Sorbonne Universités

SCIENCES SORBONNE UNIVERSITÉ

École doctorale de sciences mathématiques de Paris centre

THÈSE DE DOCTORAT

Discipline : Mathématiques

présentée par

BÉNARD Léo

Reidemeister torsion on character varieties

---

dirigée par Julien MARCHÉ

Soutenue le 14 mars 2018 devant le jury composé de :

M. Nicolas BERGERON	Sorbonne Université	examinateur
M. Stefan FRIEDL	Universität Regensburg	examinateur
Mme Catherine GILLE	Université Paris Diderot	examinatrice
M. Michael HEUSENER	Université Clermont-Auvergne	rapporteur
M. Julien MARCHÉ	Sorbonne Université	directeur
M. Joan PORTI	Universitat Autònoma de Barcelona	rapporteur

Institut de mathématiques de Jussieu-Paris Rive gauche.
UMR 7586.
Boîte courrier 247
4 place Jussieu
75 252 Paris Cedex 05

Sorbonne Université.
Campus Pierre et Marie Curie.
École doctorale de sciences mathématiques de Paris centre.
Boîte courrier 290
4 place Jussieu
75 252 Paris Cedex 05

Remerciements

Ces remerciements seront sûrement incomplets, je m'excuse par avance auprès de toutes celles et tous ceux que j'aurais oublié.es.
En premier lieu, je remercie Julien Marché pour avoir encadré mes deux mémoires de master puis ma thèse avec tant d'entrain. Ce fut un privilège de côtoyer et d'être formé par un tel mathématicien pendant toutes ces années. Je me souviens encore du jour où, venant le rencontrer pour lui demander de me proposer un sujet de mémoire de M1, je lui posais des questions sur sa recherche ; il dessina un noeud de trêfle sur son célèbre tableau Velleda. Je commençais alors à étudier la topologie, il me dit : "j'essaye de comprendre le groupe fondamental du complémentaire dans $\mathbb{R}^3$ de ce genre d'objets". Plus de cinq années plus tard, je dois avouer m'être demandé à cet instant quel était l'intérêt de ces objets d'apparence anodine. Je peux affirmer désormais m'être forgé ma propre réponse à cette question, cette thèse en est un témoignage.
Il a été un directeur extrêmement disponible, m'a appris à ne pas me décourager devant un problème trop difficile mais à persister, quitte à modifier mon angle d'approche, à tirer partie de mes échecs, à mettre en valeur mes réussites. Il m'a transmis son infinie curiosité mathématique, son goût d'apprendre sans cesse. S'il m'a formé en tant que mathématicien, j'ai énormément appris de son humilité également et j'ai bénéficié de ses conseils toujours pertinents en toutes circonstances.
Je remercie Michael Heusener et Joan Porti d'avoir bien voulu rapporter cette thèse. C'est un grand honneur qu'ils me font, leurs travaux ont été tout au long de cette thèse une source d'inspiration et d'admiration pour moi. J'ai beaucoup apprécié les visites que je leur ai rendues à Clermont-Ferrand et à Barcelone et cette thèse doit beaucoup aux conversations que nous avons eues alors, et aux suggestions qu'ils m'ont faites.
C'est un grand honneur également de compter Nicolas Bergeron, Stefan Friedl et Catherine Gille dans mon jury. Nicolas fut un interlocuteur privilégié pour moi, d'abord comme enseignant, puis comme collègue. Son rôle en tant que directeur

adjoint de l'IMJ-PRG et sa préoccupation constante de nous prodiguer, à nous doctorants, des conditions de recherche optimales, dépassent largement la gratitude que je peux lui exprimer ici. Les travaux de Stefan ont énormément apporté à ma compréhension du sujet de cette thèse comme immergé dans un environnement mathématique vaste et fascinant, et la qualité des exposés que j'ai eu la chance de le voir donner à plusieurs reprises est un modèle du genre. Catherine m'a donné l'opportunité de parler au séminaire de topologie chaque fois que j'en avais le souhait, en particulier j'y ai donné mon premier exposé de recherche. Plus généralement je la remercie d'avoir contribué à créer, au sein de l'équipe de Topologie et Géométrie Algébrique, un environnement propice à l'épanouissement des jeunes topologues.

En tant qu'étudiant, puis doctorant, j'ai profité de la qualité exceptionnelle de l'environnement scientifique à l'Université Pierre et Marie Curie et au sein de l'IMJ-PRG. Parmi tous, j'ai pu compter lors de mon année de L3 sur les soutiens de Jean-Yves Chemin et d'Adrien Deloro pour ma candidature aux bourses de master de la fondation PGSM, et je les remercie pour cela. Je remercie également la fondation PGSM de m'avoir fait profiter de ce programme pendant mes deux années de master ; je remercie également Andrei Iordan qui fut mon tuteur au sein de la fondation.

Je remercie les membres de l'équipe Analyse Complexe et Géométrie pour m'avoir accueilli à bras ouverts, notamment Gilles Courtois, Elisha Falbel, Andrés Sambarino et Maxime Wolff pour nos nombreuses discussions, mathématiques ou non.

Je remercie également les membres des équipes Analyse Algébrique et Topologie et Géométrie Algébrique, en particulier Christian Blanchet, Antoine Ducros, Gregor Masbaum et Marco Maculan en TGA, et Laurent Charles, Hélène Eynard-Bontemps, Emmanuel Ferrand, Gregory Ginot, Antonin Guilloux, Vincent Humilière, Frédéric Le Roux, Alexandru Oancea et Fabrice Rouillier en AA (et merci pour vos bons pots d'équipe, sur ce plan vous êtes incontestablement au top !).

Une pensée pour M. Ferry, qui a su transformer mon goût des maths en études supérieures, et merci Hélène Benhamou pour ce premier contact avec la rigueur et la précision mathématique.

J'ai eu la chance de côtoyer des camarades d'études exceptionnels, en particulier merci à Jules pour les années à Jussieu, les canapés à Toulouse et les virées en conférence. Merci aussi à Marine, Henri, Berit, David, Léo, Sacha, Théo, Vincent, Jean-Michel et Cyril.

À mon arrivée en thèse, j'ai rencontré en premier lieu mon co-bureau Adrien. Quelle chance de partager depuis lors un morceau de ce bureau et un morceau de vie avec

lui ! Puis sont arrivés Johannes et Jesua, et l'année suivante Davide, pour former une équipe inénarrable, merci à vous tous pour votre patience, votre écoute, votre bonne humeur pendant ces quelques années ; quel bureau mémorable !
Louis, Nicolina et Macarena formèrent rapidement le noyau dur de notre groupe de doctorants, merci à vous pour les discussions passionnnées, les belotes (encore plus passionnées), les balades en Espagne et les Libanais place Monge.
Le groupe de doctorant.es qui nous a accueilli alors et nous a permis de prendre nos marques dans ce nouvel environnement mériterait de plus grandes louanges, pour leur disponibilité envers les nouveaux arrivants, l'esprit d'équipe et l'odeur de fondue qui régnait alors dans le couloir, merci à François, Clément, Maylis, Olivier, Juliette, Thibault, Florian, Liana entre autres. J'ajoute ici Mathilde qui complète qualitativement le fameux binôme des GG's, essentiel à toute soirée rillette/rince-cochon.
Les arrivant.es suivants ne furent pas en reste, merci à Hugo pour avoir chapeauté toute cette marmaille, Justin, Thomas, et tous les autres qui ont participé à la vie du couloir des doctorants de Jussieu et au BDD : Arthur, Miguel, Ruben, Xavier, Adrien S., Robin, Arthur-César, Valentin, Leon...
Dans le groupe des topologues quantiques, d'abord Renaud et Ramanujan m'ont accueilli à bras ouverts, puis n'ont jamais refusé de répondre à mes nombreuses questions sur le sujet, un grand merci à eux. Merci également à Marco pour la franche camaraderie qui commença un jour de randonnée en montagne dans les Pyrénées, à Christina et à Daniel.
Plus généralement, les nombreuses conférences auxquelles il m'a été donné d'assister ont été prétexte à de nombreuses rencontres (une pensée particulière pour La Llagonne-Matemale, merci Thomas pour faire vivre cette conférence incroyable dans cet endroit magique), merci à Guillem, Jéremy, Sélim, Delphine, Louis-Hadrien, Fathi, Giulio, Anthony, Xavier, Livio, Filip, Mounir, Brian, et tant d'autres. Tant de bons moments à vos côtés, il m'est impossible de me les rappeler tous, mais merci !
Enfin, merci aux organisateurs du séminaire Aromaths, pour ceux que je n'ai pas encore cité : Ayman, David, Ella, Etienne, Neige, Rémi et tous ceux qui ont contribué et qui contribueront à ce séminaire. Merci également à toute la bande de la Fête de la Science, en particulier Frédérique, Marie et Mathilde.
En dehors des mathématiques, il y a une place dans cette liste de remerciement pour tous les copains d'Arkose Montreuil, en particulier, Mattias et Laura pour les bons moments grimpe, les sorties Fontainebleau et les week-ends en vadrouille ; et Anne-So, Antoine, Gwen, JB, tous les autres...

Merci à toutes celles et à tous ceux que j'ai côtoyé de près ou de loin à l'époque de Quedubon, en particulier un grand merci pour toutes ces années ensemble à Alexandra, Bastien, Thang, Carole... Merci aussi à Camille pour les relectures !
Certaines rencontres ont profondément changé ma vie, c'est le cas de celle que j'ai faite avec Raquel et Philippe, merci pour tout. Merci également à Jacques-Alain, qui sait ce que cette thèse lui doit.
La fin approche, et mes amis de toujours m'excuseront de les avoir gardés jusque là. Merci Valou et Aurélie, Amine et Émilie, votre réussite fait ma fierté. Merci Sofiane pour toutes ces années de vie commune, de travail, d'apprentissage, dans ton talent de cuisinier je retrouve avec émotion nos balbutiements communs d'une autre époque.
Merci à ma (grande) famille, qui compte plus que tout pour moi. Parmi eux, merci à mes grands-parents maternels Françoise et Charles pour ce que vous m'avez transmis. Mes frères et soeurs Philippe et Sophie, Aline, Marius et Clémentine, merci d'être là, de près ou de loin, chacun de vous m'est indispensable. Certains sont arrivés plus récemment, une pensée pour Enora, Alicia, Déliah, Rewan, Basile et Charles, par ordre d'apparition :)
Laurence, Pierre et Isabelle ont un rôle particulier dans ma vie, merci à vous pour avoir été mes parents quand il le fallait. Cette richesse ne comblait pas de carence, bien au contraire, et je remercie ma mère Laure et mon père Gilles pour tout ce qu'ils ont fait pour moi, votre soutien permanent m'est essentiel et votre courage m'inspire et m'inspirera pour longtemps.
Enfin, Léa, tout ce qui se dit dans cette thèse n'est rien comparé au bonheur que j'ai de t'avoir rencontrée, et de vivre chaque jour auprès de toi. MERCI.

Résumé

L'objet de cette thèse est l'étude de la torsion de Reidemeister définie globalement sur la variété des caractères de variétés de dimension 3 à bord torique. Cette étude se divise en deux parties, selon le complexe cohomologique que l'on choisit d'associer à une représentation du groupe fondamental de la variété M dans le groupe $\mathrm{SL}_2(\mathbb{C})$.
Dans le premier cas, la torsion est définie comme une forme différentielle sur un revêtement double de la variété des caractères, on montre que cette forme différentielle n'a pas de pôles, et on relie l'apparition de zéros à différentes propriétés de la variété des caractères, certaines de nature algébrique, d'autres de nature topologique.
Dans un deuxième temps on étudie le comportement asymptotique de la torsion, qu'on relie à la topologie de surfaces incompressibles dans la variété M, construites via la théorie de Culler-Shalen. On en déduit une relation entre la topologie de la variété des caractères et celle de ces surfaces.
Dans le deuxième cas, on introduit une fonction sur la variété des caractères, qui s'avère être une spécialisation d'un polynôme d'Alexander tordu très étudié. Sous certaines conditions liées à l'étude des surfaces précédentes, on prouve que cette fonction admet des pôles à l'infini, en particulier qu'elle est non constante, et que ce polynôme l'est aussi.

Mots-clés

Torsion de Reidemeister, variétés de caractères, théorie de Culler-Shalen, théorie des noeuds.

Abstract

In this PhD dissertation we study the Reidemeister torsion as a globally defined invariant on character varieties of 3-manifolds with toral boundary. Given a representation of the fundamental group of a 3-manifold M into the group $\mathrm{SL}_2(\mathbb{C})$, two cohomological complexes arise naturally from the action of $\pi_1(M)$ either on $\mathrm{sl}_2(\mathbb{C})$ or on $\mathbb{C}^2$. This choice divides this thesis in two parts.
In the first part, the torsion is defined as a rational differential form on a double cover of the character variety. We show that this differential form has no poles, and we study its zero locus. We relate it with algebraic and topological properties of the character variety. Then we study the asymptotical behavior of the torsion, with the help of some incompressible surfaces in M constructed by the Culler-Shalen theory. We deduce a relation between the topology of the character variety and the topology of those surfaces.
In the second part, we define a function on the character variety, that turns out to be a specialization of a well-known twisted Alexander polynomial. Under some conditions on the former incompressible surfaces, we prove that this regular function has poles at infinity. In particular it implies that this function is non-constant, and so is this Alexander polynomial.

