Timeless games

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Abstract. Two models of classical linear logic are set up. First our recent version of AJM games model which will be our source model. Then the target model, *polarized pointed relations*, a variant of the plain relational model which is constructed in two steps: first the model of pointed relations, then the additional polarization structure which yields a proper duality. Then the natural time-forgetting map is shown to generate a lax functor from the source to the target. Finally a further refinement of the target model using *bipolarities* is sketched, giving a closer link with the games model for the interpretation of syntax. Thus a bridge is constructed that goes from a dynamic model to a static model of evaluation.

1 Introduction

The basic mathematical reflex was to model types, programs, evaluation with sets, functions, composition. As years went by, denotational semantics pictured the syntactic triple with increasingly sophisticated tools: lattices, posets, concrete data structures, coherence spaces, hypercoherences, etc. The drive behind was to model more and more accurately. On the other hand, operational or natural semantics, which was the basic computer science reflex, was becoming more abstract. The drive here was to legibility, portability. There would obviously be great mutual benefit if both theories could trade freely. It seems now that games semantics is a middle ground where a general agreement could be found for this trade. Games semantics pictures the syntactic triple as games, strategies and plays. The point is that evaluation is now modeled as a communication, a composition that takes time. In a recent series of papers, [6, 5], the link between games models and abstract machines was explored. And the conclusion is that, indeed, suitable and especially elegant abstract machines can be used to effectively describe communication between strategies.

But to be a middle ground one needs to relate to both parties. In this paper we undertake the first general exploration of the link between games models and functional models. To map communication to composition should only be a matter of forgetting time. This paper tries to zero in on a precise statement of that obvious intuition.

Two models of classical linear logic are set up. First our recent version of AJM games model which will be our source model. Then the target model, *polarized pointed relations*, a variant of the plain relational model which is constructed in two steps: first the model of pointed relations, then the additional polarization structure. Finally the natural time-forgetting map is shown to generate a lax functor from the source to the target. At the very end of the paper we introduce a further refinement, *bipolarized* pointed relations. The game interpretation of syntax is shown to be sent onto the bipolarized intrepretation through the time-forgetting map.

There are of course other solutions one could conceive of by varying the choice of the models. Other kinds of target models including an additional coherence structure or a partial ordering reflecting the prefix-ordering of plays could come up with a better fit with games. Constraining the source model to memoryless strategies or another analogous restricted class could also be useful in tightening the link.

There are also other investigations [11] along the same lines which were already lead in the intuitionistic case. Specifically, Ehrhard's extensional collapse of sequential algorithms, which Lamarche [10] and Curien showed could be thought of as games model, to strongly stable functions in [7, 8]. To track down the relation between the present attempt and those former investigations should serve well the general purpose of bridging both kinds of models.

Let us recall that the category we described in [4] is a simple adaptation of AJM games category [2, 3] which models classical MELL for relevant reduction (reduction without weakening steps). This was obtained by relaxing conditions on strategies and more particularly by giving up determinism. One motivation underlying the conception of the present paper was to formally find a bound for this loss.

We feel that our games model of [4] is indeed a good starting point to investigate the link between games models and functional models mentioned before because it offers the advantages first of simplicity — morphisms are so-called *saturated strategies* which are easy to define —; secondly of a real duality — the 'Opponent moves first' limitation has been removed —; and finally of a general notion of composition of strategies.

The target model introduced with the time-forgetting map, called *pointed relations model*, is an unexpected variant of the relational model: we replace binary relations on sets by pointed relations on pointed sets. A pointed set is a set with a distinguished bottom element corresponding to the empty play; composition of relations must be modified in order to constrain degenerate composition (composition with an empty communication).

Interestingly enough, this model turns out to be less degenerate than the traditional relational one — for instance though we still have the same space for $A \otimes B$ and $A \parr B$, at the level of morphisms it is no more the case that $f \otimes g = f \parr g$.

Then, keeping the same notion of composition we add a polarity to the elements (corresponding to the author of the first move of the source play), which brings up a real non-degenerate model. One step further consists in adding a second polarity (corresponding to the author of the last move of the source play), which tightens the link with the game model.

A good way to sum up the difference between the pointed relations model and the games model is to say that the former deals with composition of *positions* (states) and the latter with composition of *plays* (successions of positions).

Finally it seems that the time-forgetting map applied to strategies doesn't lose as much information as could have been expected at first sight: this is due to the presence of the bottom element which maintains on the set of positions a part of the prefix-ordering from the set of plays. The general problem of which strategies are identified through this map is still open.

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2 Preliminaries on games

Let us recall the usual definitions on games (introduced in [1, 2]) together with particular notions and results of [4] that will be needed in the sequel.

Games are between two agents (players): *Player* P and *Opponent* O. A *game* A will consist in the following data:

- a set of *moves* M_A;
- a map λ_A from M_A to $\{O,P\}$ associating to each move the player it belongs to;
- a prefix-closed subset P_A of *plays*, that is finite alternating sequences of moves;
- an equivalence relation $\equiv_A$ on plays satisfying the following conditions:
  - if $s \equiv_A s'$ then s and s' should have the same length and if they are not empty start with a move by the same player.
  - prefix-closure: if $s \equiv_A s'$, then t, t' being prefixes respectively of s and s' of the same length, we have: $t \equiv_A t'$.
  - extendability: if two plays sa, s' satisfy $s \equiv_A s'$ then there exists a move a' such that $s'a'$ is a valid play and $sa \equiv_A s'a'$.

  As an example the equivalence on atomic games might simply be the identity; these properties are then preserved by the further constructions.

2.1 Constructions on games

Let us review the three basic AJM constructions one needs to build MELL games.

Orthogonal. Given a game A, the game $A^\perp$ is defined as:

- M_A;
- $\bar\lambda_A$, with $\bar\lambda(m) = P$ if $\lambda(m) = O$ and $\bar\lambda(m) = O$ if $\lambda(m) = P$;
- P_A;
- $\equiv_A$.

Linear negation is here seen as the exchange of players. This operation is clearly involutive.

Times. Given A and B, $A \otimes B$ is defined as:

- $M_A + M_B$;
- $\lambda_A + \lambda_B$;
- $s \in P_{A \otimes B}$ if $s_A \in P_A$, $s_B \in P_B$ and O only switches components in s;
- $s \equiv s'$ if for all n, s and s' n-th moves are in the same component, and both $s_A \equiv s'_A$ and $s_B \equiv s'_B$.

Above s_A stands for the projection of s on A, that is the subsequence of s where only moves in A are kept. Same for s_B. Subgames A and B are often referred to as *components* of $A \otimes B$.

Bang. Given A, $!A$ is defined as:

- $\omega \times M_A$;
- $\lambda_{!A}(im) = \lambda_A(m)$;
- $s \in P_{!A}$ if $s_i \in P_A$ and O only may switch components (s_i stands for the projection of s where we only keep moves m when im is in s);
- $s \equiv s'$ if there is a permutation θ on ω such that for all n, s n-th move is in the $\theta(i)$ component iff s' n-th move is in the i component, and for all i, $s_{\theta(i)} \equiv_A s'_i$.

Given a MELL formula $F[X_1, \ldots, X_n]$ and games $A_1, \ldots, A_n$ one can build a game $F[A_1, \ldots, A_n]$ following the constructions above and:

$$\begin{aligned} A \parr B &= (A^\perp \otimes B^\perp)^\perp, \\ ?A &= (!A^\perp)^\perp. \end{aligned}$$

If Γ is a MELL sequent $A_1, \ldots, A_n$, then the associated game is by convention $A_1 \parr \ldots \parr A_n$. If Δ is a subsequent of Γ and $s \in P_\Gamma$, then s_Δ or s/Δ, the *projection* of s on Δ, means the subsequence of s with moves in M_Δ. Hereafter we simply write $s \in \Gamma$ in place of $s \in P_\Gamma$.

