))$.

Given a multilinear-free (poly-)term φ and a multilinear-free substitution Φ, we say that the pair (φ, Φ) is *adapted* if $\mathcal{V}(\varphi) \subseteq \mathsf{Dom}\,\Phi$, and no element of $\mathcal{V}(\Phi)$ is bound in φ. In that situation, we can apply the substitution Φ to the term φ, getting a (poly-)term $\Phi\varphi$ which is clearly also multilinear-free.

Let φ be a multilinear-free (poly-)term and let $p : \mathcal{V}(\varphi) \to \mathcal{V}$ be a function. We use $p\varphi$ for the (poly-)term obtained by substituting each variable y occurring in φ with $p(y)$, in the most naive way (that is, without renaming captured variables).

Let σ be a (poly-)term, we say that (φ, p) *represents* σ if $p\varphi = \sigma$, a situation which can be pictured as follows:



Example. The simple term $\sigma = \langle z \rangle \, (z(\lambda y\, y)^2)$ is represented by the pair (φ, p) where

$$\varphi = \langle z_1 \rangle \, (z_2(\lambda y\, y_1)(\lambda y\, y_2)) \quad \text{and} \quad \left\{ \begin{array}{l} p(z_1) = p(z_2) = z \\ p(y_1) = p(y_2) = y \end{array} \right. .$$

Clearly, if both (φ, p) and (ψ, q) represent σ, there is a (generally not unique) bijection $f : \mathcal{V}(\varphi) \to \mathcal{V}(\psi)$ such that $qf = p$ and $f\varphi = \psi$. This can be proved by induction on σ. If σ is the simple term of the example above, there are two such bijections f.

If $p : \mathcal{V} \to \mathcal{V}$ is a partial function, we use $\mathfrak{S}_p$ for the subgroup of $\mathfrak{S}_{\mathsf{Dom}\,p}$ of all bijections f on $\mathsf{Dom}\,p$ such that $pf = p$: it is a finite product of symmetric groups. If φ is a multilinear-free (poly-)term and $p : \mathcal{V}(\varphi) \to \mathcal{V}$, we use $\mathsf{Iso}(\varphi, p)$ for the subgroup of $\mathfrak{S}_p$ whose elements f satisfy $f\varphi = \varphi$, since it is the isotropy group of φ for the action of $\mathfrak{S}_p$ on the multilinear-free simple (poly-)terms having the same free variables as φ.

Example. Consider the following closed simple term:

$$\sigma = \lambda x \langle x \rangle (\lambda y \langle x \rangle y^2)^2 .$$

We represent this terms by the pair (φ, p) where

$$\varphi = \lambda x \langle x_1 \rangle (\lambda y \langle x_2 \rangle y_1 y_2)(\lambda y \langle x_3 \rangle y_3 y_4) \quad \text{and} \quad \begin{cases} p(x_1) = p(x_2) = p(x_3) = x \\ p(y_1) = \cdots = p(y_4) = y \end{cases} .$$

We have $\mathfrak{S}_p \simeq \mathfrak{S}_{\{x_1,x_2,x_3\}} \times \mathfrak{S}_{\{y_1,y_2,y_3,y_4\}}$ (a group with 144 elements). Then $\mathsf{Iso}(\varphi, p)$ is the subgroup generated by the two transpositions which swap respectively y_1, y_2 and y_3, y_4, and by the permutation f given by $f(x_1) = x_1$, $f(x_2) = x_3$, $f(x_3) = x_2$, $f(y_1) = y_3$, $f(y_2) = y_4$, $f(y_3) = y_1$ and $f(y_4) = y_2$. This subgroup has 8 elements, as easily checked. Observe by the way that $\mathsf{m}(\sigma) = 2 \times 2^2 = 8$.

Combinatorial interpretation. Here is the announced combinatorial interpretation of the multiplicity coefficients.

Lemma 13 *Let σ be a (poly-)term, let φ be a multilinear-free (poly-)term and $p : \mathcal{V}(\varphi) \to \mathcal{V}$ be a function such that (φ, p) represents σ. Then $|\mathsf{Iso}(\varphi, p)| = \mathsf{m}(\sigma)$.*

This is a simple proof by induction on σ. This formula for the cardinality of $\mathsf{Iso}(\varphi, p)$ is completely standard if we observe that this group is a wreath product of symmetric groups.