Keywords

Reidemeister torsion, character varieties, Culler-Shalen theory, knot theory.

Contents

Chapter 0

Introduction

The Reidemeister torsion has been introduced as a topological invariant of homological complexes in 1935 by both Reidemeister (in [Rei35]) and Franz (in [Fra35]) independently. It also appears in a more algebraic context in the seminal work of Cayley (see [GKZ94, Appendix B]) in 1848.

The Reidemeister torsion is a generalization of the notion of determinant: consider an endomorphism d of a finite dimensional vector space V over a field k. Any choice of bases $e = \{e_1, e_2, \ldots, e_n\}$ and $f = \{f_1, f_2, \ldots, f_n\}$ allows us to compute the determinant of the linear map d in those bases, namely, this determinant $\det(d, e, f)$ is the volume of the polytope $d(e)$ with the normalization that turns the volume of the polytope f equal to 1. Equivalently it is the determinant of the matrix whose columns are the vectors of $d(e)$ in the basis f. While $\det(d, e, f)$ depends on the map d in a crucial way, it also depends on the bases e and f. Yet it would not be affected if one would modify the basis e, for instance into the basis $e' = \{\lambda e_1, \lambda^{-1} e_2, e_3, \ldots, e_n\}$ for some $\lambda \in k^*$.

One could generalize this definition when the map d is no longer injective, it gives rise to an exact sequence of k-vector spaces

$$0 \to C^{-1} \xrightarrow{d_{-1}} C^0 \xrightarrow{d_0} C^1 \to 0$$

with $\dim C^{-1} = n_{-1}, \dim C^0 = n_0, \dim C^1 = n_1 = n_0 - n_{-1}$.

To do so, one has to fix some bases c^i of the vector spaces C^i, for $i = -1, 0, 1$. We define the torsion of this based exact sequence $\mathrm{tor}(C^*, c^*)$ to be the volume of the polytope $d_{-1}(c^{-1}), \bar{c}^1$ in C^0, with $\bar{c}^1$ any lift to C^0 of the basis c^1, and with the normalization that turns the volume of the polytope given by c^0 equal to 1. Equivalently, it is the determinant of the matrix $d_{-1}(c^{-1}), \bar{c}^1$ in the basis c^0. One also has to check that this does not depend on the choice of a lift $\bar{c}^0$, which is routine linear algebra. More interestingly, one could provide an alternative definition of $\text{tor}(C^*, c^*)$, in terms of the determinants of some restriction of the maps d_{-1} and d_0: let $\bar{d}_{-1}$ be the map d_{-1} co-restricted to the vector space $C' \subset C^0$ generated by the n_{-1} first vectors of the basis c^0. Up to a permutation of the vectors of the basis c^0, one can assume that this new map d_{-1} is an isomorphism. Since the sequence is exact, the restriction $\bar{d}_0$ of the map d_0 to the last n_1 vectors of the basis c^0 turns to an isomorphism too, and the claim is that $\text{tor}(C^*, c^*) = \frac{\det \bar{d}_0}{\det \bar{d}_{-1}}$.

It is important to remark that there is a sign indeterminacy in the second definition: when permuting the basis vectors, the determinant could have been multiplied by -1. In this thesis we will always consider the torsion to be defined up to sign, and we will never try to solve this ambiguity, because we will study some vanishing properties of this torsion.

The previous definition generalizes whenever the exact sequence is not a short exact sequence any more: one can define the torsion $\text{tor}(C^*, c^*)$ of a based exact complex $C^0 \to C^1 \to \ldots \to C^n$ with basis $c^0, \ldots, c^n$. A more complicated task is to generalize this definition in cases where the complex C^* is not an exact complex. It will depend on a choice of bases of the complex, but also on bases of its cohomology groups $H^i(C^*)$. This is the core of Section 1.5 of this dissertation.

At this point, one should emphasize the fact that the torsion of a cohomological complex carries much more informations than the cohomology of this complex, at least it provides some topological information even when the complex is acyclic. The definition of the torsion may seem rather involved, but it was motivated by the fact that Reidemeister used it to distinguish non-homeomorphic lens spaces with isomorphic fundamental groups. To be concise, lens spaces $L_{(p,q)}$, for p, q relatively prime, are three-dimensional closed manifolds that are somewhat "simple" in 3-dimensional topology: for instance they have rational homology of the 3-sphere, and finite fundamental groups $\mathbb{Z}/p\mathbb{Z}$. In [Rei35] it is shown that the computation of the torsion of some acyclic homological complex of a lens space is a topological invariant, and that it involves the second integer q in a crucial way. This results have prompted a vast increase in work and publications on torsion theory, notably Whitehead torsion, Milnor torsion...

An very nice instance of a generalization of this invariant is the Ray-Singer analytic torsion. In the article "R-torsion and the Laplacian on Riemannian manifolds" ([RS71]), D.B. Ray and I.M. Singer define a topological invariant in terms of the action of the Laplacian on differential forms. Let us say a word about this construc-

tion in the case of an acyclic complex $C^* = C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} C^n$. In this case the Reidemeister torsion is defined from a choice of bases c^* of the C^i's that allows us to write each C^i as a direct sum $C^i = \ker d_i \oplus K_i$. Then we consider the restriction maps $\bar{d}_i : K_i \to \mathrm{im}(d_i) = \ker(d_{i+1})$ and we compute the Reidemeister torsion $\mathrm{tor}(C^*, c^*) = \prod \det(\bar{d}_i)^{(-1)^i}$. An other way to do so is the following: consider the adjoint maps $d_{i-1}^* : C^i \to C^{i-1}$, namely they are obtained by taking the transpose of the matrices of each d_i in the basis c^i, c^{i+1}. We define on C^i the natural scalar product that turns the base c^i into an orthogonal base, hence the orthogonal splitting $C^i = \ker(\bar{d}_i) \oplus^\perp \mathrm{im}(\bar{d}_i^*)$. Now we can define the combinatorial Laplacians to be the operators $\Delta_i : C^i \to C^i$ given by the formula $\Delta_i = d_i^* d_i + d_{i-1} d_{i-1}^*$, and a careful examination shows that can express the torsion as $\mathrm{tor}(C^*, c^*) = \prod \det(\Delta_i)^{\frac{i}{2}(-1)^{i+1}}$. Without more informations, it may seem to be just a more complicated way to compute this invariant, but the striking point is that we can now define such an invariant in terms of the metric Laplacian acting on the De Rham complex of a closed manifold M: there is still a natural hermitian product and the formula above makes sense. The Laplacian turns into an infinite dimensional self-adjojnt operator, and we can define its determinant as the (infinite) product of its non-zero eigenvalues. This product can be shown to be well-defined with the use of some zeta functions, and this invariant $T_W = \prod \det(\Delta_i)^{\frac{i}{2}(-1)^{i+1}}$ has been conjectured in [RS71] to be equal to the Reidemeister torsion. This has been proved independently by Jeff Cheeger ([Che77]) in 1977 and Werner Müller ([Mul78]) in 1978, then generalized by Muller ([Mul93]) and Bismut-Zhang ([BZ92]).
Assume that C^* is a complex of modules over a PID R, such that this complex is acyclic as a complex of vector spaces over the fraction field $K = \mathrm{Frac}(R)$. In other words all the cohomology groups are torsion modules. Then the torsion $\mathrm{tor}(C^* \otimes_R K)$ carries many informations about those cohomology modules. In particular, in the case of Alexander modules of a 3-manifold M, Thang Le ([Le14]) has proved from this observation that the torsion of the homological complex of the maximal abelian covering $\bar{M}$ (namely the Alexander polynomial) was somehow related with the growth of the torsion part of the first homology groups of finite abelian covers M_n of M. We should mention the work of Nicolas Bergeron and Akshay Venkatesh ([BV13]) where those relations are discussed in a much more general setting, and explicitely expressed in terms of the Ray-Singer analytic torsion.

Notation. In this dissertation we are interested in the following situation: M is a 3-manifold, compact, connected and orientable, with boundary $\partial M = \mathbb{S}^1 \times \mathbb{S}^1$ a torus.

Although some of the results presented in this text do not need this assumption, we assume that the first Betti number of M is 1. Given $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ a representation, we construct two cohomological complexes, arising from the action of ρ on $\mathbb{C}^2$ (the standard action) or on $\mathrm{sl}_2(\mathbb{C})$ (the adjoint action).
More generally, we study twisted cohomology groups given by some representations $\rho : \pi_1(M) \to \mathrm{SL}_2(K)$, where K is the function field of some variety defined over an algebraically closed field k. We denote by $\mathcal{O}_v \subset K$ valuation subrings of K with k their residual field. The representation Ad will denote the adjoint representation $\mathrm{Ad} : \mathrm{SL}_2(K) \to \mathrm{Aut}(\mathrm{sl}_2(K))$ induced by matrix conjugation. We use the following notations:

1. By $H^*(M, \mathrm{Ad} \circ \rho)$ (respectively $H^*(M, \rho)$), we denote the twisted cohomology groups with $\pi_1(M)$ acting on $\mathrm{sl}_2(K)$ through $\mathrm{Ad} \circ \rho$ (resp. on K^2 through ρ).
2. Whenever $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathcal{O}_v)$, we denote by $H^*(M, \mathrm{Ad} \circ \rho)_v$ (respectively $H^*(M, \rho)_v$) the twisted groups with coefficients in $\mathrm{sl}_2(\mathcal{O}_v)$ (resp. $\mathcal{O}_v^2$).
3. In this case, we denote by $\bar{\rho} : \pi_1(M) \to \mathrm{SL}_2(k)$ the composition of ρ with the residual map $\mathcal{O}_v \to k$, and by $H^*(M, \mathrm{Ad} \circ \bar{\rho})$ (resp. $H^*(M, \bar{\rho})$) the twisted groups for the action of $\pi_1(M)$ on $\mathrm{sl}_2(k)$ (resp. on k^2).
4. For $\lambda : \pi_1(M) \to k^*$, we denote by $H^*(M, \lambda)$ the twisted cohomology groups with action of λ by multiplication on the field k. Of course, when λ is constant equal to 1, we keep the notation $H^*(M, k)$.

The torsion of twisted cohomological complexes by a representation $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ for M a hyperbolic 3-manifold has been studied for a long time. Let us mention the seminal work of Joan Porti in his PhD dissertation [Por97] (see the survey [Por15] too) for the adjoint torsion, and of Teruaki Kitano [Kit94, Kit96] for the acyclic torsion and the relation with the twisted Alexander polynomial. Many other authors have substantially contributed to the topic in the past ten years, including Jérome Dubois, Nathan Dunfield, Stefan Friedl, Takahiro Kitayama, Yoshikazu Yamaguchi.
In the first chapter of this thesis, we give the main definitions that will be used in the rest of this work. In particular we discuss various refinements of the character varieties theory, its relations with incompressible surfaces in 3-manifolds via the Culler-Shalen theory, and define the Reidemeister torsion in a general setting. In the second chapter, we study the adjoint torsion on the character variety, we define it as a rational differential form, and we study its poles and zeros. In the third chapter,

we focus on the acyclic torsion function, and prove that under mild hypothesis on M the torsion is not constant.

The adjoint torsion

In Chapter 2, we focus on the adjoint torsion, and present the results of the article [Ben16].

The Reidemeister adjoint torsion is a topological invariant $\mathrm{tor}(M, \mathrm{Ad} \circ \rho)$ that we may interpret as a volume element in the twisted cohomology, that is an element in

$$\mathrm{Det}(H^*(M, \mathrm{Ad} \circ \rho)) = \bigotimes_{i=0}^{3} \mathrm{Det}(H^i(M, \mathrm{Ad} \circ \rho))^{(-1)^i}.$$

Moreover as soon as ρ and ρ' are conjugated representations, there is a natural isomorphism $\mathrm{Det}(H^*(M, \mathrm{Ad} \circ \rho)) \simeq \mathrm{Det}(H^*(M, \mathrm{Ad} \circ \rho'))$ that preserves the torsion. Hence it is natural to define the Reidemeister adjoint torsion as a section of some line bundle over the character variety.

