Linear logic, Ludics, Implicit Complexity, Operator Algebras

Geometry of Interaction

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Disclaimer

Syntax/Semantics

Syntax = finite (recursive) sets
Semantics = embedding of syntax into abstract (nonrecursive) framework

- in model theory: formula $\rightsquigarrow$ sets, or elements of a (infinite) boolean algebra
- in denotational semantics: proofs $\rightsquigarrow$ morphism between domains
- etc.

1987: birth of the GoI

The three levels of logic

$$\begin{aligned} \text{The level of formula (truth)} &\rightsquigarrow \text{model theory} \\ \text{The level of proofs (provability)} &\rightsquigarrow \text{denotational semantics} \\ \text{The level of interaction (cut elimination)} &\rightsquigarrow \text{geometry of interaction} \end{aligned}$$

GoI programme = build a semantics for cut elimination.

GoI in litterature

"Instances" of GoI:

- a game semantics interpretation (Abramsky-Jagadeesan-Malacaria game model) $\rightsquigarrow$ traced monoidal categories
- the semantics of sharing reduction (Abadi-Gonthier-Lévy context semantics)
- an abstract machine (Danos-Regnier):
  - a regular paths computing device
  - a reversible automaton
  - Krivine's machine
- an abstract version of proof-nets experiments
- a precursor of ludics: GoI admits *localisation* (that's kind of the problem)

The basic schema of GoI

Formula $A \rightsquigarrow$ space $\mathcal{S}(A)$

Proof $\Pi : A \rightsquigarrow$ operator π acting on $\mathcal{S}(A)$ (notation: $\pi : A$)

Cut $\rightsquigarrow$ given $\pi_1 : A \multimap B$ and $\pi_2 : B \multimap C$, define $\mathrm{Ex}(\pi_1, \pi_2) : A \multimap C$

Note: π_1, π_2 are not functions, $\mathrm{Ex}(\pi_1, \pi_2)$ is not composition

Simplified (but not so much) version:

Given $\pi_1 : A$ and $\pi_2 : A^\perp$, define $\mathrm{Ex}(\pi_1, \pi_2) : \perp$

$\rightsquigarrow \pi_1 \perp \pi_2$ if $\mathrm{Ex}(\pi_1, \pi_2) \in \perp$ (see ludics...)

Basic schema: the a priori typed variant

Categorical flavor, implicit use of duality (*eg* game semantics)

- $\mathcal{S}(A)$ is built by induction on A:
  $\mathcal{S}(A \otimes B) = \mathcal{S}(A \multimap B) = \mathcal{S}(A) + \mathcal{S}(B)$
- $\pi : A$ is an operator on $\mathcal{S}(A)$ (satisfying. . .)
- Possibly get a definability theorem: $\pi : A$ if Π actually is a proof of A $\rightsquigarrow$ full abstraction of AJM game model

What about pure lambda-calculus (or system F)?

Remark

$$\pi : A \multimap B = \begin{pmatrix} \pi^{A^\perp, A^\perp} & \pi^{A^\perp, B} \\ \pi^{B, A^\perp} & \pi^{B, B} \end{pmatrix}$$

Basic schema: the a posteriori typed variant

Girard's symmetric realisability construction (at work in: LL strong normalization, phase semantics, ludics, quantum coherent spaces...)

- Fix a given (universal) space S (*eg* $S = \ell^2$) $\rightsquigarrow \mathcal{S}(A) = S$ for all A: all operators act on S
- Fix a duality, *eg* $\pi \perp \pi'$ iff $\pi\pi'$ is nilpotent
- $\mathcal{T}(A)$ is a set of operators defined by induction on A:
  - $\mathcal{T}(A^\perp) = \mathcal{T}(A)^\perp$
  - $\mathcal{T}(A \otimes B) = \{\pi_1 + \pi_2, \pi_1 \in \mathcal{T}(A), \pi_2 \in \mathcal{T}(B)\}^{\perp\perp}$

  Thus $\mathcal{T}(A \multimap B) = (\mathcal{T}(A) \otimes \mathcal{T}(B^\perp))^\perp$
- Adequation lemma: if Π proof of A then $\pi \in \mathcal{T}(A)$

Remark

What is $\pi_1 + \pi_2$*?*

The multiplicative case: MLL

Follow the a priori typed scheme: operators = partial permutations on finite sets

Formula A	$\rightsquigarrow$	$\mathcal{S}(A) = \{$*occurrences of* atoms in $A\}$ $\mathcal{S}(A \otimes B) = \mathcal{S}(A \multimap B) = \mathcal{S}(A) + \mathcal{S}(B)$
Proof	$\rightsquigarrow$	axiom links permutation on $\mathcal{S}(A)$
Cut	$\rightsquigarrow$	identify atoms in A to their dual in $A^{\perp}$;

The multiplicative case (continued)