2.2 Strategies and their composition

A *strategy* on A will be a non-empty prefix-closed set of plays in P_A. By convention, we ask a P-strategy to be closed by O-moves, and dually, an O-strategy to be closed by P-moves. A P-strategy will simply be termed a strategy in the sequel.

Recall that the identity strategy, denoted by *id*, is the Player *copy-cat* strategy, that is the prefix-closure of the set: $\{s \in A \parr A^\perp : s_A = s_{A^\perp}\}$.

Composition of plays is defined using the following lemma:

Lemma 1 (Zipping) *Let s_1 and s_2 respectively be in Γ, A and $A^\perp, \Delta$ and be such that $s_{1A} = s_{2A^\perp} \neq \epsilon$. There is a* unique *$s \in \Gamma, A \otimes A^\perp, \Delta$ such that:*

- *$s_{\Gamma,A} = s_1$, $s_{A^\perp,\Delta} = s_2$,*
- *P doesn't switch between Γ and Δ and*

– $s_{A\otimes A^\perp}$ *is played by id on the O-side (Opponent follows the copy-cat strategy).*

We define $s_1;s_2$ *by* $s_1;s_2 = s_{\Gamma,\Delta}$.

Let us give a sketch of proof for the existential part of the statement (a complete proof may be found in [4]).

As $s_{1A} = s_{2A^\perp} \neq \epsilon$ one of these two projections starts with a P-move. Let us assume it is $s_{2A^\perp}$.

We consider in s_1 (resp. s_2) the subsequences of consecutive moves in A (resp. $A^\perp$) which we call *interfaces* of s_1 (resp. s_2). Therefore s_1 (resp. s_2) is the alternated concatenation of interfaces and sequences of consecutive moves in Γ (resp. Δ), which we denote by $s_1^1, \ldots, s_1^n$ (resp. $s_2^1, \ldots, s_2^m$). Note that, since the first move in s_{1A} is an O-move, by the switching conditions, it must be that the first interface of s_1 is s_1^1.

We now build s a valid play satisfying the expected properties by 'sewing' s_1 and s_2 by their interfaces (see figure 1).

As $s_{1A} = s_{2A^\perp}$ the first interfaces of s_1 and s_2 are in prefix-ordering relation: $s_1^1 \leq s_2^2$ or $s_2^2 \leq s_1^1$. Say for instance $s_1^1 \leq s_2^2$, and let us denote $s_1^1 = m_1 \ldots m_k$.

Now let us denote by s_1' the sequence $s_1 \backslash s_1^1$ (s_1 without its prefix s_1^1) and s_2' the sequence $s_2 \backslash s_2^1 s_1^1$. There are two cases.

– The last move in s_{1A} is the last move of s_1^1 (thus also the last move of s_2^2) and we have $s_1^1 = s_2^2$. If it is a P-move in s_1 then by the switching condition $s_1' = \epsilon$, $s_2' = s_2^3$ and we set $s = s_2^1 m_1 m_1 \ldots m_k m_k s_2^3$ where each m_i is alternatively played by Player in A (resp. $A^\perp$) then by Opponent in $A^\perp$ (resp. A) so that Opponent indeed follows the copy-cat strategy in $A \otimes A^\perp$. If it is a P-move in s_2 then similarly, $s_1' = s_1^2$, $s_2' = \epsilon$ and we set $s = s_2^1 m_1 m_1 \ldots m_k m_k s_1^2$.
– Otherwise it is clear that s_1' and s_2' satisfy similar hypotheses to s_1 and s_2 so that by induction we can build an s'. Now by concatenating $s_2^1 m_1 m_1 m_2 m_2 \ldots m_k m_k$ and s' we obtain the desired s. □

Let s_1 and s_2 be in Γ, A and $A^\perp, \Delta$ respectively, we extend the notation introduced in the lemma, to the *passive composition* case: $s_1;\epsilon = s_1$ if $s_{1A} = \epsilon$ and $\epsilon; s_2 = s_2$ if $s_{2A^\perp} = \epsilon$.

Definition 1 *If* σ *is a strategy in* Γ, A *and* τ *a strategy in* $A^\perp, \Delta$ *we define:*

$$\sigma;\tau = \{s_1;s_2, \quad s_1 \in \sigma,\ s_2 \in \tau,\ \textit{and } s_1;s_2 \textit{ is defined } \}$$

This gives a games/strategies category as was shown in [4].

Definition 2 *Set:*

$$[\sigma] = \{s,\ s.t.\ \exists s' \in \sigma, \quad s \equiv s'\}.$$

Say a strategy is saturated *if* $[\sigma] = \sigma$.

The extendability and prefix-closure properties of the equivalence relation are needed to guarantee that $[\sigma]$ is still prefix-closed and closed by valid O-moves (resp. P-moves if σ is an O-strategy).

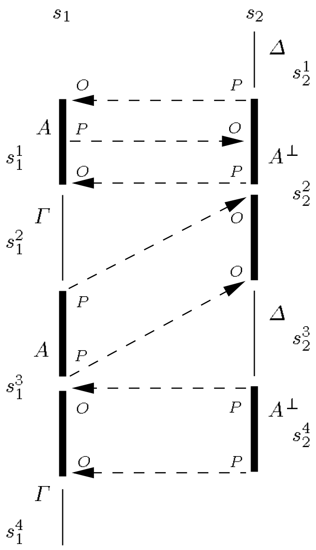

Fig. 1: Sewing s_1 and s_2

The saturation property is preserved by composition but the category of saturated strategies denoted by $\mathbb{G}$ is not exactly a subcategory of the raw category, because it does not have the same units.

Let us finally recall the strategies used in [4] to interpret MELL proof-nets:

- the *identity* (P)-strategy $id_A : A^\perp, A$ has already been given; an axiom is interpreted by $[id]$.
- interpretation of *multiplicatives*: let us just define the $\otimes$-functor. Given σ_i in Γ_i, A_i, $i = 1, 2$, one defines:

$$\begin{aligned}\sigma_1 \otimes \sigma_2 \quad = \quad &\{s \in \Gamma_1, \Gamma_2, A_1 \otimes A_2, \text{ s.t. } s/\Gamma_i, A_i \in \sigma_i, \text{ and} \\ &\quad \text{O switches between } \Gamma_1, A_1 \text{ and } \Gamma_2, A_2\}.\end{aligned}$$

  Note that the restriction that O switches between the Γ_i, A_i's is useless in the intuitionistic case.
- *dereliction*: $[d]$ is the saturation of the strategy d in $?A^\perp, A$ generated by the following partial injection:
  $d(m) = 0m$ for $m \in M_A^O$, and inversely
  $d(0m) = m$ for $m \in M_A^P$.
- *contraction*: take $l + r$ a bijection from $\omega + \omega$ to ω, then $[c]$ is the saturation of the strategy in $?A^\perp, !A \otimes !A$ generated by the following partial injection:
  $c(im) = l(i)m$ for $m \in M_A^O$ and im played in the left component of $!A \otimes !A$,
  $c(im) = r(i)m$ for $m \in M_A^O$ and im played in the right component of $!A \otimes !A$.
  Inversely, when $m \in M_A^P$ we have for any i, $c(l(i)m) = im$ played in the left component of $!A \otimes !A$ and $c(r(i)m) = im$ played in the right component of $!A \otimes !A$.