In his Phd thesis [Por97], Joan Porti defined the torsion as an analytic function on a Zariski open subset of the character variety depending on a choice of a boundary curve. Many computations have been performed by J. Dubois and al. [Dub06], [DHY09] and the torsion has been extended to the whole character variety in [DG16]. We will follow in this article the approach of [Mar15], where the Reidemeister torsion of any 3-manifold with boundary is interpreted as a rational volume form on the character variety. More precisely, if the boundary of M is a torus, the torsion is a rational volume form on the *augmented character variety* which is the following 2-fold covering of the character variety:

$$\bar{X}(M) = \{(\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C}), \lambda : \pi_1(\partial M) \to \mathbb{C}^*), \mathrm{Tr}\, \rho|_{\pi_1(\partial M)} = \lambda + \lambda^{-1}\} // \mathrm{SL}_2(\mathbb{C}).$$

In this thesis we assume that $X(M)$ is 1-dimensional and *reduced*, in the sense of schemes. The first assumption is guaranteed by the assumption that M is *small*, that is without closed incompressible oriented surfaces not parallel to the boundary. Let $\bar{X}$ be an irreducible component of $\bar{X}(M)$ containing the character of an irreducible representation and let Y be its smooth projective model. It is a smooth compact curve obtained from $\bar{X}$ by desingularizing and adding a finite number of points at infinity: we call the latter points ideal points of Y and the others are finite points. We denote by v an element of Y that can be viewed as a valuation on the function field $\mathbb{C}(Y) = \mathbb{C}(\underline{X})$. Its local ring at v will be denoted by $\mathcal{O}_v$. The torsion will be denoted by $\mathrm{tor}(M, \mathrm{Ad} \circ \rho)$ and seen as an element of $\Omega_{\mathbb{C}(Y)/\mathbb{C}}$. The first result is the following theorem:

Theorem (Theorem 2.0.1)**.** *Let v be a finite point of Y, then tor$(M, \mathrm{Ad} \circ \rho)$ has no pole at v. More precisely*

1. *If v projects to an irreducible character in $X(M)$, then the vanishing order of* $\mathrm{tor}(M, \mathrm{Ad} \circ \rho)$ *at v is the length of the torsion part of the module $\Omega_{\mathbb{C}[X]/\mathbb{C}} \otimes \mathcal{O}_v$. This integer is an invariant of the local singularity which can be computed explicitly. In particular if v projects to a smooth point of $X(M)$ then the torsion does not vanish at v.*
2. *Suppose that M is a knot complement and m is a meridian. If v projects to a reducible character $\lambda + \lambda^{-1}$ in $X(M)$ then $\lambda(m)^2$ is a root of the Alexander polynomial of M of order $r \geq 1$. Under some technical hypothesis detailed in Section 2.3,* $\mathrm{tor}(M, \mathrm{Ad} \circ \rho)$ *vanishes at v at order bounded by $2r - 2$. In particular it does not vanishes if λ^2 is a simple root.*

If v is an ideal point of Y, then the Culler-Shalen theory associates to v an action of $\pi_1(M)$ on the Bass-Serre tree of $\mathrm{SL}_2(\mathcal{O}_v)$ which itself produces an incompressible surface Σ in M. We say that Σ is associated to the ideal point v.

Theorem (Theorem 2.0.2)**.** *Let v be an ideal point of Y and Σ be an incompressible non-Seifert surface associated to v. We suppose that Σ is a union of parallel connected copies $\Sigma_1 \cup \ldots \Sigma_n$ and that both components of $M \setminus \Sigma_i$ are handlebodies. Let us also assume that Y contains the character of a representation whose restriction to Σ is irreducible. Then the torsion* $\mathrm{tor}(M, \mathrm{Ad} \circ \rho)$ *has vanishing order at v bounded by $-n(\chi(\Sigma) + 1)$ if the ideal point does not correspond to the character of an abelian representation of $\pi_1(\Sigma)$, and by $-n\chi(\Sigma) - m$ else, where m depends only on the restriction of ρ to $\pi_1(\Sigma)$.*

We say that a surface $S \subset M$ is *free* if its complement is a union of handlebodies. Many natural constructions yield such surfaces. For example, for a knot diagram and consider the checkerboard surfaces (for an example of such a surface, see Figure 1 on the left). If one of them, say Σ, is an incompressible non orientable surface in M, then the boundary of a neighborhood of Σ is orientable, remains incrompressible and does split M into two handlebodies, as can be easily seen (both part of its complement retract onto a graph). In fact, it is the case for every incompressible surfaces as soon as M is small. On the other hand, every incompressible suface whose class in $H_2(M, \partial M)$ is non zero will be splitting M. The distinction with the abelian case at the end of the theorem comes from the fact that this situation appears naturally in basic examples as the figure eight knot. We deduce from this theorem an unexpected relation between the genus of the character variety of M and the genus of the incompressible surfaces in M. More precisely, suppose that M is a knot complement whose character variety is one dimensional. Then, let us pick a smooth component of the variety, and assume that each ideal point $y \in Y$ corresponds to an incompressible surface Σ_y that verifies the hypothesis of the theorem. Let us further assume that the Alexander polynomial of M has only simple roots. Then

$$-\chi(Y) \leq \sum_y -n_y \chi(\Sigma_y) - m_y$$

where m_y is defined as in Theorem 2.0.2. In the simple case where the surfaces Σ are connected, it turns into

$$-\chi(Y) \leq \sum_y (-\chi(\Sigma_y) - 1).$$

Example 0.0.1. We know from [HT85] that the knot 5.2 has two incompressible surfaces in its complement: Σ_1 whose Euler characteristic is -4, and Σ_2 whose Euler characteristic is -2 (see figure 1).
The (geometric component of the) character variety has 3 ideal points, two of them corresponding to Σ_2, and the third to Σ_1. The torsion vanishes at order 1 on the Σ_2's ideal points, and at order 3 at the other. Hence on the augmented variety Y, one obtains $-\chi(Y) = 4 \times 1 + 2 \times 3 = 10$. The covering map $Y \to X(M)$ ramifies on six points, hence $-\chi(X) = \frac{1}{2}(-\chi(Y) - 6) = 2$, and the genus of X is 2, as it can be computed directly.

Question 0.0.2. Is the inequality of Theorem 2.0.2 an equality?

In all the examples we have listed in Section 2.2.3 for connected incompressible surfaces, it happens to be an equality. A careful examination of the proof shows that it has to be generically the case. The lack of equality should be interpreted as a non-transversal situation.
Furthermore, it also seems reasonable that the vanishing order of the torsion is always positive, and one may wonder:

Question 0.0.3. For which knot complements is the torsion a regular differential form on a component of the character variety?

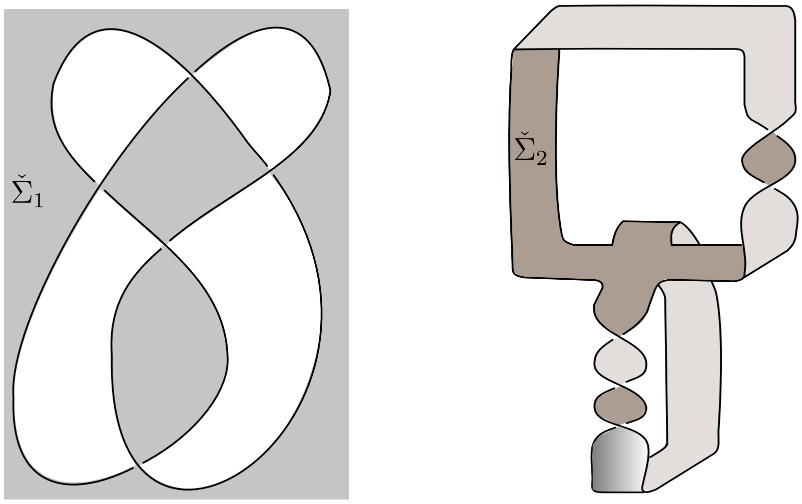


Figure 1 – Incompressible surfaces in the complement of the knot 5.2. The surface Σ_1 is the orientation covering of the non oriented surface here colored on the left, that can be thought as the boundary of a tubular neighborhood of this non orientable surface. The surface Σ_2 is again the orientation covering of the surface colored on the right, that can be obtained as follows: consider two parallel copies of each twisted bands above and below the square in the middle, and plumb them along this square. The result is connected because the bands below have an odd number of twists, and this is our surface Σ_2

From Theorem 2.0.1, it is so on the affine part of $\bar{X}(M)$. Yet the torsion could have a pole at infinity. In fact, examples of torus knots (p, q) provide such a situation: each component of the character variety is isomorphic to $\mathbb{C}$ with an unique ideal point, which corresponds to an essential annulus in the knot complement. Although Theorem 2.0.2 cannot apply since the fundamental group of an annulus have no non abelian representations, one can compute directly that the torsion has a pole of order one at those points.

Further work could concern reducible characters that are limits of irreducible characters, such as in Theorem 2.0.1, item 2. It is most likely that one could eventually weaken the hypothesis of this theorem. But this requires a better understanding of the relation between the first twisted cohomology group and the tangent space at this point. This complex problem is partially answered in [FK91, HPSP01].

The acyclic torsion

In Chapter 3, we focus on the acyclic torsion, and present the results of the article [Ben17]. The acyclic torsion $\mathrm{tor}(M, \rho)$ is defined as a non zero rational function on the character variety. More precisely, we take X to be a one-dimensional component of the character variety, and the torsion function is seen as an element of the function field $k(X)^*$. While it is not usually defined in the way it is here, the torsion function has been long established as such. The question of how to compute this function and whether or not it vanishes is still under investigation. It is known to be a constant function on the character varieties of torus knots. The first non constant computation was done by Kitano in [Kit94] on the geometric component of the figure eight knot's character variety . Since then, because of its proximity with the twisted Alexander polynomial, there has been many more studies of this torsion. In [DFJ12], the authors address several questions on the twisted Alexander polynomial. The acyclic torsion is the evaluation at $t = 1$ of this polynomial. As an application of our techniques, we prove that under some mild hypothesis, this torsion is non-constant on the geometric component of the character variety. In particular the twisted Alexander polynomial is non constant. The first result of this chapter is the following:

Theorem (Theorem 3.0.1)**.** *Let X be a geometric component of the character variety $X(M)$ of a hyperbolic manifold M. Then $\mathrm{tor}(M, \rho)$ is a regular function on X, that vanishes at a character χ if and only if the vector space $H^1(M, \bar{\rho})$ is non trivial, where $\bar{\rho}$ is a representation $\bar{\rho} : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ whose character is χ.*

The fact that the torsion has no poles on X was expected. Yet, to our knowledge, there has been no such formal statement to date, and it is interesting in and by itself. The characterization in terms of jump of dimension of the vector space $H^1(M, \bar{\rho})$ is also notalbe, because it relates the torsion with the deformation theory of semi-simple representations in $\mathrm{SL}_3(\mathbb{C})$. We provide more detailed explanations about this at the beginning of Chapter 3. Similarly to Chapter 2, we treat separately the case of ideal points in X. When saying that a curve $\gamma \in \pi_1(\Sigma)$ has trivial eigenvalue, we will mean that the representation of the fundamental group of the incompressible surface Σ corresponding to the ideal point maps γ on a matrix with eigenvalues equal to 1.

Theorem (Theorem 3.0.2)**.** *Let $x \in \hat{X}$ be an ideal point in the smooth projective model of X, and assume that an associated incompressible surface Σ is a union of parallel homeomorphic copies Σ_i such that $M \setminus \Sigma_i$ is a (union of) handlebodie(s). If the curve $\gamma = \partial\Sigma \in \pi_1(M)$ has trivial eigenvalues, then the torsion function $\mathrm{tor}(M, \rho)$ has a pole at x.*

We deduce the following corollary:

Corollary 0.0.4. *Let M be a hyperbolic manifold and X be a geometric component of its $\mathrm{SL}_2(\mathbb{C})$ character variety. Assume that an ideal point of X detects an incompressible surface which is connected or union of parallel free copies, and such that the eigenvalue of its boundary curve is 1. Then the torsion function is not constant on the component X.*

Here again, the hypothesis that the complement of any connected component of Σ is union of handlebodies is automatically satisfied if the manifold M is small. If M is a complement of a knot in an homology sphere, the hypothesis on the eigenvalue of the boundary curve is automatically satisfied for Σ a Seifert surface (a surface that bounds the knot), and we discuss it more generally in Chapter 3.
Since we expect the torsion to be non constant on the geometric component of any small hyperbolic three manifold, we adress the following question:

Question 0.0.5. Is it true that a geometric component of a small hyperbolic three manifold detects necessarily an incompressible surface whose boundary curve has eigenvalue 1?

Two-bridge knots are known to have non-Seifert incompressible surfaces with only 2 boundary components, in case which the eigenvalue of the boundary curve is ± 1. Yet, as mentioned above, any Seifert surface's boundary curve has eigenvalue 1. It thus seems reasonable to consider this question.

Chapter 1

Character varieties and Reidemeister torsion

In this first chapter we introduce the main objects that will be studied in this thesis. The first section deals with the theory of character varieties, the second section is a review about twisted cohomology, the third section is a short overview of the Culler-Shalen theory, and provides the tools that will be used in the upcoming chapters, the fourth section gives an insight on the Alexander module theory and its relation with the character varieties, and we the fifth section is an introduction to the Reidemeister torsion.

1.1 Character varieties

In this section we furnish the definitions relative to character varieties of a finitely generated group, we first compare the classical definition coming from Geometric Invariants Theory with the "trace functions" definition, then we state a theorem of Kyoji Saito. We use this theorem to define the tautological representation. We end this section with examples.

1.1.1 Representation variety

Let Γ be a finitely generated group, with $S = \{\gamma_1, \dots, \gamma_n\}$ a system of generators of Γ, and k be an algebraically closed field of characteristic zero.