- $\pi_1 : A \multimap B$, $\pi_2 : B \multimap C$,
  $\rightsquigarrow \pi = \pi_1 + \pi_2 : (A \multimap B) \otimes (B \multimap C)$
- $\sigma : (A \multimap B) \otimes (B \multimap C)$ partial permutation on $\mathcal{S}(A^\perp) + \mathcal{S}(B) + \mathcal{S}(B^\perp) + \mathcal{S}(C)$ exchanging dual (occurrences of) atoms in B and $B^\perp$

Remark

π and σ are partial symetries: π^2 and σ^2 are projectors

Execution formula

Matrix representation:

$$\pi = \pi_1 + \pi_2 = \begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{pmatrix} = \begin{pmatrix} {\pi_1}^{A^\perp, A^\perp} & {\pi_1}^{A^\perp, B} & 0 & 0 \\ {\pi_1}^{B, A^\perp} & {\pi_1}^{B,B} & 0 & 0 \\ 0 & 0 & {\pi_2}^{B^\perp, B^\perp} & {\pi_2}^{B^\perp, C} \\ 0 & 0 & {\pi_2}^{C, B^\perp} & {\pi_2}^{C,C} \end{pmatrix}$$

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The execution formula

$$\mathrm{Ex}(\pi_1, \pi_2) = (1 - \sigma^2)\pi \left(\sum_{k \geq 0} (\sigma\pi)^k \right) (1 - \sigma^2)$$

GoI and experiments

Hint: let A be a MLL formula. A *point* in $|A|$ may be viewed as a *set* (*ie* a vector over $\mathbb{F}_2$) of *localisations*(moves) in $\mathcal{S}(A)$

For example:

- $a = (a_1, a_2) \in |A_1 \otimes A_2| = |A|_1 \times |A|_2$
- $\alpha \in \mathcal{S}(A_1 \otimes A_2) = \mathcal{S}(A_1) + \mathcal{S}(A_2) \rightsquigarrow \alpha = (i, \alpha_i)$, $i = 1$ or 2, $\alpha_i \in \mathcal{S}(A_i)$
- $(a_1, a_2) = a_1 \oplus a_2 = \{(1, \alpha_1), \alpha_1 \in a_1)\} \cup \{(2, \alpha_2), \alpha_2 \in a_2)\}$

Theorem

If $\pi : A$ is the GoI of $\Pi : A$ then $a \in [\![\Pi]\!]$ iff $\pi(a) = a$

The multiplicative case in the a posteriori typed scheme

- Fixed space = $\mathbb{N}$ (or ℓ^2)
- $p, q : \mathbb{N} \hookrightarrow \mathbb{N}$ (iso $\mathbb{N} + \mathbb{N} \simeq \mathbb{N}$), *eg* $p(k) = 2k$ and $q(k) = 2k + 1$
- $\pi_1 + \pi_2 = p\pi_1 p^* + q\pi_2 q^*$

  $(\pi_1 + \pi_2)(2k) = 2\pi_1(k)$ and $(\pi_1 + \pi_2)(2k + 1) = 2\pi_2(k) + 1$

NB Operators act now on infinite space

Relating the two schemes

A priori typed

$$\pi : A \multimap B = \begin{pmatrix} \pi^{A^\perp, A^\perp} & \pi^{A^\perp, B} \\ \pi^{B, A^\perp} & \pi^{B,B} \end{pmatrix}$$

A posteriori typed

$$p\pi^{A^\perp, A^\perp} p^* + q\pi^{A^\perp, B} p^* + q\pi^{B, A^\perp} p^* q\pi^{B,B} q^*$$

Adding exponentials: the a priori typed scheme

- $\mathcal{S}(!A) = \mathbb{N} \times \mathcal{S}(A)$, adding a copy index
- From $\pi : A \multimap B$ construct $!\pi : !A \multimap !B$:
  - $!\pi^{!A,!A}(k, a) = (k, \pi^{A,A}(a))$
  - $!\pi^{!A,!B}(k, a) = (k, \pi^{A,B}(a))$
  - $!\pi^{!B,!A}(k, b) = (k, \pi^{B,A}(b))$
  - $!\pi^{!B,!B}(k, b) = (k, \pi^{B,B}(b))$

Exponentials continued

$$
\begin{array}{rrcll}
d : & !A & \multimap & A & : d^* \\
d^{!A,A} : & (0, a) & \leftrightarrow & a & : d^{A,!A} \\
\\
c : & !A & \multimap & !A_1 \otimes !A_2 & : c^* \\
c^{!A,!A\otimes !A} : & (p(k), a) & \leftrightarrow & (k, a)_1 & : c^{!A\otimes !A,!A} \\
 & (q(k), a) & \leftrightarrow & \qquad (k, a)_2 & : c^{!A\otimes !A,!A} \\
\\
dig : & !A & \multimap & !!A & : dig^* \\
dig^{!A,!!A} : & (\tau(k, k'), a) & \leftrightarrow & (k, (k', a)) & : dig^{!!A,!A}
\end{array}
$$

where $\tau : \mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$

Exponentials: the a posteriori typed scheme

- Fixed space = $\mathbb{N}$ (or ℓ^2)
- Use $\langle ., . \rangle : \mathbb{N} \times \mathbb{N} \simeq \mathbb{N}$ for exponentials: $(k, a) \rightsquigarrow \langle k, a \rangle$
- Define $\pi \perp \pi'$ if $\pi\pi'$ nilpotent (*ie* computation is finite)
- $\mathcal{T}(!A) = \{!\pi, \pi \in \mathcal{T}(A)\}^{\perp\perp}$