– *weakening*: $[w]$ only contains the empty play in $?A^{\perp}$.
– *promotion*: take '$\lceil,\rceil$' a bijection from $\omega \times \omega$ to ω, and let $s \in ?\Gamma, !A$. We extend our notation pertaining to plays in a !/? game by defining s_i to be obtained by, first, selecting the subsequence of s consisting of moves im in $M_{!A}$ and moves $\lceil i,j \rceil m$ in $M_{?\Gamma}$ for any $j \in \omega$, and second, replacing im's by m's and $\lceil i,j \rceil m$'s by jm's.
  Intuitively $\sigma^!$ records the current s_i where the last O-move was taking place, if any, and plays in it as σ would have; it is only O that changes from one copy, that is from one s_i, to another.
  For any i, the one move play im in $!A$ is in $\sigma^!$ if the one move play m in A is in σ. Similarly, the one move play $\lceil i,j \rceil m$ in $?\Gamma$ is in $\sigma^!$ if σ plays jm in $?\Gamma$.
  Suppose $s \in \sigma^!$ ends with a P-move and im is an O-move in $!A$. Then $\sigma^!$ extends it by a P-move according to the following rules:

$$\frac{s_i, m, m' \in \sigma \quad m, m' \in M_A}{s, im, im' \in \sigma^!} \qquad \frac{s_i, m, jm' \in \sigma \quad m \in M_A, m' \in M_\Gamma}{s, im, \lceil i,j \rceil m' \in \sigma^!}$$

  Suppose $s \in \sigma^!$ ends with a P-move and $\lceil i,j \rceil m$ is an O-move in $?\Gamma$. Then $\sigma^!$ extends it with a P-move according to the following rules:

$$\frac{s_i, jm, j'm' \in \sigma \quad m, m' \in M_\Gamma}{s, \lceil i,j \rceil m, \lceil i,j' \rceil m' \in \sigma^!} \qquad \frac{s_i, jm, m' \in \sigma \quad m \in M_\Gamma, m' \in M_A}{s, \lceil i,j \rceil m, im' \in \sigma^!}$$

  Now, promotion on σ is interpreted by $[\sigma^!]$.

2.3 Relation to the AJM model

Consider the intuitionistic version $\mathbb{IG}$ of the saturated strategies model:

– Plays are asked to satisfy the following extra condition: their first move must be an O-move.
– Games are defined as before up to this notion of play.
– Constructions on games include: $\otimes$, $\multimap$, !. Actually the switching condition for these connectives is a consequence of alternacy, projecting and 'O starts' conditions (see [3]).
– Strategies, composition and saturated strategies are defined as before.

Notice that the identity morphism $id_A : A \multimap A$ of $\mathbb{IG}$ is not the same as the one of the former category since all its non-empty plays are required to start with an O-move in the right-hand side component.

Proposition 2 *$\mathbb{IG}$ is a model of IMELL (intuitionistic multiplicative linear logic).*

$\mathcal{G}$ will stand for the AJM model of IMELL without the question/answer features which are not needed here ([2,3]). Let us recall that in this setting:

– Games are these specified above for $\mathbb{IG}$.

- Strategies are prefix-closed sets of valid plays required to be *history-free*: a being an O-move, ($sab \in \sigma$, $ta \in \sigma$) implies that tab is a valid play and $tab \in \sigma$; furthermore if $tac \in \sigma$ then $c = b$. The *function of strategy* associated to σ will be the partial function $f_\sigma : M_A^O \to M_A^P$ defined by: $f_\sigma(a) = b$ iff there exists a play sab in σ.
- A partial equivalence $\sim$ is defined on the set of strategies on A by: $\sigma \sim \sigma'$ iff we both have: ($sab \in \sigma$, $\lambda_A(a) = O$, $sa' \in \sigma'$, $sa \equiv s'a'$) implies that there exists b' such that $s'a'b' \in \sigma'$ and $s'a'b' \equiv sab$; and the symmetrical condition.
- $\mathcal{G}[A, B]$ is the set of partial equivalence classes $\langle\sigma\rangle$ for strategies σ on $A \multimap B$ satisfying $\sigma \sim \sigma$.
- Composition of strategies coincides with that described in section 2.2 because of the 'opponent moves first' condition on plays.

In $\mathcal{G}$ and $\mathbb{IG}$ we consider basic games defined by the set of plays $\{\epsilon,\ *,\ *a\}$ where $*$ (resp. a) is an O-move (resp. P-move), and the identity as equivalence relation on this set.

Given IMELL formulas A and B, we define an application $F_{A,B}$ (abbreviated to F):

$$\begin{array}{ccc} \mathcal{G}[A,B] & \to & \mathbb{IG}[A,B] \\ \langle\sigma\rangle & \to & [\sigma] \end{array}$$

It is well-defined since if $\sigma \sim \sigma'$ we have $[\sigma] = [\sigma']$.

Lemma 3 *If σ is a (history-free and self-equivalent) strategy on $A \multimap B$ of the model $\mathcal{G}$ we have:*

$$F(\langle\sigma\rangle) = \bigcup \langle\sigma\rangle$$

Proof:

Let us start with the right-to-left inclusion. If s is an element of $\bigcup\langle\sigma\rangle$ there exists $\sigma' \sim \sigma$ such that s' belongs to σ'. By induction on the length of s, using the extendability property of $\equiv$ and the definition of $\sigma' \sim \sigma$ we get:

$$\exists t \in \sigma /\ t \equiv s.$$

So s belongs to $[\sigma]$.

As to the converse we need to introduce an alternative way to handle equivalence between plays: we associate to each game A a set of bijections referred to as *automorphisms* of A and defined by:

- on a basic game id_{M_A} is the only automorphism.
- on a multiplicative game $A_1 m A_2$, an automorphism ϕ is given by a pair of automorphisms ϕ_i on the A_i's: $\phi = [\phi_1, \phi_2]$.
- on an exponential game $\Box A$, an automorphism is given by a permutation θ on ω and a family $(\phi_i)_{i\in\omega}$ of automorphisms on A: $\phi(im) = \theta(i)\phi_i(m)$.

ϕ being an automorphism on A let ϕ^* denote the application on sequences of moves of A defined by: $\phi^*(m_1 ... m_k) = \phi(m_1)...\phi(m_k)$. Then ϕ^* preserves validity and $s \equiv t$ can be rephrased as: there exists an automorphism ϕ on A such that $t = \phi^*(s)$.

Now let s be a play in $[\sigma]$: $\exists t \in \sigma / \ s \equiv t$. Let ϕ be an automorphism on A such that $t = \phi^*(s)$. Then it can be checked that the partial function $f = \phi \circ f_\sigma \circ \phi^{-1}$ is the function of a valid history-free strategy σ' and that $\sigma' \sim \sigma$. Since t belongs to σ' we get $t \in \bigcup \langle \sigma \rangle$. $\square$

Proposition 4 *F is a functor from $\mathcal{G}$ to $\mathbb{IG}$.*

Notice that F is not full since $\mathbb{IG}[A, B]$ contains saturation of non history-free strategies and of history-free but not self-equivalent strategies.

3 Pointed relations

We consider *pointed sets* A with one distinguished element $\perp_A$. In the sequel when it is not ambiguous we will write simply $\perp$ for $\perp_A$. The intuition behind this element is that it is meant to interpret empty plays.

A *pointed relation* $f : A \to B$ is a subset of $A \times B$ containing $(\perp_A, \perp_B)$.

Composition of elements: given elements of $A_1 \times A_2$ and $A_2 \times A_3$ we define their composition by:

$$\begin{array}{rcll} (x_1, x_2); (x_2, x_3) & = & (x_1, x_3) & \quad \text{when } x_2 \neq \perp \\ (x_1, \perp_{A_2}); (\perp_{A_2}, \perp_{A_3}) & = & (x_1, \perp_{A_3}) & \\ (\perp_{A_1}, \perp_{A_2}); (\perp_{A_2}, x_3) & = & (\perp_{A_1}, x_3) & \end{array}$$

where the x_i's are any elements of the A_i's.

The last two compositions are called *passive* and the first one *active*, as for the composition of plays.

Composition of pointed relations: f and g being respectively pointed relations on $A_1 \to A_2$ and $A_2 \to A_3$, define:

$f; g = \{(x, y); (y, z), \text{ s.t. } (x, y) \in f, \ (y, z) \in g \text{ and this composition is defined } \}.$

This composition is associative and has identity morphisms given by:

$$id_A = \{(x, x), x \in A\}.$$

When checking associativity the only delicate case is:

$$(x_1, \perp); ((\perp, x_2); (x_2, \perp)) \in f; (g; h)$$

where $x_1, x_2 \neq \perp$.