Definition 1.1.1. A *representation* is a group homomorphism $\rho : \Gamma \to \mathrm{SL}_2(k)$. We define the *representation variety* $R(\Gamma) = \mathrm{Hom}(\Gamma, \mathrm{SL}_2(k)) = \{\rho : \Gamma \to \mathrm{SL}_2(k)\}$. It is an affine algebraic variety defined over k (and even over $\mathbb{Q}$). For any choice of S there is an embedding:

$$\begin{aligned}\iota_S : R(\Gamma) &\to \mathrm{SL}_2(k)^n \hookrightarrow k^{4n}\\ \rho &\mapsto (\rho(\gamma_1), \dots, \rho(\gamma_n))\end{aligned}$$

that does not depend on S in the following way: for any other $S' = \{\gamma'_1, \dots, \gamma'_m\}$, there is a group isomorphism $f_{S,S'} : \Gamma \to \Gamma$ that turns the generators γ_i into words $w_i(\gamma'_1, \dots, \gamma'_m)$. It induces an algebraic map $F_{S,S'} : k^{4n} \to k^{4m}$ such that $F_{S,S'} \circ \iota_S = \iota_{S'}$ and the restriction of $F_{S,S'} : \iota_S(R(\Gamma)) \to \iota_{S'}(R(\Gamma))$ is an isomorphism.
The algebra of functions of the representation variety is

$$k[R(G)] = k[X_\gamma^{i,j}, 1 \le i,j \le 2, \gamma \in \Gamma]/(X_e - \mathrm{Id}, X_\gamma X_\delta - X_{\gamma\delta}, \gamma, \delta \in \Gamma)$$

where for any $\gamma \in \Gamma, X_\gamma$ denotes the 4-tuple $\begin{pmatrix} X_\gamma^{1,1} & X_\gamma^{1,2} \\ X_\gamma^{2,1} & X_\gamma^{2,2} \end{pmatrix}$ and the multiplication is induced by the two-by-two matrices multiplication. This algebra is finitely generated, any choice of S provides a finite set of generators.

The group $\mathrm{SL}_2(k)$ acts by conjugation on the representation variety $R(\Gamma)$, but the topological quotient is not an algebraic variety in general, nor even a Hausdorff space, as is emphasized in the following example.

Example 1.1.2. Let $\Gamma = \mathbb{Z}$. The representation variety $R(\mathbb{Z})$ is isomorphic to $\mathrm{SL}_2(k)$ via $\rho \mapsto \rho(1)$. The one-parameter subgroup $\left\{M_t = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right\}_{t \in k^*}$ acts on $R(\Gamma)$ by conjugation, in particular $M_t \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} M_t^{-1} = \begin{pmatrix} 1 & t^2 \\ 0 & 1 \end{pmatrix}$, hence the whole family given by $\left\{\rho_t : 1 \mapsto \begin{pmatrix} 1 & t^2 \\ 0 & 1 \end{pmatrix}\right\}_{t \in k^*}$ lies in the same $\mathrm{SL}_2(k)$-orbit. In particular $\rho_0 : 1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \overline{\mathrm{SL}_2(k) \cdot \rho_1}$, despite ρ_0 and ρ_1 are not conjugated. In this exaple, the topological quotient $R(\mathbb{Z})/\mathrm{SL}_2(k)$ is not Hausdorff. The following section explains how to solve this issue.

1.1.2 Algebro-geometric quotient

The action of the group $\mathrm{SL}_2(k)$ on $R(\Gamma)$ induces a natural action on its algebra of functions $k[R(\Gamma)]$ by pre-composition. The sub-algebra of invariant functions $k[R(\Gamma)]^{\mathrm{SL}_2}$ is known to be finitely generated, although this is a delicate problem first answered by Hilbert in the late nineteenth century. There is an amount of good references on the topic, let us just mention [MFK93] and [KP96].

Recall that any finitely generated k-algebra A is the quotient of a polynomial algebra, in other words there is an exact sequence $0 \to I \to k[X_1, \ldots, X_n] \to A \to 0$. By Hilbert's basis theorem, the ideal I is finitely generated, hence A defines an algebraic set $V(I) \subset k^n$, namely the zero-locus of any generating set of polynomials for I. Up to isomorphism, this set does not depend on the presentation of A, hence we denote by $\operatorname{Spec}(A)$ the affine algebraic variety defined by A.

Remark 1.1.3. In general, the use of the term variety is reserved to irreducible and reduced algebraic sets. An *irreducible* set is a set which is not a reunion of two proper closed subsets. An *irreducible component* is a maximal irreducible subset. Given a ring R, $\operatorname{Spec}(R)$ is said to be *reduced* if R does not contain any nilpotent element. In particular, by mean of an irreducible component we will assume it to be reduced. We will call many algebraic objects varieties despite they have no reason to be irreducible, nor reduced.

Definition 1.1.4. We define the *character variety* $X(\Gamma) = R(\Gamma) /\!/ \operatorname{SL}_2(k)$ as the spectrum $\operatorname{Spec}(k[R(\Gamma)]^{\operatorname{SL}_2})$ of the sub-algebra of invariants.

It is usually called the algebro-geometric quotient of $R(\Gamma)$ by $\operatorname{SL}_2(k)$, let us list without proof some of its properties:

- It comes with a projection map $\pi : R(\Gamma) \to X(\Gamma)$ that satisfies the following universal property: for any SL_2-invariant morphism $f : R(\Gamma) \to Y$, with Y an algebraic variety, there is an unique map $f' : X(\Gamma) \to Y$ such that $f = f' \circ \pi$.
- The k-points of this quotient are in bijection with the closed orbits of $\operatorname{SL}_2(k)$ acting on $R(\Gamma)$, or with conjugacy classes of semi-simple (or completely reducible) representations of Γ into $\operatorname{SL}_2(k)$. In other words, orbits whose closure intersect in $R(\Gamma)$ are identified in $X(\Gamma)$.
- It is the biggest Hausdorff quotient of the topological quotient $R(\Gamma)/\operatorname{SL}_2(k)$.

Example 1.1.5. Back to $\Gamma = \mathbb{Z}$, recall that the $\operatorname{SL}_2(k)$-orbit of $\rho_1 : 1 \mapsto \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ contains the trivial representation ρ_0 in its closure. We are going to see how the theory of algebraic quotients solves this problem.
Since the variety $R(\mathbb{Z})$ is isomorphic to $\operatorname{SL}_2(k)$, what we want is to compute the subalgebra of invariant of the algebra $k[\operatorname{SL}_2] = {}^{k[X^{1,1}, X^{1,2}, X^{2,1}, X^{2,2}]}/_{(X^{1,1}X^{2,2} - X^{1,2}X^{2,1} - 1)}$. Assume $f \in k[\operatorname{SL}_2]$ is an invariant. Then for any diagonalizable $A \in \operatorname{SL}_2(k)$, there is some $\lambda \in k^*$ and $g \in \operatorname{SL}_2(k)$ such that $A = g \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) g^{-1}$ hence $f(A) = p(\lambda)$ for some polynomial $p \in k[t, t^{-1}]$ that does not depend on A. Moreover, the matrix $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ is conjugated to $\left(\begin{smallmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{smallmatrix}\right)$ hence $p \in k[t + t^{-1}]$. Since diagonalizable matrices

form a dense subset of $\mathrm{SL}_2(k)$, we conclude that $f \in {}^{k[X^{1,1}+X^{2,2}]}/_{(X^{1,1}X^{2,2}-1)}$, that is $f \in k[\mathrm{Tr}]$. Reciprocally, it is clear that the trace function is an invariant, hence the algebra of invariant is isomorphic to $k[\mathrm{Tr}]$.
In other words, any representation $\rho : \mathbb{Z} \to \mathrm{SL}_2(k)$ is equivalent to $1 \mapsto \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ for $\lambda \in k^*$, such that $\lambda + \lambda^{-1} = \mathrm{Tr}(\rho(1))$. In particular $\{\rho_t\}_{t \in k^*}$ and ρ_0 are identified in $X(\mathbb{Z})$, which is isomorphic to k.

It should be clear from this example that the computation of the algebra of invariants is in general somewhat difficult, on the other hand it provides a motivation to pay attention on a particular kind of functions, namely trace functions, which will play a key role in the next paragraph, and in the rest of this thesis.

Notation. When Γ is the fundamental group of a manifold M, its character variety will be denoted by $X(M)$ instead of $X(\Gamma)$.

1.1.3 Traces and skein algebra

Definition 1.1.6. For any $\gamma \in \Gamma$, we define the *trace function* $t_\gamma : R(\Gamma) \to k$ by $t_\gamma(\rho) = \mathrm{Tr}(\rho(\gamma))$. Those functions are invariant under the action of SL_2, thus they define functions on the quotient $X(\Gamma)$, that we still denote by t_γ.

The following lemma is straightforward, but crucial:

Lemma 1.1.7. *For any $\gamma, \delta \in \Gamma$, the identity $t_\gamma t_\delta = t_{\gamma\delta} + t_{\gamma\delta^{-1}}$ holds on $R(\Gamma)$.*

Proof. For any $A \in \mathrm{SL}_2(k)$, the Cayley-Hamilton theorem gives $A^2 - \mathrm{Tr}\, A \cdot A + \mathrm{Id} = 0$. After right-multiplying by $A^{-1}B$ and taking the trace, one obtains $\mathrm{Tr}(AB) - \mathrm{Tr}(A)\,\mathrm{Tr}(B) + \mathrm{Tr}(A^{-1})\,\mathrm{Tr}(B) = 0$, and the result follows. □

Theorem 1.1.8 ([Pro87])**.** *The algebra given by $B[\Gamma] = k[Y_\gamma, \gamma \in \Gamma]/(Y_e - 2, Y_\gamma Y_\delta - Y_{\gamma\delta} - Y_{\gamma\delta^{-1}}, \gamma, \delta \in \Gamma)$ is isomorphic to the algebra of invariants $k[X(\Gamma)]$.*

The proof of this theorem is rather abstract, one can see [PS00, Mar15] for instance. On the other hand, it is enlightening to convince oneself that those algebras define the same points as an algebraic set, that is conjugacy-class of semi-simple representations are fully determined by their traces. But the algebras can hold more information than the algebraic sets they define, for instance if they contain nilpotent elements. Despite we will not give a proof of this theorem, we shall state and prove the following proposition:

Proposition 1.1.9. *[CS83] Let Γ be a finitely generated group, then the algebra $B[\Gamma]$ is finitely generated.*

Proof. Let $\gamma_1, \ldots, \gamma_n$ be a set of generators of Γ, we define the sub-algebra $B_0 = k[Y_{\gamma_{i_1}\cdots\gamma_{i_r}}, i_k \in \{1, \ldots, n\}, i_k \neq i_l] \subset B[\Gamma]$. We want to prove that $B_0 = B[\Gamma]$.

— Consider the case when $Y_\gamma \in B[\Gamma]$ is of the form $\gamma = \gamma_{i_1}^{m_1} \ldots \gamma_{i_r}^{m_r}$ with the i_k's distincts. Define $K_j = -m_j$ if $m_j \leq 0$, and $K_j = m_j - 1$ if $m_j > 0$. We prove that $Y_\gamma \in B_0$ by induction on $v = \sum_{j=1}^{r} K_j$.
If $v = 0$, then the m_j's all lie in $\{0, 1\}$, hence $Y_\gamma \in B_0$ by definition. If $v > 0$, as $Y_{\gamma\delta} = Y_{\delta\gamma}$, we can assume that $m_r \neq 0$. Moreover, conjugation by γ_{i_r} leaves v invariant, hence we assume that $m_r \notin \{0, 1\}$, and then the trace formula allows us to write Y_γ as a sum of two terms with strictly smaller v's, and we conclude by induction.

— If not all i_k's are distincts, again we can assume that $i_s = i_r$ for some $s < r$. Then we cut Y_γ with $\alpha = \gamma_{i_1}^{m_1} \ldots \gamma_{i_s}^{m_s}$ and $\beta = \gamma_{i_{s+1}}^{m_{s+1}} \ldots \gamma_{i_r}^{m_r}$ by writing $Y_\gamma = Y_\alpha Y_\beta - Y_{\alpha\beta^{-1}}$, and step by step, we decrease v to fall in the former case.

□

Another motivation to introduce the algebra $B[\Gamma]$ is its relation with the so-called skein algebra.

Definition 1.1.10. Let M be a compact three-manifold. A *framed link* in M is a smooth embedding of several copies of an annulus $i : \mathbb{S}^1 \times [0,1] \sqcup \ldots \sqcup \mathbb{S}^1 \times [0,1] \hookrightarrow M$. Two links are equivalent if they are isotopic in M. The *Kauffman skein module* $\mathcal{S}_A(M)$ is the quotient of the $k[A, A^{-1}]$-module generated by isotopy classes $[L]$ of framed links in M, up to the so-called *Kauffman relations* depicted in Figure 1.1.