Isotropy group of a substitution. More generally, if Φ is a multilinear-free substitution and if $p : \mathsf{Dom}\,\Phi \to \mathcal{V}$ and $q : \mathcal{V}(\Phi) \to \mathcal{V}$ are functions, we define the group

$$\mathsf{Iso}(p, \Phi, q) = \{g \in \mathfrak{S}_q \mid \exists f \in \mathfrak{S}_p \; g\Phi = \Phi f\} .$$

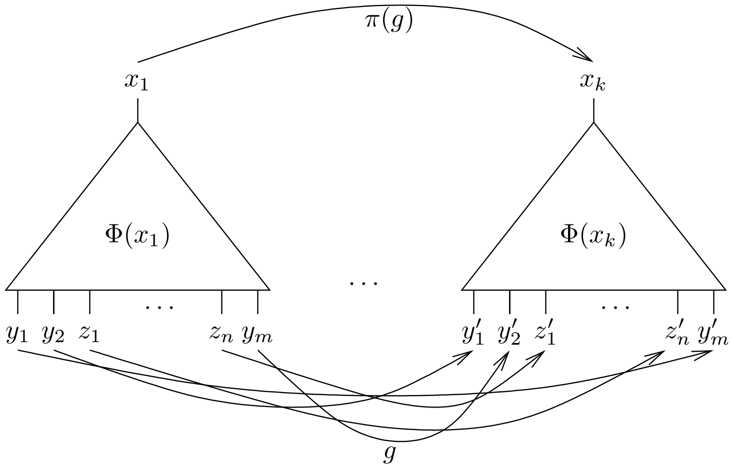
Observe that this is simply the isotropy group in $\mathfrak{S}_q$ of the following tuple of multilinear-free poly-terms:

$$\Big(\prod_{y \in p^{-1}(z_i)} \Phi(y) \Big)_{i=1,\dots,l}$$

where $(z_1, \dots, z_l)$ is an enumeration without repetitions of the range of p.

Due to the injectivity of Φ as a function, the bijection f associated to g in the definition above is uniquely determined, and clearly the map $g \mapsto f$ is a group homomorphism. In other words, $\mathsf{Iso}(p, \Phi, q)$ comes equipped with a group homomorphism π such that

$$\forall g \in \mathsf{Iso}(p, \Phi, q) \quad g\Phi = \Phi\pi(g) .$$



Assume that we are given φ, Φ, p and q as above, with (φ, Φ) adapted. Then there is yet another set of permutations which will play an important role in the sequel, and this set is not a group in general, namely:

$$\mathsf{Iso}(\varphi, p, \Phi, q) = \{f \in \mathfrak{S}_p \mid \exists g \in \mathfrak{S}_q \; g\Phi\varphi = \Phi f \varphi\} .$$

The following inclusion always holds.

$$\pi(\mathsf{Iso}(p, \Phi, q))\, \mathsf{Iso}(\varphi, p) \subseteq \mathsf{Iso}(\varphi, p, \Phi, q) . \tag{6}$$

Indeed, let $g \in \mathsf{Iso}(p, \Phi, q)$ and let $f \in \mathsf{Iso}(\varphi, p)$. Then $\Phi\pi(g)f\varphi = g\Phi\varphi$ and so $\pi(g)f \in \mathsf{Iso}(\varphi, p, \Phi, q)$. We shall see that when the pair (φ, p) is "uniform", the converse inclusion holds as well. The crucial step for proving this is the forthcoming factorization property, Lemma 14.

Uniform pairs. We define when a pair (F, p) is *uniform*, F being a multilinear-free poly-term and $p : \mathcal{V}(F) \to \mathcal{V}$ a naming function. We shall see in Lemma 15 that this notion is equivalent to the concept of uniformity we have already defined using the coherence relation on poly-terms, but we give first the following self-contained definition, very suitable to our present combinatorial considerations. The definition is by induction. The pair (F, p) is uniform in one of the following situations:

- $F = x_1 \dots x_n$ where the x_i's are variable and $p(x_i) = p(x_j)$ for all i, j;
- $F = (\lambda y\, \varphi_1) \dots (\lambda y\, \varphi_n)$ and $(\varphi_1 \dots \varphi_n, p)$ is uniform;
- $F = (\langle \varphi_1 \rangle\, G_1) \dots (\langle \varphi_n \rangle\, G_n)$ and $(\varphi_1 \dots \varphi_n, l)$ and $(G_1 \dots G_n, r)$ are uniform, where l and r are the obvious restrictions of p.