The composition evaluates to $(x_1, \perp)$. Note that since $x_2 \neq \perp$ the composition $((x_1, \perp); (\perp, x_2)); (x_2, \perp)$ is not defined. Now as we consider *pointed* relations we have $(\perp, \perp) \in g$, $(\perp, \perp) \in h$ and so:

$$(x_1, \perp) = ((x_1, \perp); (\perp, \perp)); (\perp, \perp) \in (f; g); h.$$

so that we can conclude.

So we get a category $\mathbb{PR}$ of pointed sets/pointed relations.

3.1 Monoidal structure

The tensor bifunctor. Let A and B be pointed sets. Then $A \otimes B$ is $A \times B$ and $\perp_{A \otimes B} = (\perp_A, \perp_B)$. For $i = 1, 2$, if $f_i : A_i \to B_i$, then $f_1 \otimes f_2 : A_1 \otimes A_2 \to B_1 \otimes B_2$ is given by:

$$\begin{aligned} f_1 \otimes f_2 \quad = \quad & \{((x_1, x_2), (y_1, y_2)), \text{ s.t. } (x_i, y_i) \in f_i \text{ and } y_1, y_2 \neq \perp\} \\ & \cup \{((x_1, \perp), (y_1, \perp)), \text{ s.t. } (x_1, y_1) \in f_1\} \\ & \cup \{((\perp, x_2), (\perp, y_2)), \text{ s.t. } (x_2, y_2) \in f_2\}. \end{aligned}$$

Remark. An alternative definition for $f_1 \otimes f_2$ is that $((x_1, x_2), (y_1, y_2))$ belongs to $f_1 \otimes f_2$ iff (x_i, y_i) belongs to f_i and when $y_i = \perp$, then either $x_i = \perp$, or x_{i+1} and y_{i+1} are $\perp$ ($i+1$ must be understood modulo 2). This condition will be referred to as the *connectedness* condition.

Lemma 5 *Tensor is a bifunctor and has as unit: $I = \{\perp_I\}$.*

Let us only check that for $f_i : A_i \to B_i$ and $g_i : B_i \to C_i$ we do have:

$$(f_1; g_1) \otimes (f_2; g_2) \subseteq (f_1 \otimes f_2); (g_1 \otimes g_2).$$

Let $((a_1, a_2), (c_1, c_2)) \in (f_1; g_1) \otimes (f_2; g_2)$. There exist b_i's such that:

$$(a_i, b_i); (b_i, c_i) = (a_i, c_i) \in f_i; g_i.$$

- Assume that $c_1 \neq \perp$ and $c_2 \neq \perp$. Then we know that $((b_1, b_2), (c_1, c_2)) \in (g_1 \otimes g_2)$. Do we have $((a_1, a_2), (b_1, b_2)) \in (f_1 \otimes f_2)$? Suppose $b_i = \perp$, then by definition of composition as $(a_i, c_i) \in (f_i; g_i)$ and $c_i \neq \perp$ we have $a_i = \perp$. So $((a_1, a_2), (b_1, b_2)) \in (f_1 \otimes f_2)$ and $(b_1, b_2) = (\perp, \perp)$ implies $(a_1, a_2) = (\perp, \perp)$. So $((a_1, a_2), (c_1, c_2)) \in (f_1 \otimes f_2); (g_1 \otimes g_2)$.
- Assume c_1 is $\perp$. Then by definition of the tensor of morphisms (connectedness condition): $a_1 = \perp$ or ($a_2 = \perp$ and $c_2 = \perp$). Suppose for instance $a_1 = \perp$. Using the fact that the relations contain $(\perp, \perp)$ we have: $((\perp, b_2), (\perp, c_2)) \in (g_1 \otimes g_2)$ and $((\perp, a_2), (\perp, b_2)) \in (f_1 \otimes f_2)$. Moreover their composition is possible since if $b_2 = \perp$ then we must have: $a_2 = \perp$ or $c_2 = \perp$. So finally: $((a_1, a_2), (c_1, c_2)) \in (f_1 \otimes f_2); (g_1 \otimes g_2)$.

Proposition 6 *The category $\mathbb{PR}$ is $*$-autonomous.*

The symmetric monoidal structure is easily checked. As to the closedness, given the pointed set B, let T_B denote the functor given by:

- $T_B(A) = A \otimes B$;
- for $f \in \mathbb{PR}[A, A']$, $T_B(f) = f \otimes id_B$.

Then T_B has a right adjoint given by:

- $F_B(A) = B \multimap A$, given by $B \times A$ and $\perp_{B \multimap A} = (\perp_B, \perp_A)$.
- for $f \in \mathbb{PR}[A, A']$:

$$\begin{array}{rcl} F_B(f) & = & \{((a,b),(a',b)), \text{ s.t. } (a,a') \in f, a \neq \perp, b \neq \perp\} \\ & & \cup \{((\perp,\perp),(a',\perp)), \text{ s.t. } (\perp,a') \in f\} \\ & & \cup \{((\perp,b),(\perp,b)), \text{ s.t. } b \in B\} \end{array}$$

The application $\alpha : \mathbb{PR}[T_B(A), C] \longrightarrow \mathbb{PR}[A, F_B(C)]$ given by:

$$\alpha(f) = \{(x,(y,z)), \text{ s.t. } ((x,y),z) \in f\}$$

is well-defined and satisfies the expected naturality condition w.r.t. A and C.

Considering as dualizing object the set reduced to one element: $D = \{\perp\}$ we get $A \simeq ((A \multimap D) \multimap D)$ and the category obtained is $*$-autonomous.

Now $\parr$ can be defined by:

- $A \parr B = (A^\perp \otimes B^\perp)^\perp$
- if $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$ then $f_1 \parr f_2 : A_1 \parr A_2 \to B_1 \parr B_2$ is:

$$f_1 \parr f_2 = {}^t({}^t f_1 \otimes {}^t f_2),$$

where ${}^t f_i : B_i^\perp \to A_i^\perp$ is obtained from f_i in the expected way.

Notice that up to isomorphism, $A \parr B = A \otimes B$ but:

$$\begin{array}{rcl} f_1 \parr f_2 & = & \{((x_1,x_2),(y_1,y_2)), \text{ s.t. } (x_i,y_i) \in f_i \text{ and } x_1, x_2 \neq \perp\} \\ & & \cup \{((x_1,\perp),(y_1,\perp)), \text{ s.t. } (x_1,y_1) \in f_1\} \\ & & \cup \{((\perp,x_2),(\perp,y_2)), \text{ s.t. } (x_2,y_2) \in f_2\} \\ & \neq & f_1 \otimes f_2. \end{array}$$

Thus the isomorphism between $A \parr B$ and $A \otimes B$ is not natural.

3.2 Exponential structure

Given a pointed set A, the set $!A$ is the set of multisets of elements of $A \backslash \{\perp_A\}$ and we define $\perp_{!A} = [\,]$.

We interpret the exponential rules in the following way:

– *promotion*: if $f : !\Gamma \to A$ then $f^! : !\Gamma \to !A$ is defined by:

$$\begin{array}{rcl} f^! & = & \{(u_1 + \cdots + u_n, [x_1, \ldots, x_n]), \text{ s.t. } (u_i, x_i) \in f \text{ and } x_i \neq \bot_A\} \\ & & \cup \{(u, []), \text{ s.t. } (u, \bot_A) \in f\}. \end{array}$$

so that $f^!$ is a kind of union of all n-ary tensors $f \otimes f \ldots \otimes f$, with some contraction (see below) performed on the context part.