The following is well-known to specialists:

Proposition 1.1.11. *If M is a 3-manifold, and $\Gamma = \pi_1(M)$, then the morphism of k-algebras $B[\Gamma] \simeq \mathcal{S}_A(M) \otimes_{k[A,A^{-1}]} {}^{k[A,A^{-1}]}/_{(A+1)}$ that maps Y_γ to $-[\gamma]$ is an isomorphism.*

Proof. We denote the latter by $\mathcal{S}_{-1}(M)$, note that the algebra structure is induced by disjoint union of links, and is well-defined since the relation $A = A^{-1} = -1$ implies that crossings above or below are equivalent. Then the following pictures (Figure 1.2) show that the trace relation and the relation induced by the Kauffman relations on the algebra structure of $\mathcal{S}_1(M)$ are equivalent, hence those reciprocal maps are well-defined. □

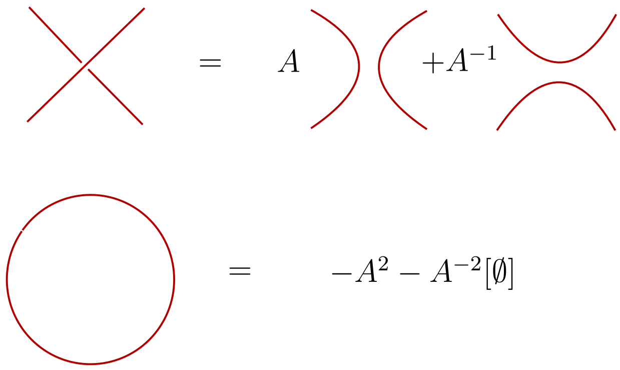


Figure 1.1 – Kauffman skein relations. The indicated moves are performed locally, the rest of the link remains unchanged. The strands depict bands of the framed links.

1.1.4 Characters, reducibility and irreducibility

From the previous definition of the character variety through trace functions, we have the following definition of points of the character variety, which enlightens on the terminology.

Definition 1.1.12. A *k-character* is a k-point of the character variety in the sense of algebraic geometry, that means a morphism $\chi : B[\Gamma] \to k$. Any representation $\rho : \Gamma \to \mathrm{SL}_2(k)$ induces a character $\chi_\rho : B[\Gamma] \to k$ that sends Y_γ to $\mathrm{Tr}(\rho(\gamma))$.

Remark 1.1.13. This definition generalizes to R-characters for any k-algebra R.

Definition 1.1.14. A representation $\rho : \Gamma \to \mathrm{SL}_2(k)$ is *reducible* if there exists an invariant line in k^2, and *irreducible* if not.

The following standard lemma tells us that this notion can be defined directly on characters. For any elements $\alpha, \beta \in \Gamma$, we will denote the commutator $\alpha\beta\alpha^{-1}\beta^{-1}$ by $[\alpha, \beta]$.

Lemma 1.1.15. *Let $k \subset \mathbb{K}$ be a field extension (we take $\mathbb{K}$ to be either k, either a transcendental extension of k). A representation $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{K})$ is absolutely irreducible (irreducible in an algebraic closure) iff there exists $\alpha, \beta \in \Gamma$ such that* $\mathrm{Tr}(\rho(\alpha\beta\alpha^{-1}\beta^{-1})) \neq 2$.

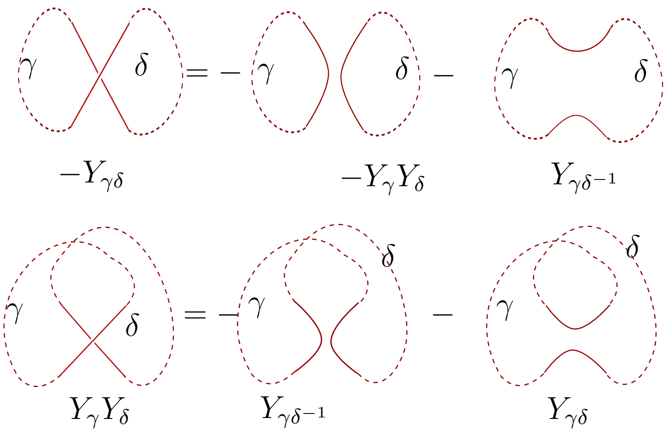


Figure 1.2 –

Proof. If ρ is reducible on $\bar{\mathbb{K}}$, then one can write ρ as a subgroup of $\mathrm{SL}_2(\bar{\mathbb{K}})$ with every element of Γ mapped to an upper-triangular matrix. It is clear thus that the commutator of any two elements have trace equal to 2. Conversely, if any commutator's trace is 2, then any commutator fixes a non trivial subspace of $\bar{\mathbb{K}}^2$. Assume $[\alpha, \beta]$ and $[\delta, \gamma]$ have two distinct fixed lines, they provide a basis of $\bar{\mathbb{K}}^2$ such that $\rho([\alpha, \beta]) = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ and $\rho([\delta, \gamma]) = \left(\begin{smallmatrix} 1 & 0 \\ y & 1 \end{smallmatrix}\right)$ for some non-zero $x, y \in \bar{\mathbb{K}}$. Now we compute $\mathrm{Tr}(\rho([[\alpha, \beta], [\delta, \gamma]])) = 2 + (xy)^2$, a contradiction. Hence the image of the commutator subgroup has an invariant subspace in $\bar{\mathbb{K}}^2$. We prove that it implies that ρ is reducible ; denote by F the invariant subspace of $[\Gamma, \Gamma]$, let $\alpha \in \Gamma \setminus [\Gamma, \Gamma]$ and $\beta \in [\Gamma, \Gamma]$ with $\rho(\beta) \neq \mathrm{Id}$. Then $[\rho(\beta)^{-1}, \rho(\alpha)^{-1}].F = F$ and $\rho(\beta).F = F$, hence $\rho(\beta)$ stabilizes $\rho(\alpha).F$, and $\rho(\alpha).F = F$. □

Definition 1.1.16.

- For any $\alpha, \beta \in \Gamma$, we denote by $\Delta_{\alpha,\beta} \in B[\Gamma]$ the function $Y_\alpha^2 + Y_\beta^2 + Y_{\alpha\beta}^2 - Y_\alpha Y_\beta Y_{\alpha\beta} - 4 = Y_{[\alpha,\beta]} - 2$. For any k-algebra R, we will say that an R-character χ is *irreducible* if there exists $\alpha, \beta \in \Gamma$ such that $\chi(\Delta_{\alpha,\beta}) \neq 0$. If not, we say that it is *reducible*. A character χ will be said *central* if $\chi(Y_\gamma)^2 = 4$ for any $\gamma \in \Gamma$.
- Since being reducible is a Zariski closed condition, any irreducible component $X \subset X(\Gamma)$ that contains only reducible characters will be said *of reducible type*. A component that contains an irreducible character (equivalently, a dense open subset of irreducible characters) will be said *of irreducible type*.

1.1.5 Saito's Theorem and some consequences

From the equivalence of the two definitions of the character variety, we know that any character $\chi \in X(\Gamma)$ is the character of a representation $\rho : \Gamma \to \mathrm{SL}_2(k)$. The following theorem of Kyoji Saito [Sai94] generalizes this fact and has crucial applications for us. We detail the proof in Appendix A, because it is not available in the literature.

Theorem 1.1.17. *Let R be a k-algebra, and $\chi : B[\Gamma] \to R$ an R-character. Assume that $\chi(\Delta_{\alpha,\beta})$is invertible for some $\alpha, \beta \in \Gamma$, and let $A, B \in \mathrm{SL}_2(R)$ such that $\mathrm{Tr}\, A = \chi(Y_\alpha), \mathrm{Tr}\, B = \chi(Y_\beta), \mathrm{Tr}\, AB = \chi(Y_{\alpha\beta})$. Then there exists a unique representation $\rho : \Gamma \to \mathrm{SL}_2(R)$ whose character is χ and such that $\rho(\alpha) = A$ and $\rho(\beta) = B$.*

As a application of this theorem, one can deduce the following proposition, see [Mar15].

Proposition 1.1.18. *Let $\mathbb{K}$ be either an algebraically closed field or a degree one extension of an algebraically closed field . Then the $\mathbb{K}$-irreducible characters correspond bijectively to GL_2-conjugacy classes of irreducible representations $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{K})$.*

Idea of the proof. Given an irreducible character $\chi : B[\Gamma] \to \mathbb{K}$, we fix $\alpha, \beta \in \Gamma$ such that $\chi(\Delta_{\alpha,\beta})$ is invertible and in order to use Saito's Theorem, we produce $A = \begin{pmatrix} \chi(Y_\alpha) & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -\frac{1}{u} \\ u & \chi(Y_\beta) \end{pmatrix}$ for some u in an at most quadratic extension $\hat{\mathbb{K}}$ of $\mathbb{K}$ such that $u + u^{-1} = \chi(Y_{\alpha\beta})$. Then we obtain a representation $\rho : \Gamma \to \mathrm{SL}_2(\hat{\mathbb{K}})$ whose character is χ, and general considerations about Brauer groups implies that ρ could have been picked, in fact, in $\mathrm{SL}_2(\mathbb{K})$.
The converse map is given by $\chi(Y_\gamma) = \mathrm{Tr}(\rho(\gamma))$. □

Remark 1.1.19. This proposition enlightens about the quotient map $\pi : R(\Gamma) \to X(\Gamma)$. It says that, given an irreducible k-character $\chi \in X(\Gamma)$, all representations in its fiber $\pi^{-1}\{\chi\}$ are pairwise conjugated, hence the fiber is isomorphic to the whole group $\mathrm{PSL}_2(k)$. It is often summarized saying that on the irreducible open subset of the character variety, the algebraic quotient is a principal $\mathrm{PSL}_2(k)$ bundle. On the other hand, both statement are false outside of the irreducible locus of $X(\Gamma)$: in Example 1.1.5, we have seen that the fiber of a central character contains non trivial representations, and the fibers of the map π are 2-dimensional, hence are not isomorphic to $\mathrm{PSL}_2(k)$.

1.1.6 The augmented variety

In this subsection we define a two-fold cover of the character variety when Γ is the fundamental group of a 3-manifold with boundary a single torus, which has been previously considered by several authors, see for instance [DG16]. Both the definition and the motivation arise from the following case. This augmented variety will be used in Chapter 2.

Definition 1.1.20 (The character variety $X(\partial M) = X(\mathbb{Z}^2)$)**.** Let $\chi \in X(\partial M)$ be a k-character, it is the character of a semi-simple representation $\rho : \mathbb{Z}^2 \to \mathrm{SL}_2(k)$. Since $\mathbb{Z}^2$ is abelian, it can be written $\rho(\gamma) = \begin{pmatrix} \lambda(\gamma) & 0 \\ 0 & \lambda^{-1}(\gamma) \end{pmatrix}$, and $\chi(Y_\gamma) = \lambda(\gamma)+\lambda^{-1}(\gamma)$ for some $\lambda \in H^1(\partial M, k^*)$. Moreover, there is an involution σ of $H^1(\partial M, k^*)$ that sends λ to λ^{-1}, and clearly what we have just described is a morphism $X(\partial M) \to H^1(\partial M, k^*)/\sigma \simeq (k^*)^2/\sigma$ that turns out to be an isomorphism.
On the functions algebra side, there is a map from $B[\mathbb{Z}^2]$ into the algebra $C[\mathbb{Z}^2] = k[Z_\gamma, \gamma \in \mathbb{Z}^2]/(Z_\gamma Z_\delta - Z_{\gamma\delta})$, the latter being isomorphic to $k[X, X^{-1}, Y, Y^{-1}]$, that sends $Y_\gamma \mapsto Z_\gamma + Z_{\gamma^{-1}}$. Its image lies in the σ-invariant part $C[\mathbb{Z}^2]^\sigma \simeq k[X+X^{-1}, Y+Y^{-1}, XY+(XY)^{-1}]$, for $\sigma : Z_\gamma \mapsto Z_{\gamma^{-1}}$ and we get an isomorphism $B[\mathbb{Z}^2] \simeq C[\mathbb{Z}^2]^\sigma$. The double cover we are going to define is $H^1(\partial M, k^*)$ in this example, it corresponds to a choice of an eigenvalue function Z_γ for $\gamma \in \partial M$ a boundary curve.

Definition 1.1.21. We define the *augmented representation variety* $\bar{R}(M)$ to be the subvariety of $R(M) \times H^1(\partial M, k^*)$ given by the pairs $\{(\rho, \lambda), \rho : \pi_1(M) \to \mathrm{SL}_2(k), \lambda : \pi_1(\partial M) \to k^*, \lambda(\gamma) + \lambda(\gamma)^{-1} = \mathrm{Tr}\,\rho(\gamma)\ \forall \gamma \in \pi_1(\partial M)\}$.
Again $\mathrm{SL}_2(k)$ acts on $\bar{R}(M)$, trivially on the second factor, thus we define the *augmented character variety* to be the quotient $\bar{X}(M) = \bar{R}(M) /\!/ \mathrm{SL}_2(k)$.

The advantage of this two-fold covering is the following: on one hand the functions of $X(M)$ are trace functions, on the other hand on $\bar{X}(M)$ we have at our disposal, for any $\gamma \in \pi_1(\partial M)$, two *eigenvalue functions* $Z_{\gamma^{\pm 1}}$ that maps the pair (ρ, λ) to an eigenvalue $\lambda(\gamma)$ for $\rho(\gamma)$.
The following definition provides a direct description of the algebraic quotient.

Definition 1.1.22. Recall that the algebra $C(\pi_1(\partial M)$ is defined by $k[Z_\gamma, \gamma \in \pi_1(\partial M)]/(Z_\gamma Z_\delta - Z\gamma Z\delta)$. Denote by $\bar{B}[\Gamma] = B[\Gamma] \otimes_{B[\pi_1(\partial M)]} C[\pi_1(\partial M)]$ and we define the augmented character variety as the fibered product:

$$\bar{X}(M) = X(M) \times_{X(\partial M)} H^1(\partial M, k^*)$$

that is, $\bar{X}(M) = \mathrm{Spec}\,\bar{B}[\Gamma]$.