When φ is a multilinear-free simple term, we say that (φ, p) is uniform if (φ, p) is uniform, φ being considered as a singleton poly-term.

The main property of uniform pairs is the following factorization lemma.

Lemma 14 (factorization) *Let (φ, p) be a uniform pair and let Φ and Φ' be two multilinear-free substitutions of domain $\mathcal{V}(\varphi)$. If $\Phi\varphi = \Phi'\varphi$, then there exists $f \in \mathsf{Iso}(\varphi, p)$ such that $\Phi' = \Phi f$.*

Proof. We can restrict our attention to the case where φ is a poly-term, and the only interesting case in the inductive definition above of uniformity is obviously the last one. With the notations of that definition, we can find $g \in \mathsf{Iso}(\varphi_1 \dots \varphi_n, l)$ such that $\Lambda' = \Lambda g$ and $h \in \mathsf{Iso}(G_1 \dots G_n, r)$ such that $P' = Ph$ where Λ, Λ' and P, P' are the restrictions of Φ, Φ' to $\mathcal{V}(\varphi_1 \dots \varphi_n)$ and $\mathcal{V}(G_1 \dots G_n)$ respectively. Taking the union f of these two bijections g and h, we obtain an element f of $\mathfrak{S}_p$, and it remains to show that $fF = F$. For this, it will be sufficient to show that there is an index i such that $g\varphi_1 = \varphi_i$ and $hG_1 = G_i$. We know that there is an i such that $g\varphi_1 = \varphi_i$ since $g \in \mathsf{Iso}(\varphi_1 \dots \varphi_n, l))$ (and this i is unique since each φ_j contains at least one variable, and all these variables are distinct). We know moreover that $\Phi(\langle \varphi_1 \rangle\, G_1 \dots \langle \varphi_n \rangle\, G_n) = \Phi'(\langle \varphi_1 \rangle\, G_1 \dots \langle \varphi_n \rangle\, G_n)$ and hence there is a (uniquely determined) j such that $\Phi' \langle \varphi_1 \rangle\, G_1 = \Phi \langle \varphi_j \rangle\, G_j$, hence $\Lambda'\varphi_1 = \Lambda\varphi_j$, that is $\Lambda g \varphi_1 = \Lambda \varphi_j$. This implies that $g\varphi_1 = \varphi_j$ (because Λ is an injective partial function from variables to simple terms), hence $\varphi_i = \varphi_j$ and so we must have $j = i$. Therefore $\Phi' \langle \varphi_1 \rangle\, G_1 = \Phi \langle \varphi_i \rangle\, G_i$, hence $P'G_1 = PG_i$, that is $PhG_1 = PG_i$. If $G_1 = 1$ then $G_i = 1$ and $hG_1 = G_i$ holds trivially. Otherwise we conclude again using the injectivity of P. $\square$

The uniformity hypothesis is essential: take for φ the poly-term xy, for p the identity map on $\{x, y\}$, and define Φ and Φ' by $\Phi(x) = x$, $\Phi(y) = y$ and $\Phi'(x) = y$, $\Phi'(y) = x$. Then $\Phi\varphi = \Phi'\varphi = \varphi$ but $\Phi \neq \Phi'$ and the only element of $\mathsf{Iso}(\varphi, p)$ is the identity. The problem is of course that the pair (φ, p) is not uniform.

We state now the equivalence between the two notions of uniformity introduced so far.

Lemma 15 *Let σ be a (poly-)term. Let φ be a multilinear-free (poly-)term and $p : \mathcal{V}(\varphi) \to \mathcal{V}$ be a function such that $\sigma = p\varphi$. Then σ is uniform (that is $\sigma \frown \sigma$) iff the pair (φ, p) is uniform.*

The proof is a straightforward induction on σ.

The equation. Let φ be a multilinear-free simple term, Φ be a multilinear-free substitution with $\mathsf{Dom}\,\Phi = \mathcal{V}(\varphi)$, $p : \mathcal{V}(\varphi) \to \mathcal{V}$ and $q : \mathcal{V}(\Phi) \to \mathcal{V}$ be functions. Assume that the pair (φ, Φ) is adapted and that the pair (φ, p) is uniform.