– *dereliction*: $der_A : !A \to A$ is defined by

$$der_A = \{([x], x), \text{ s.t. } x \neq \bot_A\} \cup \{([], \bot_A)\}.$$

– *contraction*: $cont_A : !A \to (!A \otimes !A)$ is defined by:

$$cont_A = \{(u + v, (u, v)), \text{ s.t. } u, v \in !A\}.$$

– *weakening*: $weak_A : !A \to I$ is given by:

$$weak_A = \{([], \bot_I)\}.$$

Proposition 7 *The category $\mathbb{PR}$ is a model of MELL for relevant reduction (reduction without weakening steps).*

Proof:

That is that $\mathbb{PR}$ satisfies the following equations:

$$\begin{array}{rcl} f^!; der_A & = & f \\ f^!; cont_A & = & cont_\Gamma; (f^! \otimes f^!) \\ f^!; g^! & = & (f^!; g)^! \end{array}$$

for any $f : !\Gamma \to A$ and $g : !A, !\Delta \to B$

We only deal here with the case of the contraction reduction step, which should though give the main ideas for the other cases (dereliction and commutative cut reduction steps). Let us consider the case $f : !A \to B$. First see that: $f^!; cont_B \subseteq cont_A; (f^! \otimes f^!)$.

Let $(u, (v, w))$ belong to $f^!; cont_B$; then $(u, v + w) \in f^!$. This implies that u can be split as $u = u' + u''$ so that: $(u', v) \in f^!$, $(u'', w) \in f^!$. Moreover: if v and w are not both $[]$ then if v (resp. w) is $[]$ then u'(resp. u'') can be taken $[]$; if u and v are both $[]$ then u' or u'' can be taken $[]$. This guarantees that $((u', u''), (v, w)) \in (f^! \otimes f^!)$ and so $((u' + u''), (v, w)) \in cont_A; (f^! \otimes f^!)$.

The only problem for the reverse inclusion comes from the case where we consider $((u, u'), (v, v'))$ in $(f^! \otimes f^!)$ such that for instance $v = []$. Yet, by the connectedness condition we also know that then either $u = []$ or $u' = v' = []$, which implies $(u + u', v + v') \in f^!$. □

Remark. Due to the way we interpret the promotion rule (more precisely because of the subset: $\{(u, []), \text{ s.t. } (u, \bot_A) \in f\}$ of $f^!$), the weakening reduction step is not valid in this semantics. This is part of the price we pay to get proposition 19 and theorem 23 relating the game interpretation of syntax with the $\mathbb{PPR}$ and $\mathbb{BPR}$ interpretations. Indeed, these results rely in particular on the fact that $\widehat{[\sigma^!]} \subseteq \widehat{[\sigma]}^!$.

4 Polarized pointed relations

Let $Pol = \{O, P, N\}$ be the set of polarities. A *polarized pointed set* is a pointed set A together with a polarity function λ_A from A to Pol such that only $\perp_A$ has polarity N. Intuitively polarities correspond in the pointed relations setting to the author of the first move in the games setting.

Morphisms. A subset f of $A_1 \times A_2$ is a *pointed polarized relation* if f is a pointed relation and f satisfies the *polarity preservation* property: for any (x_1, x_2) in f,

$$\lambda_{A_1}(x_1) = P \text{ implies } \lambda_{A_2}(x_2) = P \text{ or } N.$$

Composition of points and of pointed polarized relations are defined as in $\mathbb{PR}$. Let us check they are compatible with the preservation property. Suppose (x_1, x_3) belongs to $f; g$ with f, g respectively elements of $\mathbb{PPR}[A_1, A_2]$, $\mathbb{PPR}[A_2, A_3]$. There exists x_2 such that: $(x_1, x_3) = (x_1, x_2); (x_2, x_3)$. Assuming $\lambda_{A_1}(x_1) = P$ we get that:

- either $\lambda_{A_2}(x_2) = N$, which means that $x_2 = \perp_{A_2}$ and by the passive composition restriction it follows that $x_3 = \perp_{A_3}$
- or $\lambda_{A_2}(x_2) = P$ and then $\lambda_{A_3}(x_3) = P$ or $\lambda_{A_3}(x_3) = N$.

Associativity and existence of identities are shown as for $\mathbb{PR}$, so that we get a category of pointed polarized sets and relations, denoted by $\mathbb{PPR}$.

4.1 Monoidal structure

The tensor bifunctor. We define $A_1 \otimes A_2$ to be the set of couples (x_1, x_2) that don't have polarity (P, P) and we set $\perp_{A_1 \otimes A_2} = (\perp_{A_1}, \perp_{A_2})$. For any $(x_1, x_2) \in A_1 \otimes A_2$, the polarity $\lambda_{A_1 \otimes A_2}(x_1, x_2)$ is given by the following table:

$x_1 \backslash x_2$	O	P	N
O	O	P	O
P	P		P
N	O	P	N

We shall denote by $\lambda_\otimes$ the partial function from $Pol^2 \to Pol$ defined by the table above.

Lemma 8 *If p_1, p_2, p_3 are polarities then we have:*
$\lambda_\otimes(p_1, \lambda_\otimes(p_2, p_3)) = \lambda_\otimes(\lambda_\otimes(p_1, p_2), p_3)$.

Let us denote $\lambda_\otimes(p_1, \ldots, p_n) = \lambda_\otimes(\ldots \lambda_\otimes(p_1, p_2), \ldots, p_n)$.

For $i = 1, 2$, let $f_i : A_i \to B_i$. We define $f_1 \otimes f_2 : A_1 \otimes A_2 \to B_1 \otimes B_2$ by:

$$\begin{aligned} f_1 \otimes f_2 \quad = \quad & \{((x_1, x_2), (y_1, y_2)), \text{ s.t. } (x_i, y_i) \in f_i, \quad y_1, y_2 \neq \perp \\ & \quad \text{and } (y_1, y_2) \in B_1 \otimes B_2\} \\ & \cup \{((x_1, \perp), (y_1, \perp)), \text{ s.t. } (x_1, y_1) \in f_1\} \\ & \cup \{((\perp, x_2), (\perp, y_2)), \text{ s.t. } (x_2, y_2) \in f_2\}. \end{aligned}$$

Note that the definition is the same than in the $\mathbb{PR}$ case but for the simple new constraint that in the case where none of y_1, y_2 are $\perp$, we ask that $(y_1, y_2) \in B_1 \otimes B_2$.

Let us verify that this is well-defined: only the first subset requires a check; we want that $(x_1, x_2) \in A_1 \otimes A_2$ and that $(\lambda_{(A_1 \otimes A_2)}(x_1, x_2), \lambda_{(B_1 \otimes B_2)}(y_1, y_2)) \neq (P, O)$.

Suppose that (x_1, x_2) is (P, P), then as $(x_i, y_i) \in f$ and $y_i \neq \perp$ we know that y_i is P for $i = 1, 2$, which contradicts $(y_1, y_2) \in B_1 \otimes B_2$. Furthermore if (x_1, x_2) has polarity P then x_1 or x_2 is P, so as $y_1 \neq \perp$ and $y_2 \neq \perp$ and by definition of morphisms, y_1 or y_2 is P, which gives that (y_1, y_2) is not O.

Proposition 9 *$\otimes$ is a bifunctor on this category.*

The proof of the equality $(f \otimes g); (f' \otimes g') = (f; g) \otimes (f'; g')$ is similar to the one of the previous section and follows from the definitions of composition and of the tensor of morphisms. $(id_A \otimes id_B) = id_{A \otimes B}$ is obvious.

Closing and duality Let $\bar{\cdot}$ be defined on *Pol* by: $\bar{O} = P$, $\bar{P} = O$, $\bar{N} = N$.

Given a $\mathbb{PPR}$ object A, its dual $A^\perp$ is defined as the same pointed set endowed with the polarity function: $\lambda_{A^\perp}(x) = \bar{\lambda}_A(x)$.