1.1.7 The tautological representation

Let X be an irreducible component of $X(M)$ of irreducible type (in the sense of Definition 1.1.16).

The component X corresponds to a minimal prime ideal $\mathfrak{p}$ of $B[\Gamma]$ such that $k[X] = B[\Gamma]/\mathfrak{p}$ is the function algebra of X. Denote by $k(X)$ the fraction field of $k[X]$, and by χ_X the composition $B[\Gamma] \to k[X] \to k(X)$, it is irreducible as a $k(X)$-character. The following is an immediate consequence of Proposition 1.1.18, since one-dimensional varieties over k have a function field which has transcendance degree 1 over k.

Proposition 1.1.23. *Let X be a one-dimensional component of irreducible type of $X(M)$. Then there is a representation $\rho_X : \Gamma \to \mathrm{SL}_2(k(X))$, called the tautological representation, defined up to conjugation, whose character is χ_X.*

Remark 1.1.24. The whole paragraph above can be rephrased in terms of the augmented character variety, with $\bar{X}$ a one-dimensional component of irreducible type of $\bar{X}(M)$ and $\rho_{\bar{X}} : \Gamma \to \mathrm{SL}_2(k(\bar{X}))$ the tautological representation. The component $\bar{X}$ is said of irreducible type if it projects on a component of irreducible type of $X(M)$. The striking point of the augmented construction is that the restricted tautological representation $\rho_\partial : \pi_1(\partial M) \to \pi_1(M) \to \mathrm{SL}_2(k(\bar{X}))$ is now diagonalizable.

1.1.8 Examples

The character variety of $\Gamma = \mathbb{Z}$

We already computed the character variety $X(\mathbb{Z}) \simeq k$ with parameter $t = \mathrm{Tr}\,\rho(1)$, it has a double cover similar to the augmented variety which is k^* with parameter u, where $u + u^{-1} = t$. The tautological representation is $\rho : \mathbb{Z} \to \mathrm{SL}_2(k(u))$ defined by $\rho(1) = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$. Remark that, as predicted by Proposition 1.1.18, the tautological exists in $\mathrm{SL}_2(k(t))$ too, one can see it by sending 1 to $\begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix}$, but this is less convenient for computations.

The free group with two generators $\Gamma = \langle \alpha, \beta \rangle$

Using the trace formula, it is clear that the algebra $B[\Gamma]$ is generated by $Y_\alpha, Y_\beta, Y_{\alpha\beta}$. Moreover, those functions form a basis: given a relator $P(Y_\alpha, Y_\beta, Y_{\alpha\beta}) = 0$, one proves that $P = 0$ by setting $\rho(\alpha) = \begin{pmatrix} X & 1 \\ -1 & 0 \end{pmatrix}$ and $\rho(\beta) = \begin{pmatrix} 0 & -Z^{-1} \\ Z & Y \end{pmatrix}$. One gets $P(X, Y, Z + Z^{-1}) = 0$, hence P is trivial in $k(X, Y, Z)$. The function ring is then $k[X, Y, Z + Z^{-1}]$, and the character variety is isomorphic to k^3. The representation

ρ defined below is the tautological representation in $\mathrm{SL}_2(k(X,Y,Z))$ and there is no reason that it exists in $\mathrm{SL}_2(k(X,Y,Z+Z^{-1}))$, because the Brauer group of this field is not trivial.

The trefoil knot

Here M is the exterior of the trefoil knot in $\mathbb{S}^3$, $\pi_1(M) = \langle a, b|\ a^2 = b^3\rangle$. Denote by $z = a^2 = b^3$, it generates the center of Γ. Hence any irreducible representation ρ needs to map z onto $\pm\,\mathrm{Id}$. If $\rho(z) = \mathrm{Id}$, then $\rho(a) = -\,\mathrm{Id}$ and necessarily ρ becomes abelian, thus we fix $\rho(z) = -\,\mathrm{Id}$. Up to conjugacy, we fix $\rho(b) = \left(\begin{smallmatrix} -j & 0 \\ 0 & -j^2 \end{smallmatrix}\right)$. One can still conjugate ρ by diagonals matrices without modifying $\rho(b)$, thus one can fix the right-upper entry of $\rho(a)$ to be equal to 1 ; and as $\rho(a)^2 = -\,\mathrm{Id}$, the Cayley-Hamilton theorem implies that $\mathrm{Tr}\,\rho(a) = 0$, hence $\rho(a) = \left(\begin{smallmatrix} t & 1 \\ -(t^2+1) & -t \end{smallmatrix}\right)$, for some $t \in k$. As $(j-j^2)t = \mathrm{Tr}(ab^{-1})$, the function field of the component of irreducible type X is $k(t)$; and $X \simeq k$. The latter representation ρ is the tautological representation.
The augmented character variety is obtained by picking an eigenvalue of any boundary curve. Consider the meridian ab^{-1}, its trace is $(j-j^2)t$, hence the field extension $u + u^{-1} = (j-j^2)t$ provides a double covering $\bar{X} \to X$, that ramifies twice, when $(j-j^2)^2t^2 = 4$. Note that $\bar{X}$ is isomorphic to k^*.

The figure-eight knot

Here M denotes the exterior of the figure-eight knot in $\mathbb{S}^3$, $\pi_1(M) = \langle u, v|\ vw = wu\rangle$ with $w = [u, v^{-1}]$. Note that the meridians u and v are conjugated, hence they define the same trace functions. Denote by $x = Y_u = Y_v$, and by $y = Y_{uv}$, then $B[\pi_1(M)] = k[x,y]/(P)$ where $P(x,y) = (x^2-y-2)(2x^2+y^2-x^2y-y-1)$ is obtained by expanding the relation $\mathrm{Tr}\,vw = \mathrm{Tr}\,wu$ with the help of the trace relation. The first factor of P is the equation of the component of reducible type, and we denote by X the curve defined by the second factor of P. It is a smooth plane curve of degree 3, and the Plücker formula implies that its compactification has genus 1. The augmented variety $\bar{X} \to X$ is described as follows: add the equation $\alpha + \alpha^{-1} = x$, one obtain a curve in k^3, that ramifies at four points $\{x^2 = 4, y^2 - 5y + 7 = 0\}$ on X, hence the compactification of $\bar{X}$ has genus 3 by the Riemann-Hurwitz formula. The tautological representation $\rho : \Gamma \to k(\bar{X})$ can be defined by

$$\rho(u) = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha^{-1} \end{pmatrix}, \rho(v) = \begin{pmatrix} \alpha & 0 \\ y - \alpha^2 - \alpha^{-2} & \alpha^{-1} \end{pmatrix}$$

1.2 Twisted cohomology

In this section we recall the basics of twisted cohomology that we will use in the sequel of the manuscript. Given a group Γ, we will denote by $\mathbb{Z}[\Gamma]$ the group ring of Γ, whose elements are formal sums $\sum_{\gamma\in\Gamma} n_\gamma e_\gamma, n_\gamma \in \mathbb{Z}$ with multiplication $e_\gamma e_\delta = e_{\gamma\delta}$ extended by linearity.
We fix an abelian group A together with a left-action $\Gamma \times A \to A, (\gamma, a) \mapsto \gamma \cdot a$. It induces a right-action by $a \cdot \gamma = \gamma^{-1} \cdot a$. It turns A into a $\mathbb{Z}[\Gamma]$-module.

Definition 1.2.1 (Group homology and cohomology)**.** Let $E\Gamma$ be the *full simplex set* over Γ, that is, the vertices of $E\Gamma$ are in bijection with Γ, and any finite part of Γ of cardinal n defines an $n-1$ simplex in $E\Gamma$. This space is contractile, and the natural transitive action of Γ on itself by left multiplication induces an action on $E\Gamma$, denote by $B\Gamma = E\Gamma/\Gamma$.
Define the *complex of twisted chains* as the $\mathbb{Z}[\Gamma]$-modules:

$$\begin{aligned} &C_k(\Gamma, A) = A \otimes_{\mathbb{Z}[\Gamma]} C_k(E\Gamma) \\ &= \{a \otimes [\gamma_0, \ldots, \gamma_k] : \forall \gamma \in \Gamma, a \cdot \gamma \otimes [\gamma_0, \ldots, \gamma_k] = a \otimes [\gamma\gamma_0, \ldots, \gamma\gamma_k]\} \end{aligned}$$

and the *complex of twisted co-chains* as the $\mathbb{Z}[\Gamma]$-modules:

$$\begin{aligned} &C^k(\Gamma, A) = \mathrm{Hom}_{\mathbb{Z}[\Gamma]}(C_k(E\Gamma), A) \\ &= \{f : \Gamma^{k+1} \to A : f(\gamma\gamma_0, \ldots, \gamma\gamma_k) = \gamma \cdot f(\gamma_0, \ldots, \gamma_k)\} \end{aligned}$$

with boundary maps given by $\partial(a \otimes [\gamma_0, \ldots, \gamma_k]) = \sum_i (-1)^i a \otimes [\gamma_0, \ldots, \hat{\gamma}_i, \ldots, \gamma_k] \in C_{k-1}(\Gamma, A)$ and $df(\gamma_0, \ldots, \gamma_{k+1}) = \sum_i (-1)^i f(\gamma_0, \ldots, \hat{\gamma}_i, \ldots, \gamma_{k+1}) \in C^{k+1}(\Gamma, A)$. Recall that the hat notation means that the term with a hat does not appear.
As usual we denote by

$$Z_k(\Gamma, A) = \ker \partial : C_k(\Gamma, A) \to C_{k-1}, Z^k(\Gamma, A) = \ker d : C^k(\Gamma, A) \to C^{k+1}(\Gamma, A)$$

and

$$B_k(\Gamma, A) = \mathrm{im}\, \partial : C_{k+1}(\Gamma, A) \to C_k(\Gamma, A), B^k(\Gamma, A) = \mathrm{im}\, d : C^{k-1}(\Gamma, A) \to C^k(\Gamma, A)$$

We define the twisted homology $\mathbb{Z}$-modules as

$$H_k(\Gamma, A) = Z_k(\Gamma, A)/B_k(\Gamma, A)$$

and the twisted cohomology $\mathbb{Z}$-modules as

$$H^k(\Gamma, A) = Z^k(\Gamma, A)/B^k(\Gamma, A)$$

Remark 1.2.2. When $A = \mathbb{Z}$ and Γ acts trivially, then what we defined is the homology (resp cohomology) of the space $B\Gamma$. We will generalize this observation later.

Remark 1.2.3. For computation, we will use the following non homogeneous convention for the group homology and cohomology: the homological complex will be

$$C_i(\Gamma, A) = \{a \otimes [\gamma_1, \ldots, \gamma_i] \mid a \in A, \gamma_k \in \Gamma\}$$

with boundary

$$\delta(a \otimes [\gamma_1, \ldots, \gamma_i]) = a.\gamma_1 \otimes [\gamma_2, \ldots, \gamma_i] - a \otimes [\gamma_1\gamma_2, \ldots, \gamma_i] + \ldots + (-1)^i a \otimes [\gamma_1, \ldots, \gamma_{i-1}]$$

and the cohomological complex will be

$$C^i(\Gamma, A) = \{f : G^i \to A\}$$

with boundary

$$df(\gamma_1, \ldots, \gamma_i) = \gamma_1.f(\gamma_2, \ldots, \gamma_i) - f(\gamma_1\gamma_2, \ldots, \gamma_i) + \ldots + (-1)^i f(\gamma_1, \ldots, \gamma_{i-1})$$

It is a straightforward computation to show that those conventions define isomorphic complexes, hence the same homology.

In the sequel we fix a finite connected CW-complex K with universal cover $\tilde{K}$, which will be in general a cell decomposition of a compact manifold M and we denote by $\Gamma = \pi_1(M)$ its fundamental group. If $\{e_1^k, \ldots, e_{r_k}^k\}$ is the family of k-dimensional cells of K, we denote by $\{\tilde{e}_1^k, \ldots, \tilde{e}_{r_k}^k\}$ a choice of lifts in $\tilde{K}$, it provides a $\mathbb{Z}[\Gamma]$-basis of the $\mathbb{Z}[\Gamma]$-module $C_k(\tilde{M})$.
Again we fix an abelian group A, and a left-action of Γ on A.

Definition 1.2.4 (Twisted homology and cohomology)**.** We define the homological and cohomological twisted complexes

$$C_k(M, A) = A \otimes_{\mathbb{Z}[\Gamma]} C_k(\tilde{M}) \text{ and } C^k(M, A) = \mathrm{Hom}_{\mathbb{Z}[\Gamma]}(C_k(\tilde{M}), A)$$

with boundary maps $\partial(a \otimes e_i^k) = a \otimes \partial e_i^k$ and $df(e_i^k) = (-1)^i f(\partial e_i^k)$. Finally, the twisted homology and cohomology of M are the $\mathbb{Z}$-modules $H_k(M, A)$ and $H^k(M, A)$.