Let us first check that

$$\pi(\mathsf{Iso}(p, \Phi, q))\, \mathsf{Iso}(\varphi, p) = \mathsf{Iso}(\varphi, p, \Phi, q)\,.$$

Let $f \in \mathsf{Iso}(\varphi, p, \Phi, q)$, that is $f \in \mathfrak{S}_p$ and there exists $g \in \mathfrak{S}_q$ such that $g\Phi f\varphi = \Phi\varphi$. Since the pair (φ, p) is uniform, we can apply Lemma 14 and hence there exists $f' \in \mathsf{Iso}(\varphi, p)$ such that $g\Phi f = \Phi f'$. This means that $g \in \mathsf{Iso}(p, \Phi, q)$ and $\pi(g) = f' f^{-1}$. Hence $f = \pi(g^{-1}) f' \in \pi(\mathsf{Iso}(p, \Phi, q))\, \mathsf{Iso}(\varphi, p)$. We have already seen that the converse inclusion holds (inclusion (6)).

Since $|\pi(\mathsf{Iso}(p, \Phi, q))| = |\mathsf{Iso}(p, \Phi, q)| \,/\, |\ker \pi|$, applying Lemma 12 we obtain

$$|\mathsf{Iso}(\varphi, p, \Phi, q)| = \frac{|\mathsf{Iso}(p, \Phi, q)|\, |\mathsf{Iso}(\varphi, p)|}{|\ker \pi|\, |\pi(\mathsf{Iso}(p, \Phi, q)) \cap \mathsf{Iso}(\varphi, p)|}\,.$$

To conclude, we need to evaluate the expression $|\pi(\mathsf{Iso}(p, \Phi, q)) \cap \mathsf{Iso}(\varphi, p)|$.

Let $g \in \mathsf{Iso}(\Phi\varphi, q)$. Since the pair (φ, p) is uniform, by Lemma 14 again, there exists $f \in \mathsf{Iso}(\varphi, p)$ such that $g\Phi = \Phi f$. In other words $\mathsf{Iso}(\Phi\varphi, q) \subseteq \mathsf{Iso}(p, \Phi, q)$ and $\pi(\mathsf{Iso}(\Phi\varphi, q)) \subseteq \mathsf{Iso}(\varphi, p)$. So $\pi(\mathsf{Iso}(\Phi\varphi, q)) \subseteq \pi(\mathsf{Iso}(p, \Phi, q)) \cap \mathsf{Iso}(\varphi, p)$. But the converse implication holds as well. Indeed, let $g \in \mathsf{Iso}(p, \Phi, q)$ be such that $\pi(g) \in \mathsf{Iso}(\varphi, p)$. Then $g\Phi\varphi = \Phi\pi(g)\varphi = \Phi\varphi$ hence $g \in \mathsf{Iso}(\Phi\varphi, q)$.

Last observe that obviously $\ker \pi \subseteq \mathsf{Iso}(\Phi\varphi, q)$. So

$$|\pi(\mathsf{Iso}(p, \Phi, q)) \cap \mathsf{Iso}(\varphi, p)| = |\pi(\mathsf{Iso}(\Phi\varphi, q))| = \frac{|\mathsf{Iso}(\Phi\varphi, q)|}{|\ker \pi|}\,.$$

So we have proved the following result which will be essential in the sequel.

Theorem 16 (Uniform plugging equation) *If φ is a multilinear-free simple term, Φ a multilinear-free substitution with (φ, Φ) adapted, if $p : \mathcal{V}(\varphi) \to \mathcal{V}$ and $q : \mathcal{V}(\Phi) \to \mathcal{V}$ are functions and if the pair (φ, p) is uniform, then the following equation holds:*

$$|\mathsf{Iso}(\varphi, p, \Phi, q)| = \frac{|\mathsf{Iso}(p, \Phi, q)|\, |\mathsf{Iso}(\varphi, p)|}{|\mathsf{Iso}(\Phi\varphi, q)|}\,.$$

The uniformity hypothesis is necessary. Take indeed for φ the non uniform poly-term $\varphi = x_1(\langle x_2\rangle\, 1)$ (p being the constant function $x_i \mapsto x$ where x is a fixed element of $\mathcal{V}$). Then $|\mathsf{Iso}(\varphi, p)| = 1$. Define Φ by $\Phi(x_1) = \langle y_1\rangle\, 1$ and $\Phi(x_2) = y_2$ and take for q a constant function $q(y_j) = y$. Then $|\mathsf{Iso}(p, \Phi, q)| = 1$, but $\Phi\varphi = (\langle y_1\rangle\, 1)(\langle y_2\rangle\, 1)$ so that $|\mathsf{Iso}(\Phi\varphi, q)| = 2$ and the equation above cannot hold since its left hand member must be an integer.