We define $A \multimap B$ as $(A \otimes B^\perp)^\perp$, which makes the category monoidal closed, and the dualizing object $D = \{\perp\}$ gives us:

Proposition 10 *The category $\mathbb{PPR}$ is $*$-autonomous.*

4.2 Exponential structure

Intuitively $!A$ is a generalized tensor on A. Specifically, given a polarized set A, the object $!A$ is the set of multisets $[x_1, \ldots, x_n]$ of elements of $A \backslash \{\perp_A\}$, such that at most one x_i has polarity P. We set $\perp_{!A} = []$ and $\lambda_{!A}([x_1, \cdots, x_n]) = \lambda_\otimes(\lambda_A(x_1), \cdots, \lambda_A(x_n))$

An easy consequence of this definition is:

Lemma 11 *If $u_1, \cdots, u_n \in !A$ then $(u_1 + \cdots + u_n) \in !A$ iff at most one u_i is of polarity P.*

The polarity of $(u_1 + \cdots + u_n)$ is then N (resp. O) if all the u_i's are N (resp. O), and P if at least one u_i is P.

Interpretation of exponentials is done in the same way as in the pointed semantics except that in the case of promotion, as for the tensor, we ask that $[x_1, \cdots, x_n]$ belongs to $!A$; similarly, in the contraction, (u, v) is asked to belong to $(!A \otimes !A)$.

Are these definitions valid? Let us check it for instance for the promotion: do we have $(u_1 + \cdots + u_n) \in !\Gamma$ and $(\lambda_{!\Gamma}(u_1 + \cdots + u_n), \lambda_{!A}([x_1, \cdots, x_n])) \neq (P, O)$? For the sake of simplicity, we check it in the case where Γ is reduced to one formula B.

Consider the case where (u_i, x_i) belongs to f and $[x_1, \cdots, x_n]$ belongs to $!A$. Suppose u_j and u_k are both of polarity P, then as (u_i, x_i) belongs to f we know that x_j and x_k are of polarity P. As $[x_1, \cdots, x_n]$ belongs to $!A$ this implies that $j = k$. So by lemma 11 we can induce from this that $(u_1 + \cdots + u_n) \in !B$.

Suppose now that $(u_1 + \cdots + u_n)$ is P. By lemma 11 exactly one u_i is P. So as (u_i, x_i) belongs to f, x_i is P and then $[x_1, \cdots, x_n]$ is P so that we are done.

Proposition 12 *The category $\mathbb{PPR}$ is a model of MELL for relevant reduction.*

Proof: The following lemma will be useful:

Lemma 13 *Let $f^! : !\Gamma \to !A$ and for $i = 1,\dots, n$, let $(u_i, v_i) \in f^!$. If no v_i is N and if at most one v_i is P then we have:*
$(u_1 + \cdots + u_n, v_1 + \cdots + v_n) \in f^!$.

The proof of proposition 12 follows the same line as the one for $\mathbb{PR}$ except that we have to check that the extra conditions on polarities are satisfied. Let us do it for the case of the contraction cut. Suppose $f : !A \to B$. We first check that:

$$f^!; cont_B \subseteq cont_A; (f^! \otimes f^!).$$

Let $(u, (v, w)) \in f^!; cont_B$. Then $(u, v + w) \in f^!$ and there exist u', u'' such that: $u = u' + u''$, $(u', v) \in f^!$,$(u'', w) \in f^!$. The case where $v = []$ or $w = []$ is as for $\mathbb{PR}$. Let us then assume that v and w are not $[]$. By lemma 11, as $(v + w)$ belongs to $!B$ we know that v and w are not both of polarity P and so (v, w) belongs to $!B \otimes !B$. This gives us that $((u', u''), (v, w))$ belongs to $f^! \otimes f^!$ and this implies that (u', u'') is an element of $!A \otimes !A$. So $(u, (u', u''))$ belongs to $cont_A$ and so $(u, (v, w))$ is an element of $cont_A; (f^! \otimes f^!)$.

Conversely let $(u, (v, w))$ belong to $cont_A; (f^! \otimes f^!)$. Then $u = u' + u''$ and $((u', u''), (v, w))$ belongs to $f^! \otimes f^!$. Do we have: $(u, v + w) \in f^!$?

As (v, w) belongs to $!B \otimes !B$ we know that v and w are not both P. The case where v or w is $[]$ is handled as in $\mathbb{PR}$; in the other case lemma 13 tells us: $(u' + u'', v + w) \in f^!$. So $(v + w, (v, w))$ belongs to $cont_B$ and finally: $(u, (v, w)) = (u, v + w); (v + w, (v, w)) \in f^!; cont_B$.

4.3 Restriction to intuitionistic linear logic

As shown in the previous section, the $\mathbb{PPR}$ category is obtained by forgetting time in plays of the saturated strategies model and still defines a non-degenerate model of MELL for relevant reduction. In this section we adapt the construction to the intuitionistic case (as defined in section 2.3, plays in this setting are constrained to start with an O-move).

We consider formulas built over $\multimap, \otimes, !$ and the associated $\mathbb{PPR}$ objects. We define the elements hereditarily of polarity O (*hopo*) of a formula A by: the point x is *hopo* in A if (i) $\lambda_A(x) = O$ or N, and (ii)

- if $A = A_1 \multimap A_2$ or $A_1 \otimes A_2$ then $x = (x_1, x_2)$ and x_1 (resp x_2) is *hopo* in A_1 (resp. A_2).

– if $A = !A_0$ then $x = [x_1, \ldots, x_n]$ and each x_i is *hopo* on A_0.

Notice that an element obtained by composition of *hopo* elements is *hopo*. We then define the category $\mathbb{IPPR}$ by: objects are polarized pointed sets containing only *hopo* elements, morphisms are polarized pointed relations that don't contain $(a, \bot)$ with $a \neq \bot$. In particular $id_A = \{(x, x)/x \text{ } hopo \text{ on } A\}$.

Remark. Suppose a morphism f of $\mathbb{IPPR}[A_1, A_2]$ contains an element of the form $(x_1, \bot_{A_2})$ with $x_1 \neq \bot$. Since x_1 is *hopo* and $x_1 \neq \bot$ we have $\lambda_{A_1}(x_1) = O$, thus by the polarity table of $\multimap$, we get $\lambda_{A_1 \multimap A_2}(x_1, \bot_{A_2}) = P$, so that $(x_1, \bot_{A_2})$ is not in $A_1 \multimap A_2$ contradicting the closedness of $\mathbb{IPPR}$. This is the reason for the restriction on morphisms in $\mathbb{IPPR}$.

This implies that the restriction on passive composition vanishes in this category: composition is simply relational composition.

The tensor bifunctor and the promotion are defined as before but the remark above shows that tensor is the cartesian product on relations and that $f^! : !\Gamma \to !A$ is defined starting from $f : !\Gamma \to A$ by:

$$f^! = \{(u_1 + \cdots + u_n, [x_1, \ldots, x_n]), \text{ s.t. } (u_i, x_i) \in f \text{ and } x_i \neq \bot_A\} \cup \{([], [])\}.$$

Dereliction, contraction and weakening morphisms are defined as usual (with *hopo* elements).

So $\mathbb{IPPR}$ is very close to the relational model of intuitionistic multiplicative linear logic. Contrarily to what happens in the classical case ($\mathbb{PPR}$) weakening reduction is here valid.

5 Collapsing $\mathbb{G}$

We want to define a map from the saturated strategies model on $\mathbb{PPR}$.

We assume that for any basic formula A we have a map $\widehat{.}$ from the set P_A of plays on the game A on the $\mathbb{PPR}$ object A satisfying:

– $s \equiv_A s' \Rightarrow \widehat{s} = \widehat{s'}$
– $\widehat{s}$ has polarity p iff
  - $s = \epsilon$ and $p = N$, or
  - s begins with a P-move (resp. an O-move) and $p = P$ (resp. $p = O$).