This definitions extend naturally to relative homology and cohomology in the case when M has a non-empty boundary ∂M.

Remark 1.2.5. The generalization of Remark 1.2.2 is that the homology (resp. cohomology) $H_*(\Gamma, A)$ of the group Γ is the twisted homology (resp. cohomology) $H_*(B\Gamma, A)$ of the space $B\Gamma$.

A classical fact, that is a consequence of this remark, is the following theorem, see for instance [Bro82].

Theorem 1.2.6. *If $\tilde{M}$ is contractible, then we have the natural isomorphisms*

$$H_k(M, A) \simeq H_k(\Gamma, A) \text{ and } H^k(M, A) \simeq H^k(\Gamma, A)$$

Moreover, without hypothesis on M, the following is proved in [Por97, Lemma 0.6].

Proposition 1.2.7. *For $i = 0, 1$, we have the natural isomorphisms:*

$$H_i(M, A) \simeq H_i(\Gamma, A) \text{ and } H^i(M, A) \simeq H^i(\Gamma, A)$$

We will be interested in the case when A is a free module of finite type (or a vector space) V over a k-algebra R, together with a bilinear non-degenerate invariant form $B : V \times V \to R$, hence we can define the following operation in homology and cohomology:

— The Kronecker product

$$\begin{aligned} \langle \cdot, \cdot \rangle : H^k(M, V) \times H_k(M, V) &\to R \\ (f, v \otimes e_i^k) &\mapsto B(v, f(e_i^k)) \end{aligned}$$

— The Universal Coefficient Theorems, here V, V' are assumed to be modules over a PID R :

$$0 \to H_k(M, V) \otimes_R V' \to H_k(M, V \otimes_R V') \to \mathrm{Tor}(H_{k-1}(M, V), V') \to 0$$

$$0 \to H^k(M, V) \otimes_R V' \to H^k(M, V \otimes_R V') \to \mathrm{Tor}(H^{k+1}(M, V), V') \to 0$$

$$0 \to \mathrm{Ext}(H_{k-1}(M, V), V') \to H^k(M, V') \to \mathrm{Hom}_R(H_k(M, V), V') \to 0$$

$$0 \to \mathrm{Ext}(H^{k+1}(M, V), V') \to H_k(M, V') \to \mathrm{Hom}_R(H^k(M, V), V') \to 0$$

— The Poincaré-Lefschetz duality $H_i(M, \partial M, V) \simeq H_{n-i}(M, V)^* \simeq H^{n-i}(M, V)$ where $n = \dim M$

— The cup-product:

$$\cdot \cup \cdot : H^i(\Gamma, V) \times H^j(\Gamma, V) \to H^{i+j}(\Gamma, R)$$

$$(f, g) \mapsto \big((\gamma_1, \ldots, \gamma_i, \gamma_{i+1}, \ldots, \gamma_{i+j}) \mapsto (-1)^{ij} B(f(\gamma_1, \ldots, \gamma_i), g(\gamma_{i+1}, \ldots, \gamma_{i+j}))\big)$$

1.3 Culler-Shalen theory, group acting on trees and incompressible surfaces

In this section we partially describe the so-called Culler-Shalen theory. In seminal articles [CS83, CS84], Marc Culler and Peter Shalen managed to use both tree-theoretical techniques introduced by Hyman Bass and Jean-Pierre Serre in [SB77] and character varieties to study the topology of 3-manifolds. In Subsections 1,2,3 we describe the Bass-Serre tree together with its natural SL_2 action, in Subsection 4 we explain how to use this theory in the context of character varieties.

1.3.1 The tree

Let $\mathbb{K}$ be an extension of k, we define a *k-discrete valuation* as a surjective map $v : \mathbb{K} \to \mathbb{Z} \cup \{\infty\}$ such that

— $v(0) = +\infty$

— $v(x + y) \geq \min(v(x), v(y))$

— $v(xy) = v(x) + v(y)$

— $\forall z \in k, v(z) = 0$ and k is maximal for this property.

We call $\mathcal{O}_v = \{x \in \mathbb{K} : v(x) \geq 0\}$ *the valuation ring*, and we pick $t \in \mathcal{O}_v$ an element of valuation 1, that we call a *uniformizing parameter*. The group of invertible elements $\mathcal{O}_v^*$ is the group of elements whose valuation is zero, (t) is the unique maximal ideal of $\mathcal{O}_v$, $\mathcal{O}_v/(t) \simeq k$ is the *residual field*. Remark that every ideal is of the form (t^n), for some $n \in \mathbb{N}$. The main exemple to have in mind here is the valuation ring $\mathbb{C}[[t]]$ of formal series in t, with the valuation $v : \mathbb{C}((t))^* \to \mathbb{Z}, P \mapsto \mathrm{ord}_t(P)$ given by the vanishing order at $t = 0$. The uniformizing element is t and the residual field is $\mathbb{C}$.

A *lattice* L in a two dimensional $\mathbb{K}$-vector space V is a free $\mathcal{O}_v$-module of rank two that spans V as a vector space. The group $\mathbb{K}^*$ acts on the set of lattices in V by

homothety . We denote by T the set of equivalence classes, that is $L \sim L'$ iff there exists $x \in \mathbb{K}^*$ such that $L' = xL$.
We are going to define an integer-valued distance on T. We fix a lattice L together with a basis of L, say $\{e, f\}$. For any class $[L'] \in T$ one can express a basis of $L' \in [L']$ as $\{ae + bf, ce + df\}$ with $a, b, c, d \in \mathbb{K}$; and up to homothety, we can pick in fact $a, b, c, d \in \mathcal{O}_v$, that is $L' \subset L$. Assume a has minimal valuation among $\{a, b, c, d\}$, then the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ can be transformed into $\begin{pmatrix} a & 0 \\ 0 & d-\frac{bc}{a} \end{pmatrix}$ by $\mathrm{SL}_2(\mathcal{O}_v)$ right and left multiplication that preserves the standard lattice $\mathcal{O}_v^2$. Hence $L' \simeq a\mathcal{O}_v \oplus (d - \frac{bc}{a})\mathcal{O}_v \simeq t^n\mathcal{O}_v \oplus t^m\mathcal{O}_v$ for some $n, m \in \mathbb{N}$. We define $d([L], [L']) = |n - m|$, and one can check that it defines a distance that does depend only on $[L]$ and $[L']$.
This distance turns T into a graph whose vertices are classes of lattices $[L]$ such that vertices at distance 1 are linked by an edge. This graph is connected since any two vertices admit representatives L, L' such that $L = \mathcal{O}_v \oplus \mathcal{O}_v$ and $L' = \mathcal{O}_v \oplus t^{d([L],[L'])}\mathcal{O}_v$, hence a path joining L to L' can be constructed as $L_k = \mathcal{O}_v \oplus t^k\mathcal{O}_v$ with $k = 0, \dots, d([L], [L'])$. In fact it can be shown that T is a tree, see [SB77].

1.3.2 Link of a vertex and ends of the tree

Given a vertex $[L] \in T$, one can describe the set of vertices at distance 1 of $[L]$ as follows: for each such $[L']$ there is a basis of V such that $\mathcal{O}_v^2$ is a representative of $[L]$ and that there is an unique representative L' of $[L']$ isomorphic to $\mathcal{O}_v \oplus t\mathcal{O}_v$ in this basis. Since $tL \subset L' \subset L$, it defines a map $\{[L'] : d([L], [L']) = 1\} \to k\mathbb{P}^1$ that sends L' to the line L'/tL in $L/tL \simeq k^2$, which turns out to be a bijection.
In general, there is a bijection between the set of vertices at distance n of $[L]$ and the lines in $(\mathcal{O}_v/(t^n))^2$, that is the points in the projective plane $\mathbb{P}((\mathcal{O}_v/(t^n))^2)$, hence a bijection between "half-lines" in T starting from $[L]$ and the projective space $\mathbb{P}(\hat{\mathcal{O}}_v^2) \simeq \mathbb{P}^1(\hat{\mathcal{O}}_v) \simeq \mathbb{K}\mathbb{P}^1$ (here $\hat{\mathcal{O}}_v = \varprojlim_{n\to\infty} \mathcal{O}_v/(t^n)$ denotes the completion of $\mathcal{O}_v$).

1.3.3 The SL_2 action: stabilizers of vertices, fixed points and translation length

There is a natural isometric and transitive action of $\mathrm{GL}(V)$ on T, induced by the action of $\mathrm{GL}(V)$ on V.

Definition 1.3.1. The action of a subgroup $G \subset \mathrm{GL}(V)$ on T will be said *trivial* if a vertex is fixed by the whole group.

Lemma 1.3.2. *For any $g \in \mathrm{GL}_2(\mathbb{K}), [L] \in T$, fix a basis $\{e, f\}$ of $L \in [L]$, and $n, m \in \mathbb{Z}$ such that $\{t^n e, t^m f\}$ is a basis of $g \cdot L$. Then $v(\det(g)) = n + m$.*

Proof. In this basis, $L \simeq \mathcal{O}_v^2$, and g can be written as the matrix $A \left(\begin{smallmatrix} t^n & 0 \\ 0 & t^m \end{smallmatrix}\right) B$ with $A, B \in \mathrm{GL}_2(\mathcal{O}_v)$. The result follows. □

Now we restrict to the $\mathrm{SL}(V)$ action. We say that an element $g \in \mathrm{SL}(V)$ *stabilizes* a vertex $[L] \in T$ if for any representative L we have $g \cdot L = xL$ for some $x \in \mathbb{K}^*$.

Lemma 1.3.3. *An element $g \in \mathrm{SL}(V)$ stabilizes a vertex $[L]$ iff for any representative $g \cdot L = L$.*

Proof. Assume that $g \cdot L = xL$, then by Lemma 1.3.2, $v(\det(g)) = 2v(x) = 0$ hence $x \in \mathcal{O}^*$ and $xL = L$. □

Furthermore, as $\mathrm{SL}_2(\mathcal{O}_v)$ is the stabilizer of the standard lattice $\mathcal{O}_v^2$, we deduce the proposition.

Proposition 1.3.4. *The stabilizer in $\mathrm{SL}_2(\mathbb{K})$ of any vertex of the tree T is a GL_2-conjugate of $\mathrm{SL}_2(\mathcal{O}_v)$.*

Remark 1.3.5. Since for any $g \in \mathrm{SL}(V)$, $v(\det(g)) = 0$, we know that the distance $d([L], g \cdot [L]) = |n - m|$ is even, in particular $\mathrm{SL}(V)$ acts *without inversion* on T, that is it can not fix an edge and exchange its end points.

Definition 1.3.6. Given $g \in \mathrm{SL}(V)$, we define the *translation lenght* of g to be equal to $l(g) = \min_{[L] \in T} d([L], g \cdot [L])$.

There are two ways for $g \in \mathrm{SL}(V)$ acting on the tree:

1. **Elliptic elements**

   If g has fixed points, or alternatively $l(g) = 0$, then it will be called *elliptic*. In this case, there is a basis of V such that $\mathcal{O}_v^2$ is fixed and g is an element of $\mathrm{SL}_2(\mathcal{O}_v)$. The set of fixed points T_g is a subtree of T.

2. **Hyperbolic elements**

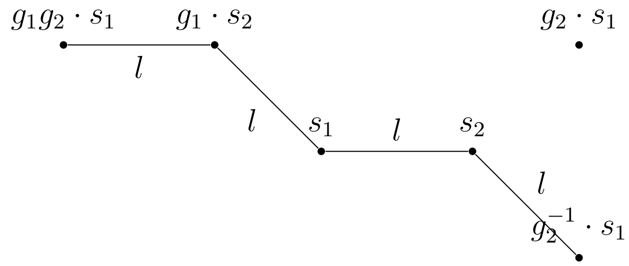
   If $l(g) > 0$, then g is called *hyperbolic* ; $A_g = \{s \in T \mid d(s, g \cdot s) = l(g)\}$ is an infinite, globally fixed, axis on which g acts by translation, and any basis of V such that the standard lattice $\mathcal{O}_v^2 \in A_g$ provides the matrix form $g = \begin{pmatrix} t^{l(g)/2} & 0 \\ 0 & t^{-l(g)/2} \end{pmatrix}$.

We prove the following lemma from [SB77, Corollaire 3, p.90].

Lemma 1.3.7. *Let G be a subgroup of* $\mathrm{SL}(V)$ *acting on the Bass-Serre tree T. If every element $g \in G$ fixes a vertex of T, then the whole group has a fixed vertex, that is the action is trivial.*

Proof. — First we prove that every two elements have a common fixed point.

Let $g_1, g_2 \in \Gamma$ which have no common fixed point, take two vertices s_1 and s_2, such that they realize the minimal distance l between the fixed subtrees T_1 and T_2. Then the segment $[s_1, s_2]$ has no other point than s_1 fixed by g_1, so it's image by g_1 is $[s_1, g_1 \cdot s_2]$. In the same way, it's image by g_2^{-1} is $[g_2^{-1} \cdot s_1, s_2]$. Consider the action of the element $g_1 g_2 \in \Gamma$.