5 Reducing the Taylor expansion of an ordinary lambda-term

With the qualitative Theorems 10 and 11 and the quantitative Theorem 16, we have the main tools for studying the beta-reduction of the Taylor expansion of an ordinary lambda-term.

If $\sigma = \sum_{\theta \in \Delta^{(!)}} a_\theta \theta$ is an element of $R\langle\Delta^{(!)}\rangle_\infty$ *which has a clique as support* (that is, the set $\mathsf{supp}(\sigma) = \{\theta \mid a_\theta \neq 0\}$ is a clique for the coherence relation defined on simple terms and on simple poly-terms in Section 3), then it makes sense to defines $\mathsf{NF}(\sigma) = \sum_{\theta \in \Delta^{(!)}} a_\theta\, \mathsf{NF}(\theta) \in R\langle\Delta_0^{(!)}\rangle_\infty$ (where $\Delta_0^{(!)}$ is the set of normal simple (poly-)terms) since indeed by Theorem 11, for each $\theta_0 \in \Delta_0^{(!)}$ there is at most one $\theta \in \mathsf{supp}(\sigma)$ such that $\theta_0 \in \mathsf{supp}(\mathsf{NF}(\theta))$. We know moreover that $\mathsf{supp}(\mathsf{NF}(\sigma))$ is a clique[8].

In particular, when M is an ordinary lambda-term, $\mathsf{NF}(M^*)$ is a well defined, generally infinite, linear combination of normal simple terms with rational coefficients whose support is a clique. We shall see that each term θ occurring in this sum occurs with $1/\mathsf{m}(\theta)$ as coefficient.

We need first to consider the case of a single big step differentiation: for dealing with this case, we apply the uniform plugging equation straightforwardly.

[8]The set $\mathcal{C}$ of all $\sigma \in R\langle\Delta^{(!)}\rangle_\infty$ such that $\mathsf{supp}(\sigma)$ is a clique is closed for the product topology, but is not a linear subspace as it is not stable under addition of (poly-)terms. We can see NF as a continuous operator from this set to itself.

Lemma 17 *Let $\sigma \in \Delta^{(!)}$ be uniform, let x be a variable and let $T \in \Delta^!$. Let $\theta \in \mathsf{supp}(\partial_x(\sigma, T))$. Then*

$$\partial_x(\sigma, T)_\theta = \frac{\mathsf{m}(\sigma)\mathsf{m}(T)}{\mathsf{m}(\theta)} .$$

Proof. Observe first that our hypotheses imply that $|T| = \mathsf{d}_x\sigma$ since otherwise the set $\mathsf{supp}(\partial_x(\sigma, T))$ would be empty. Let φ be a multilinear-free (poly-)term and $p : \mathcal{V}(\varphi) \to \mathcal{V}$ be a function such that $p\varphi = \sigma$. Then, by Lemma 15, the pair (φ, p) is uniform since σ is. By Formula (3), we can choose a multilinear-free substitution Φ and a function $q : \mathcal{V}(\Phi) \to \mathcal{V}$ in such a way that the following requirements be fulfilled:

- the pair (φ, Φ) is adapted;
- $q(\prod_{p(x')=x} \Phi(x')) = T$ (that is "(Φ, q), when restricted to $p^{-1}(\{x\})$, represents T");
- if $p(x') \neq x$ then $\Phi(x') = x'$ and $q(x') = p(x')$ (that is, the substitution Φ acts trivially on all occurrences of variables distinct from x);
- $\theta = q\Phi\varphi$.

By Formula (3), the coefficient $\partial_x(\sigma, T)_\theta$ is the number of permutations $f \in \mathfrak{S}_n$ such that

$$\sigma' \left[t_{f(1)}/x_1, \ldots, t_{f(n)}/x_n\right] = \theta ,$$

where $t_1 \ldots t_n = T$, the variables $x_1, \ldots, x_n$ are fresh and σ' is an x-linearization in $x_1, \ldots, x_n$ of σ (it can be chosen such that $\sigma' [t_1/x_1, \ldots, t_n/x_n] = \theta$ and in that case the above mentioned set of permutations is a group). So $\partial_x(\sigma, T)_\theta = |\mathsf{Iso}(\varphi, p, \Phi, q)|$. The equation follows then from Lemma 13 and Theorem 16.