Then $F[A_1, \cdots, A_n]$ being a formula containing only the atomic formulae $A_1, \ldots, A_n$, we define inductively a map $\widehat{.} : P_F \to F$ by:

$$\begin{array}{ccll} P_{A_1 m A_2} & \to & A_1 m A_2 & \text{for } m = \otimes \text{ or } \parr \\ s & \to & (\widehat{s_{A_1}}, \widehat{s_{A_2}}) & \\ P_{\Box A} & \to & \Box A & \text{for } \Box = ! \text{ or } ? \\ s & \to & [\widehat{s_{i_1}}, \ldots, \widehat{s_{i_n}}] & \end{array}$$

where $i_1, \ldots, i_n$ are the indices such that $s_{i_j} \neq \epsilon$.

Proposition 14 *If s is a play in P_A then $\lambda_A(\widehat{s})$ is the author of the first move in s. As a particular case, it is N iff s is the empty play.*

Let us only detail in the induction on A the case of the $\otimes$ connective; the other cases are handled in a similar way.

Suppose s is a play of $P_{A_1 \otimes A_2}$. The polarity of its first move can be deduced from the polarities of the first moves of the projections s_{A_1} and s_{A_2} (which are the polarities of $\widehat{s_{A_1}}$ and $\widehat{s_{A_2}}$) using the function $\lambda_\otimes$. Indeed:

- if one of the projections is empty the first move of s is the first move of the other projection if it exists (s is empty if the other projection is empty).
- if both projections start with an O-move then s starts with an O-move.
- if one projection, say s_{A_1}, starts with an O-move while the other one (s_{A_2}) starts with a P-move: the switching condition tells us that only O is allowed to change components, thus the only chance that Player gets to make the first move in A_2 is when beginning the play s in A_2. So the first move of s is a P-move.

5.1 OP plays and projections

Say s is an *OP play* if it begins with an O-move and ends with a P-move, or is empty. Dually, an s is PO in A if it begins with a P-move and ends with an O-move, or is empty (that is s is OP in $A^\perp$).

Since plays are alternating, a play is OP or PO iff it is of even length.

Intuitively, an OP play has that interesting property that each O-move in it gets an answer from P. Thus we get rid of embarrassing situations where O plays in the void. It reintroduces a local orientation for want of a global one, one once had in ILL.

We leave to the reader the proofs of the following lemmas:

Lemma 15 *Let s be in $P_{A,A^\perp}$ where P follows the axiom strategy, id_A. Then s is OP iff $s_A = s_{A^\perp}$.*

Lemma 16 *Given $\sigma : \Gamma, A$, $\tau : \Delta, B$ let $s : \Gamma, \Delta, A \otimes B$ be in $\sigma \otimes \tau$ and let $s_0 = s_{\Gamma,A}$ and $s_1 = s_{B,\Delta}$. Then $s_0 \in \sigma$ and $s_1 \in \tau$, and if s is OP then so are s_0 and s_1.*

Lemma 17 *Let $s : ?\Gamma, !A$ be in $\sigma^!$ and let s_i be as in the definition of $\sigma^!$. Then $s_i \in \sigma$ and if s is OP then so are the s_i's too.*

5.2 Collapsing strategies:

We can finally define our collapse of strategies:

Definition 3 *σ being a strategy, let $\widehat{\sigma} = \{\widehat{s},\ s \text{ is OP and } s \in \sigma\}$.*

Remark. Consider the case where we have for the basic games A an equivalence relation reduced to the identity and where the set of the $\mathbb{PPR}$ objects corresponding to these games is taken to be simply P_A.

Given an OP play s, let then σ_s be the strategy induced by s, that is the set of prefixes of s. It can then be checked that $\widehat{\sigma_s} = \widehat{\sigma_{s'}}$ iff $s \equiv s'$. To do so one might proceed in two steps: first show by induction on A that for two plays s and s' on A, if the sets $\{\widehat{t} \,/ t \text{ prefix of } s\}$ and $\{\widehat{t'} \,/ t' \text{ prefix of } s'\}$ are equal then s and s' are equivalent. Then using the hypotheses on basic games, show that if $\widehat{\sigma_s} = \widehat{\sigma_{s'}}$ then the two previous sets are equal.

That is to say that with these hypotheses $\widehat{.}$ loses no information on plays, which is a bit surprising for a 'time forgetting' map. This means that intuitively $\mathbb{PPR}$ can be seen as a kind of games model, where we are not dealing with plays anymore but with positions, that is results of plays. In fact this intuition may be generalized under the slogan that denotational semantics deals with positions, which are static objects, while games semantics deals with plays.

Now, polarization has enabled us to enrich our target model with a proper duality. What is to be expected next?

- First a functoriality property of the time-forgetting map.
- Secondly a commutation of interpretation of syntax in both models through $\widehat{.}$, namely: given a proof-net R, equality between $\widehat{\sigma_R}$ and its direct interpretation R^* in $\mathbb{PPR}$.

Unfortunately $\mathbb{PPR}$ fails to fulfill any of these two requirements... But still there are some good news to come.

5.3 Lax-functoriality:

Proposition 18 *The application* $\widehat{.}$ *is what we call a 'lax functor' from the category of games/saturated strategies to the category* $\mathbb{PPR}$*, by which we mean that if* σ *and* τ *are composable saturated strategies then we have:*

- $\widehat{\sigma;\tau} \subseteq \widehat{\sigma};\widehat{\tau}$.
- $\widehat{[id]} \subseteq id^{(PPR)}$

Proof:

We prove the first inclusion. Let $\sigma : A \multimap B$, $\tau : B \multimap C$ and s be OP in $\sigma;\tau$. By definition of composition there is $s' \in \sigma \otimes \tau :\ A^\perp, (B \otimes B^\perp), C$ such that $s = s'_{A^\perp, C}$ and $s'_B = s'_{B^\perp}$.

If $s \neq \epsilon$, put m_O for its first move. Observe that m_O is also the first move of s'. Otherwise there would be m'_P preceding m_O in s', with m'_P in $B \otimes B^\perp$ (else m'_P is in $A^\perp, C$ and is visible in s) but such an m'_P must be answered by O in $B \otimes B^\perp$, by the switching condition. Likewise the last move in s is the last move in s'. So s' is OP too.

Define $s_0 = s'_{A^\perp, B}$ and $s_1 = s'_{B^\perp, C}$, by the definition of tensor and the lemma 16, one knows $s_0 \in \sigma$ and $s_1 \in \tau$, and that both plays are OP. So $\widehat{s_0} \in \widehat{\sigma}$

and $\widehat{s_1} \in \widehat{\tau}$, and also $\widehat{s_{0B}} = \widehat{s_{1B}}$. Moreover as we already mentioned s' can go from $A^\perp, B$ to $B^\perp, C$ or *vice-versa* only through a O switching move in $B \otimes B^\perp$; so we know that if $\widehat{s_{0B}} = \perp$ or $\widehat{s_{1B}} = \perp$ then $\widehat{s_0} = (\perp, \perp)$ or $\widehat{s_1} = (\perp, \perp)$. Then the following composition is valid:

$$\widehat{s_0}; \widehat{s_1} = (\widehat{s_{0A}}, \widehat{s_{0B}}); (\widehat{s_{1B}}, \widehat{s_{0C}}) = (\widehat{s_{0A}}, \widehat{s_{1C}}) = \widehat{s}$$

and so finally $\widehat{s}$ belongs indeed to $\widehat{\sigma}; \widehat{\tau}$. $\square$

Our troubles with the interpretation of syntax start with the second inclusion: it is not the case that all positions (x, x) of $id_A^{(PPR)}$ can be lifted to plays s of $[id]$. Think for instance of the position $((a, b), (a, b))$ in $id : A \otimes B \to A \otimes B$, where a and b are (time forgotten) plays in A and B consisting in a single O-move. Since the polarity of this position is O, any corresponding play should begin with an O-move, thus in the rightmost $A \otimes B$. Such a play begins with say a^{right}. Then the only possible following move is a^{left}. After that, thanks to the switching and alternacy conditions, no move is possible so we obtain the play $a^{\text{right}} a^{\text{left}}$ which isn't mapped to the given position. Same happens if we start with b^{right}.