Theorem

if $\Pi : A$ *then* $\pi \in \mathcal{T}(A)$

This is a strong normalisation theorem.
In the setting of Hilbert spaces one can alternatively define duality by means of *weak* nilpotency, allowing to account for non terminating terms *eg* fixed points.

The GoI equationnal theory

- Monoid with 0 generated by p, q (multiplicatives), d (dereliction), r, s (contraction), t (digging)
- Involution: $0^* = 0$, $1^* = 1$, $(uv)^* = v^* u^*$
- Morphism: $!(0) = 0$, $!(1) = 1$, $!(u)!(v) = !(uv)$, $!(u)^* = !(u^*)$
- Annihilation equations: $x^* y = \delta_{xy}$ (x, y generators)
- Commutation equations:
  - $!(u)d = du$
  - $!(u)x = x!(u)$ for $x = r, s$
  - $!(u)t = t!(!(u))$

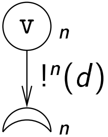
The theorem AB^*

- Orientate equations $\rightsquigarrow$ rewriting system
- Normal forms $=$ 0 or AB^*
- Inverse semigroup structure

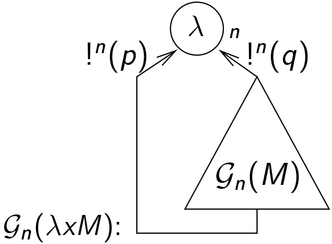
The path interpretation of GoI

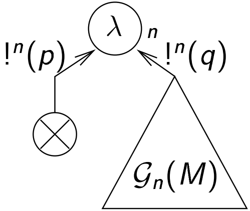
To a lambda-term M we associate a GoI weighted graph $\mathcal{G}_n(M)$:

- Variable case:

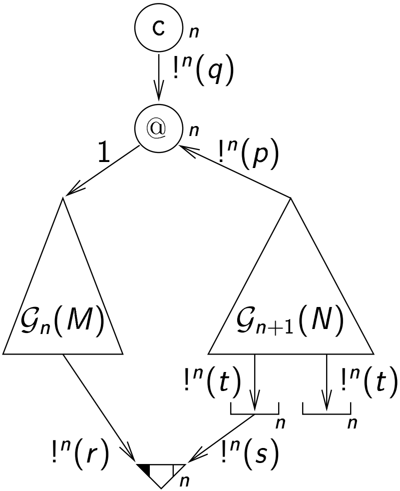


- Abstraction case:





- Application case: $\mathcal{G}_n(MN)$:



The path interpretation of GoI

Note: γ path in $\mathcal{G}_n(M)$, $w(\gamma)$ its *weight* is a GoI operator

Definition

Execution paths = invariant of beta-reduction = virtual redexes
Regular path = non null weight path ($w(\gamma) \neq 0$)

Theorem

γ is an execution path iff γ is regular

Theorem

If M is a term (thus an MELL proof) then

$$Ex(M) = \sum_{\gamma \in \mathcal{R}} w(\gamma)$$

where $\mathcal{R} = \{$regular paths $\in \mathcal{G}_0(M)\}$

Interaction Abstract Machine

Term (proof-net) $\rightsquigarrow$ weighted graph:

- token = element of S (the space in the a posteriori typed scheme)
- weighted edge = transition

$\rightsquigarrow$ Term (proof-net) = automaton: the IAM

Remark

All transitions are reversible and have disjoint domains and codomains $\rightsquigarrow$ the automaton is bideterministic

IAM

In order to make the abstract machine explicit, redefine the space S of tokens:

- token (state) $= (B, S)$ (really $B.S$):
  - B = *box stack* of *exponential signatures*
  - S = *balanced stack* of exponential signatures + multiplicative constants P and Q
  - exponential signature = binary tree with leaves in $\{\Box, R, S\}$
- Transitions = partial transformations on (B, S)

Theorem

$KAM \subset IAM$

Conclusion

A lot more to say

- GoI for additives
- Pointifixion: relating GoI/AJM games with HO
- Coherence problems: $\Pi \rightsquigarrow \Pi_0 \not\rightarrow \mathrm{Ex}(\pi) = \pi_0$
- GoI for other systems, *eg* interaction nets, π-calculus, differential nets
- ...