□

Again, the uniformity condition is absolutely essential.

Two corollaries. We derive two easy corollaries of this formula, before applying it to our main concern, which is the study of the normal forms of the terms occurring in the Taylor expansion of an ordinary lambda-term.

First, we generalize the formula to iterated big step differentiation.

Proposition 18 *Let $\sigma \in \Delta^{(!)}$ be uniform, let $x_1, \ldots, x_n$ be pairwise distinct variables and let $T_1, \ldots, T_n \in \Delta^!$ be uniform. Let $\theta \in \mathsf{supp}(\partial_{x_1,\ldots,x_n}(\sigma, T_1, \ldots, T_n))$. Then*

$$\partial_{x_1,\ldots,x_n}(\sigma, T_1, \ldots, T_n)_\theta = \frac{\mathsf{m}(\sigma)\mathsf{m}(T_1)\cdots\mathsf{m}(T_n)}{\mathsf{m}(\theta)} .$$

Proof. It will be enough to deal with the case $n = 2$. We have

$$\begin{aligned}\partial_{x_1,x_2}(\sigma, T_1, T_2)_\theta &= \partial_{x_2}(\partial_{x_1}(\sigma, T_1), T_2)_\theta \\ &= \sum_{\rho\in\Delta^{(!)}} \partial_{x_1}(\sigma, T_1)_\rho \partial_{x_2}(\rho, T_2)_\theta ,\end{aligned}$$

but since σ and T_1 are uniform, $\mathsf{supp}(\partial_{x_1}(\sigma, T_1))$ is a clique by Theorem 10 and hence there is at most one $\rho \in \mathsf{supp}(\partial_{x_1}(\sigma, T_1))$ such that $\theta \in \mathsf{supp}(\partial_{x_2}(\rho, T_2))$. Hence, since we have assumed that $\theta \in \mathsf{supp}(\partial_{x_1,x_2}(\sigma, T_1, T_2))$, there is exactly one such ρ and we know that this ρ is uniform, so we get, applying twice Lemma 17,

$$\partial_{x_1,x_2}(\sigma, T_1, T_2)_\theta = \frac{\mathsf{m}(\sigma)\mathsf{m}(T_1)}{\mathsf{m}(\rho)} \cdot \frac{\mathsf{m}(\rho)\mathsf{m}(T_2)}{\mathsf{m}(\theta)} = \frac{\mathsf{m}(\sigma)\mathsf{m}(T_1)\mathsf{m}(T_2)}{\mathsf{m}(\theta)} .$$

□

The second corollary is another version of the Taylor formula, which is now substitution-oriented instead of being application-oriented as in Lemma 8.

Theorem 19 *Let M and N be ordinary lambda-terms and let x be a variable. One has $\partial_x(M^*, N^{*n}) \to 0$ as $n \to \infty$, and the following equation holds:*

$$M\,[N/x]^* = \sum_{n=0}^{\infty} \frac{1}{n!} \partial_x(M^*, N^{*n})\,.$$

Proof. The convergence statement results from the fact that $M^{*n} \to 0$ and from the continuity of ∂_x. Just as in the proof of Lemma 8, we have

$$\sum_{n=0}^{\infty} \frac{1}{n!} \partial_x(M^*, N^{*n}) = \sum_{\substack{s \in \mathcal{T}(M) \\ T \in \mathcal{M}_{\mathrm{fin}}(\mathcal{T}(N))}} \frac{1}{\mathsf{m}(s)\mathsf{m}(T)} \partial_x(s, T)\,.$$

To conclude, observe that the family of sets $(\mathsf{supp}(\partial_x(s,T)))_{(s,T) \in \mathcal{T}(M) \times \mathcal{M}_{\mathrm{fin}}(\mathcal{T}(N))}$ is a partition of $\mathcal{T}(M\,[N/x])$ (disjointness results from Theorem 10, and the equality of sets is proved by an easy induction on M, using the Leibniz law in the case where M is an application), and then apply Lemma 17. $\square$

Proposition 20 *Let $\sigma \in \Delta^{(!)}$ be uniform and let $\theta \in \mathsf{supp}(\mathsf{NF}(\sigma))$. Then $\mathsf{m}(\theta)$ divides $\mathsf{m}(\sigma)$, and more precisely*

$$\frac{\mathsf{m}(\sigma)}{\mathsf{m}(\theta)} = \mathsf{NF}(\sigma)_\theta\,.$$

Consequently, if M is an ordinary lambda-term, then for each $\theta \in \mathsf{supp}(\mathsf{NF}(M^*))$ one has $\mathsf{NF}(M^*)_\theta = 1/\mathsf{m}(\theta)$.