This argument makes also clear that we won't have equality between $\widehat{[\sigma] \otimes [\sigma']}$ and $\widehat{[\sigma]} \otimes \widehat{[\sigma']}$. Actually we get:

$$\widehat{[\sigma] \otimes [\sigma']} \subseteq \widehat{[\sigma]} \otimes \widehat{[\sigma']}$$

By checking similar inclusions with respect to the exponential constructs and by using the proposition 18 one can state:

Proposition 19 *Let R be a MELL proof-net and σ_R, R^* be its respective interpretations as a saturated strategy and as a pointed polarized relation. We have:*

$$\widehat{\sigma_R} \subseteq R^*.$$

It can be seen on the small example we gave before that the strictness of the inclusion in proposition 19 is linked to our unability to reconstruct from a given polarized position a valid play reaching it. In the case of the $\otimes$, to build from a $A_1 \otimes A_2$ position (x_1, x_2) an appropriate play we need:

- first to lift the x_i's on plays s_i of the corresponding games A_i.
- secondly to be able to interleave s_1 and s_2 so as to respect the switching condition.

Observe that this second step cannot be made if s_1 and s_2 both end with a P-move. This leads us to take into account in the collapse not only the first move of the play, but also the last one. Therefore we introduce a third target model: bipolarized pointed relations model ($\mathbb{BPR}$), which can be seen as a symmetric version (with respect to time: beginning/end of the play) of $\mathbb{PPR}$.

6 Bipolarized pointed relations

Since they are very similar to what has been done so far we will only sketch the main constructions and state the results.

Let $Bpol = \{OO,\ OP,\ PO,\ PP,\ NN\}$ be the set of bipolarities. The bipolarity NN will also be simply denoted N. A *bipolarized pointed set* is a pointed set A together with a bipolarity function Λ_A from A to $Bpol$, such that only $\perp_A$ has bipolarity N. We define the initial polarity $\lambda^i_A(x)$ and final polarity $\lambda^f_A(x)$ of a position x by: $\Lambda_A(x) = \lambda^i_A(x)\,\lambda^f_A(x)$.

Morphisms. A subset f of $A_1 \times A_2$ containing $(\perp_{A_1}, \perp_{A_2})$ belongs to $\mathbb{BPR}[A_1, A_2]$ if for any (x_1, x_2) in f we have:

- $\lambda^i_{A_1}(x_1) = P$ implies $\lambda^i_{A_2}(x_2) = P$ or N.
- $\lambda^f_{A_2}(x_2) = P$ implies $\lambda^f_{A_1}(x_1) = P$ or N.

The usual composition and identity morphisms make $\mathbb{BPR}$ a category.

The tensor bifunctor. $A_1 \otimes A_2$ is the set of all (x_1, x_2) such that $(\lambda^i_{A_1}(x_1), \lambda^i_{A_2}(x_2)) \neq (P, P)$ and $(\lambda^f_{A_1}(x_1), \lambda^f_{A_2}(x_2)) \neq (O, O)$. The initial polarity of (x_1, x_2) in $A_1 \otimes A_2$ is given by the table of polarities of the $\otimes$ of $\mathbb{PPR}$ and the final one by:

$x_1 \backslash x_2$	O	P	N
O		O	O
P	O	P	P
N	O	P	N

The action on morphisms is defined just as for $\mathbb{PPR}$.

Interpreting linear negation by $A^\perp$ of same set as A and of bipolarity function $\Lambda_{A^\perp} = \bar{\lambda}^i_A\,\bar{\lambda}^f_A$ we build an adjunction as in the case of $\mathbb{PPR}$ and $\mathbb{BPR}$ is also $*$-autonomous.

Exponential structure is handled as for $\mathbb{PPR}$. We add the condition that $[x_1, \ldots, x_n] \in\ !A$ if at most one x_i has final polarity O. The final polarity $\lambda^f_{!A}[x_1, \ldots, x_n]$ is N if $n = 0$,and if $n \neq 0$ it is P if all x_i's have final polarity P, O otherwise.

Proposition 20 *The category $\mathbb{BPR}$ is a model of MELL for relevant reduction.*

The collapse of $\mathbb{G}$ on $\mathbb{BPR}$. We define as with $\mathbb{PPR}$ a map from the games of $\mathbb{G}$ to the corresponding objects of $\mathbb{BPR}$ abusively also denoted $\widehat{.}$. The conditions asked for the action on basic formulas are adapted to bipolarization: $\lambda^i_A(\widehat{s})$, $\lambda^f_A(\widehat{s})$ give respectively the author of the first and last move of s. Asking furthermore that for basic formulas $\widehat{.}$ is onto the set of positions we get:

Proposition 21 *For any formula A, the time-forgetting map from P_A to the set of positions of the $\mathbb{BPR}$ object A is onto:*

$$x \in A \textit{ implies } \exists s \in P_A /\ \widehat{s} = x.$$

The action on saturated strategies is defined as before and the new time-forgetting map is still a lax-functor from $\mathbb{G}$ to $\mathbb{BPR}$.

Proposition 22 *We have:*

- $\widehat{[id]} = id^{(BPR)}$
- *given* $[\sigma_i] : A_i \multimap B_i$ *for* $i = 1, 2$
  $\widehat{[\sigma_1] \otimes [\sigma_2]} = \widehat{[\sigma_1]} \otimes \widehat{[\sigma_2]}$.

Let us examine for the two equalities the less obvious inclusion: the right-to-left one.

For the first equality, it is a consequence of proposition 21: if $A(x, x) = N$ then $(x, x) = \widehat{\epsilon}$; otherwise (x, x) being a position of id_A^{BPR} there exists s in P_A such that $\widehat{s} = x$. One can then lift (x, x) on a play of the copy-cat strategy on $A \multimap A$ starting on the right-hand side (resp. left-hand side) if s starts with an O-move in A (resp. P-move).

For the second one, given $((a_1, a_2), (b_1, b_2))$ in $\widehat{[\sigma_1]} \otimes \widehat{[\sigma_2]}$, for $i = 1, 2$ there exists s_i in $[\sigma_i]$ satisfying $\widehat{s_i} = (a_i, b_i)$. As (b_1, b_2) belongs to $B_1 \otimes B_2$ by definition of the $\otimes$ in $\mathbb{BPR}$ we know that at least one $(s_i)_{B_i}$ doesn't start with a P-move, and at least one $(s_i)_{B_i}$ doesn't end with an O-move. The reader might check that this is enough to enable us to build an interleaving of the $(s_i)_{B_i}$'s which is a valid play on $B_1 \otimes B_2$. Then this construction yields a natural interleaving s of s_1 and s_2, where switches between 1 and 2 are done in $B_1 \otimes B_2$ by the Opponent, and such that $(s)_{A_i, B_i} = s_i$ for $i = 1$ and 2. This implies that s belongs to $[\sigma_1] \otimes [\sigma_2]$. □

We also get in a similar way such equalities for the exponential morphisms and the promotion construct.

Finally:

Theorem 23 *R being a MELL proof-net, the map $\widehat{.}$ on $\mathbb{BPR}$ satisfies:*

$$\widehat{\sigma_R} = R^*$$

where R^* stands for the interpretation of R in $\mathbb{BPR}$.

We obtain this result by proving it first inductively for proof-nets normal with respect to relevant reduction. Then we use the fact that relevant reduction is terminating and that it is valid in both models to conclude with equality for all proof-nets.

This theorem shows that with bipolarized relations we succeeded in defining a model without explicit time where the interpretation of syntax coincides with the former game interpretation (up to the forgetting of time in plays).

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