Proof. We proceed by induction on the size of the simple (poly-)term σ, using Lemma 4. Indeed observe that when σ is uniform, the terms to which NF is applied in the "recursive calls" of that lemma are themselves uniform (the only non-trivial case is the first one, and in that case our claim results from Theorem 10 and from the fact that any (poly-)subterm of a uniform (poly-)term is uniform).

If $\sigma = \lambda x_1 \dots x_n \langle x \rangle T_1 \dots T_k$ then $\theta = \lambda x_1 \dots x_n \langle x \rangle U_1 \dots U_k$ with $U_j \in \mathsf{supp}(\mathsf{NF}(T_j))$ for $j = 1, \dots, k$. By inductive hypothesis, $\mathsf{m}(T_j)/\mathsf{m}(U_j) = \mathsf{NF}(T_j)_{U_j}$, but $\mathsf{m}(\sigma) = \mathsf{m}(T_1) \cdots \mathsf{m}(T_k)$ and $\mathsf{m}(\theta) = \mathsf{m}(U_1) \cdots \mathsf{m}(U_k)$ and we conclude because, by multilinearity of application,

$$\mathsf{NF}(\sigma) = \sum_{V_1, \dots, V_k} \mathsf{NF}(T_1)_{V_1} \cdots \mathsf{NF}(T_k)_{V_k} \lambda x_1 \dots x_n \langle x \rangle V_1 \dots V_k\,.$$

Assume now that $\sigma = \lambda x_1 \dots x_n \langle r \rangle T_1 \dots T_k$ where $r = \langle \lambda x\, s \rangle T$. Then there exists a unique $s' \in \mathsf{supp}(\partial_x(s,T))$ such that $\theta \in \mathsf{supp}(\mathsf{NF}(\lambda x_1 \dots x_n \langle s' \rangle T_1 \dots T_k))$. By inductive hypothesis,

$$\frac{\mathsf{m}(\lambda x_1 \dots x_n \langle s' \rangle T_1 \dots T_k)}{\mathsf{m}(\theta)} = \mathsf{NF}(\lambda x_1 \dots x_n \langle s' \rangle T_1 \dots T_k)_\theta\,.$$

But $\mathsf{NF}(\sigma) = \widetilde{\mathsf{NF}}(\lambda x_1 \dots x_n \langle \partial_x(s,T) \rangle T_1 \dots T_k)$ and so $\mathsf{NF}(\sigma)_\theta = \partial_x(s,T)_{s'}\, \mathsf{NF}(\lambda x_1 \dots x_n \langle s' \rangle T_1 \dots T_k)_\theta$ (see Equation (4)). Therefore by Lemma 17 we get

$$\begin{aligned}
\mathsf{NF}(\sigma)_\theta &= \frac{\mathsf{m}(s)\mathsf{m}(T)\mathsf{m}(\lambda x_1 \dots x_n \langle s' \rangle T_1 \dots T_k)}{\mathsf{m}(s')\mathsf{m}(\theta)} \\
&= \frac{\mathsf{m}(s)\mathsf{m}(T)\mathsf{m}(T_1) \cdots \mathsf{m}(T_k)}{\mathsf{m}(\theta)} \\
&= \frac{\mathsf{m}(\sigma)}{\mathsf{m}(\theta)}\,.
\end{aligned}$$

As a last case, consider the situation where $\sigma = s_1^{p_1} \dots s_k^{p_k}$ is a uniform poly-term, with $s_i \frown s_j$ for all i, j, and s_i and s_j not α-equivalent when $i \neq j$, so that

$$\mathsf{m}(\sigma) = \prod_{j=1}^{k} p_j!\ \mathsf{m}(s_j)^{p_j}\,.$$

Then, by Theorem 11, $\mathsf{supp}(\mathsf{NF}(s_1)), \dots, \mathsf{supp}(\mathsf{NF}(s_k))$ are *pairwise disjoint* cliques and θ is of the shape $\theta = U_1 \dots U_k$ with $U_j \in \mathsf{supp}(\mathsf{NF}(s_j)^{p_j})$ for $j = 1, \dots, k$, and so the multi-sets U_j are pairwise disjoint, so that

$$\mathsf{m}(\theta) = \mathsf{m}(U_1) \cdots \mathsf{m}(U_k)\,.$$

Let $j \in \{1, \dots, k\}$, we have $\mathsf{m}(U_j) = U_j!\ \mathsf{m}^{U_j}$ so that

$$\begin{aligned} \frac{\mathsf{m}(\sigma)}{\mathsf{m}(\theta)} &= \prod_{j=1}^{k} \frac{p_j!\ \mathsf{m}(s_j)^{p_j}}{U_j!\ \mathsf{m}^{U_j}} \\ &= \prod_{j=1}^{k} [U_j]\, \frac{\mathsf{m}(s_j)^{p_j}}{\mathsf{m}^{U_j}} \end{aligned}$$

but for each j,

$$\begin{aligned} \mathsf{NF}(s_j)^{p_j} &= \left(\sum_{u \in \Delta_0} \mathsf{NF}(s_j)_u u \right)^{p_j} \\ &= \left(\sum_{u \in \Delta_0} \frac{\mathsf{m}(s_j)}{\mathsf{m}(u)} u \right)^{p_j} \quad \text{by inductive hypothesis} \\ &= \sum_{U \in \mathcal{M}_{p_j}(\Delta_0)} [U]\, \frac{\mathsf{m}(s_j)^{p_j}}{\mathsf{m}^{U}}\, U \quad \text{by the multinomial identity,} \end{aligned}$$

so

$$\begin{aligned} \mathsf{NF}(\sigma)_\theta &= \prod_{j=1}^{k} \mathsf{NF}(s_j)^{p_j}_{U_j} \\ &= \prod_{j=1}^{k} [U_j]\, \frac{\mathsf{m}(s_j)^{p_j}}{\mathsf{m}^{U_j}}\, U_j \\ &= \frac{\mathsf{m}(\sigma)}{\mathsf{m}(\theta)} \end{aligned}$$

and we are done. □

So we can summarize the situation by the following statement.

Theorem 21 *If M is an ordinary lambda-term and u is a normal simple term, there is at most one $t \in \mathcal{T}(M)$ such that $u \in \mathsf{supp}(\mathsf{NF}(t))$ and the following relation holds:* $\mathsf{m}(t)/\mathsf{m}(u) = \mathsf{NF}(t)_u$.

The unicity results from Theorem 11, and the expression for $\mathsf{NF}(t)_u$ results from Proposition 20.

We can wonder for which u such a t does exist, and how to compute it. The answer involves Böhm trees and the Krivine's machine and will be given in a forthcoming paper.

Example. Let M be the ordinary lambda-term

$$M = (\lambda f\, (f)\, \lambda x\, (f)\, \lambda d\, x)\, \lambda z\, (z)\, (z) \star$$

where $\star$ is a distinguished variable. It is easily seen that M reduces to $\star$. By the theorem above, there is at most one simple term $t \in \mathcal{T}(M)$ such that $\star \in \mathsf{supp}(\mathsf{NF}(t))$. One checks easily that

$$t = \left\langle \lambda f\, \langle f \rangle\, (\lambda x\, \langle f \rangle\, \lambda d\, x)^2 \right\rangle (\lambda z\, \langle z \rangle\, \langle z \rangle \star)(\lambda z\, \langle z \rangle\, 1)^2$$

is such a term, and more precisely that t reduces to $4\star$, in accordance with the fact that $\mathsf{m}(t) = 4$. This simple term can be seen as a "decoration" of M giving an exact quantitative account of how much each subterm of M is used during the run of the Krivine's machine starting with term M (empty environment and empty stack) and leading to the final value $\star$.

Conclusion

The main result of this paper, Theorem 21, shows that the situation is as simple and natural as one could expect. The striking fact, maybe, is not the result itself but its proof, which is based on Theorems 11 and 16, and so uses uniformity twice, and each time in a crucial way. So an essential step in the understanding of the differential extension of the functional paradigm proposed in [ER03] will be to examine the behaviour of Taylor expansions in this more general and non uniform setting.

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