We prove that it acts on the pictured axis by translation of length $2l$: by the minimality of l, $g_2^{-1} \cdot s_1$ and s_1 are at distance $2l$. It clear that s_1 and $g_1 g_2 \cdot s_1$ are at distance at most $2l$. Now if $d(s_1, g_1 g_2 \cdot s_1) < 2l$, there exists an other vertex than $g_1 \cdot s_2$ in the intersection $[s_1, g_1 \cdot s_2] \cap [g_1 \cdot s_2, g_1 g_2 \cdot s_1]$. Denote it by s. Then $g_1^{-1} \cdot s \in [s_1, s_2] \cap [s_2, g_2 \cdot s_1] \setminus \{s_2\}$, hence it is fixed by g_2, what contradicts the minimality of l.

Iterating this process, we see that $g_1 g_2$ acts as an hyperbolic element, and thus has no fixed point, which is a contradiction.

— Write $\{g_1, ... g_n\}$ a system of generators of Γ. Assume that $g_1, ..., g_k$ have a common fixed point s, and take s' a fixed point of g_{k+1} as near as possible of s. So one of the g_i's, $i = 1...k$, do not fix any other point of $[ss']$ than s ; applying the same argument as above gives that g_{k+1} fixes s too, and concludes the proof.

□

1.3.4 Curves and valuations

Examples of field extensions of k together with k-valuations are given by algebraic varieties defined over k. In particular, pick $X \subset X(\Gamma)$ an irreducible component of the character variety, its function ring $k[X]$ is a domain, and we denote by $k(X) = \mathrm{Frac}(k[X])$ its quotient field, called the *function field* of X. It is a general fact that this field is a k-valuated field, with valuations corresponding to hypersurfaces $W \subset X$. We will be interested in the case where X is one dimensional, and we refer to [Ful08] for details on what follows: there exists an unique curve $\hat{X}$, which is smooth and compact, called the *smooth projective model* of X, with a birational map $\nu : \hat{X} \dashrightarrow X$ that is an isomorphism between open subsets and induces a canonical field isomorphism $\nu^* : k(X) \xrightarrow{\sim} k(\hat{X})$. There is a homeomorphism

$$\begin{aligned} \hat{X} &\to \{k\text{-valuations on } k(X)\} \\ x &\mapsto v_x : f \mapsto \operatorname{ord}_x(f) \end{aligned}$$

where the set of valuations is endowed with the cofinite topology.

Remark 1.3.8. When the context will be clear, a curve X being given, we will often denote by v a point in the smooth projective model $\hat{X}$.

Definition 1.3.9. Let $v \in \hat{X}$ be a point in the projective model of X. We will call it an *ideal point* if ν is not defined at v, equivalently the function ring $k[X]$ is not a subring of $\mathcal{O}_v$. Otherwise we will call v a *finite point.*

Example 1.3.10. Let X be the plane curve $\{x^2 - y^3 = 0\}$ in $\mathbb{C}^2$. It is a singular affine curve, with function ring $\mathbb{C}[X] = \mathbb{C}[U, V]/(U^2 - V^3)$. The map $\mathbb{C}[X] \to \mathbb{C}[T]$ that maps U to T^3 and V to T^2 induces an isomorphism between of fields $\operatorname{Frac}(\mathbb{C}[X]) \simeq \mathbb{C}(T)$. Moreover, it defines a birational map $\nu : \mathbb{CP}^1 \to X \subset \mathbb{C}^2$ by $t \mapsto (t^3, t^2)$. Hence the smooth projective model of X is isomorphic to $\mathbb{CP}^1$ (remark that the singular point $(0, 0)$ is "smoothed" through ν), and the ideal point is ∞. As a map $\mathbb{CP}^1 \to \mathbb{CP}^2$, ν sends ∞ to $[1 : 0 : 0]$, the curve $X \cup \{\infty\} \subset \mathbb{CP}^2$ is a (non smooth) compactification of X.

1.3.5 Group acting on a tree and splitting

Let X be an irreducible component of irreducible type of $X(\Gamma)$, which is reduced and one dimensional, and let $\rho : \Gamma \to \mathrm{SL}_2(k(X))$ be the tautological representation. Let $v \in \hat{X}$ a point in the smooth projective model of X, the pair $(k(X), v)$ is a k-valuated field, and we denote by T_v the Bass-Serre tree described above. The group Γ acts simplicially on T_v as a subgroup of $\mathrm{SL}_2(k(X))$ through the tautological representation ρ. Although the representation ρ is defined up to conjugation, the action on the Bass-Serre tree is well-defined.

Proposition 1.3.11. *The action of Γ on T_v is trivial if and only if $v \in \hat{X}$ is a finite point.*

Proof. By definition, for $v \in \hat{X}$ a finite point, the ring $k[X]$ is included in $\mathcal{O}_v$, which means that $v(Y_\gamma) \geq 0$ for any $\gamma \in \Gamma$. Equivalently, $\mathrm{Tr}(\rho(\gamma)) \in \mathcal{O}_v$ for any $\gamma \in \Gamma$, and we want to prove that it is equivalent to $\rho(\gamma)$ to be conjugated to an element of $\mathrm{SL}_2(\mathcal{O}_v)$. It is clear if $\rho(\gamma) = \pm \mathrm{Id}$, if not there exists a vector $e \in k(X)^2$ such that $\{e, \rho(\gamma)e\}$ is a basis of the two dimensional vector space $k(X)^2$, and in this basis $\rho(\gamma)$ acts as the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & \mathrm{Tr}(\rho(\gamma)) \end{smallmatrix}\right)$ which lies in $\mathrm{SL}_2(\mathcal{O}_v)$. The proposition follows now from Lemma 1.3.7. □

Finite points and residual representations

If v is a finite point, Proposition 1.3.11 implies that the tautological representation can be chosen, up to conjugation, to be of the form $\rho : \Gamma \to \mathrm{SL}_2(\mathcal{O}_v)$. Such a representation will be said *convergent*. Given a convergent representation ρ, we denote by $\bar{\rho} : \Gamma \to \mathrm{SL}_2(\mathcal{O}_v) \xrightarrow{\text{mod } t} \mathrm{SL}_2(k)$ the *residual* representation. If v corresponds to the character $\chi \in X$, then the representation $\bar{\rho}$ is a lift of χ.

Ideal points and incompressible surfaces

Here M is a 3-manifold with $\partial M = \mathbb{S}^1 \times \mathbb{S}^1$, and $\Gamma = \pi_1(M)$. We pick an ideal point $v \in \hat{X}$, and we know from Proposition 1.3.11 that no representative ρ of the tautological representation converges (sends the whole group Γ into $\mathrm{SL}_2(\mathcal{O}_v)$). Now we describe quickly how to construct, from the action of Γ on T_v a surface $\Sigma \subset M$, said *dual* to the action. The reader will find many details about this delicate construction in [Sha02, Til03].

The main point is to construct a $\pi_1(M)$-equivariant map $f : \tilde{M} \to T_v$. Pick any triangulation K of M, and lift it to a $\pi_1(M)$-invariant triangulation $\tilde{K}$ of $\tilde{M}$; then pick a set of orbit representatives $S^{(0)}$ for the action of $\pi_1(M)$ on the set of 0-simplices of $\tilde{K}$, and any map $f_0 : S^{(0)} \to T_v$ from this set to the set of vertices of T_v. It induces an equivariant map from the 0-squeleton of $\tilde{K}$ to T_v, that we still denote by $f_0 : \tilde{K}^{(0)} \to T_v$. Now it is possible to extend linearly this map to the 1-squeleton, as follows: pick a set of orbit representatives $S^{(1)}$ for the action of $\pi_1(M)$ on the set of 1-simplices of $\tilde{K}$. Any edge $\sigma \in S^{(1)}$ has endpoints mapped to some given vertices through the map f_0, and we extend in the obvious way f_0 to σ. Now there is a unique $\pi_1(M)$-equivariant extension $f_1 : \tilde{K}^{(1)} \to T_v$ of f_0, it is continuous, and

can be made simplicial, up to subdivide the triangulation $\tilde{K}$. Repeat this process up to obtain the desired simplicial, equivariant map $f : \tilde{M} \to T_v$.
Now consider the set of midpoints E of the edges of T_v, the set $f^{-1}(E)$ is a surface $\tilde{S} \subset \tilde{M}$. This surface is non-empty because the action of $\pi_1(M)$ on the tree T_v is non trivial, and orientable because the map f is transverse to E. Moreover it is stable under the action of $\pi_1(M)$ on $\tilde{M}$, and hence its image through the covering map $\tilde{M} \to M$ is a surface $S \subset M$, non empty and orientable, dual to the action. It is worth to notice that it has no reason to be connected in general.

Remark 1.3.12. There is some kind of converse construction, which may explain the use of the term "dual". Given S an oriented surface in M, and $\tilde{S}$ the preimage of S in the universal cover $\tilde{M}$, one can construct a tree T_S as follows: vertices are in bijection with the connected components on $\tilde{M} \setminus \tilde{S}$, and they are joined with an edge when the corresponding components are separated by a component of $\tilde{S}$. Because the surface $\tilde{S}$ is $\pi_1(M)$-invariant, this tree T_S comes with a simplicial action of the fundamental group $\pi_1(M)$. Moreover, the stabilizers of vertices are the fundamental groups of the corresponding connected components of $M \setminus S$ and stabilizers of edges are the fundamental groups of the corresponding connected components of S.
Whenever S is a surface produced by the action of the fundamental group on a tree T, one has the following relation between T_S and T: since T_S is a retract of $\tilde{M}$, and the map $f : \tilde{M} \to T$ was constructed by extension to contractible cells, the compostion $T_S \hookrightarrow \tilde{M} \xrightarrow{f} T$ if well-defined and provides an injective $\pi_1(M)$-equivariant map $i : T_S \to T$, which can be made simplicial, after subdividing T_S is necessary (subdividing T_S turns out to add parallel copies of $\tilde{S}$ in $\tilde{M}$ in an equivariant way). The map i has no reason to be an isomorphism, but it implies that the stabilizers of vertices of T_S (namely fundamental groups of connected components of $M \setminus S$) are included in the stabilizers of some vertices of T, and similarly for the stabilizers of edges.

Definition 1.3.13. A surface Σ in a 3-manifold M is said *incompressible* if

1. Σ is oriented
2. For each component Σ_i of Σ, the homomorphism $\pi_1(\Sigma_i) \to \pi_1(M)$ induced by inclusion is injective.
3. No component of Σ is a sphere or is boundary parallel.

Remark 1.3.14. A *compression disk* $D \subset M$ is an embedded disk in M such that ∂D lies in S and is not homotopically trivial in S. The second condition above is equivalent to saying that there is no compression disk in M.

If S is a surface dual to a π_1 action on a tree T, there is a way to modify the equivariant map f in order to avoid compression disks, spherical and boundary parallel components, and hence to obtain a new surface Σ that is incompressible. We refer the reader to the references given above, where a proof of this fact can be found.

1.3.6 The split case.

Let Σ be an incompressible surface associated to an ideal point $v \in \hat{X}$. In this section we suppose that Σ is a union of n parallel copies $\Sigma_i, i = 1, \ldots, n$ and that each copy splits M into two handlebodies $M = M_1 \cup_{\Sigma_i} M_2$. Consider $V(\Sigma)$ a neighborhood of Σ homeomorphic to $\Sigma_1 \times [0, 1]$, and we consider the splitting $M = M_1 \cup_{V(\Sigma)} M_2$. We fix a basepoint $p \in \Sigma_1$, and we will denote by $\pi_1(\Sigma)$ the fundamental group of Σ_1 based in p. We identify $\pi_1(V(\Sigma))$ to $\pi_1(\Sigma)$, and the Seifert-Van Kampen Theorem provides the amalgamated product $\pi_1(M) = \pi_1(M_1) *_{\pi_1(\Sigma)} \pi_1(M_2)$. A sketchy picture is drafted in Figure 1.3.

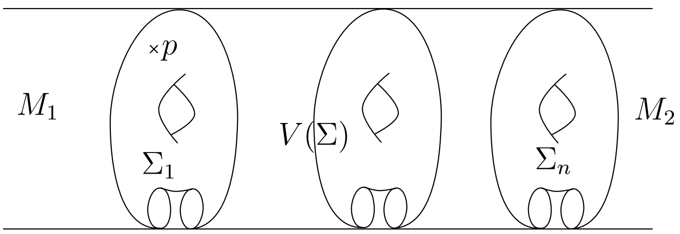


Figure 1.3 – The splitting $M = M_1 \cup_{V(\Sigma)} M_2$.

Lemma 1.3.15. *One can chose a conjugate of the tautological representation $\rho : \Gamma \to \mathrm{SL}_2(k(X))$ that restricts to representations ρ_1 and ρ_2 from $\pi_1(M_1), \pi_1(M_2)$ to $\mathrm{SL}_2(k(X))$ respectively ; such that ρ_1 is convergent and that ρ_Σ, its restriction to $\pi_1(\Sigma)$, is residually reducible. Moreover, there is a convergent representation $\rho_2' : \pi_1(M_2) \to \mathrm{SL}_2(\mathcal{O}_v)$ such that $\rho_2 = U_n \rho_2' U_n^{-1}$, with $U_n = \left(\begin{smallmatrix} t^n & 0 \\ 0 & 1 \end{smallmatrix}\right)$.*

Proof. Let $s_1 \in T_v$ be a vertex in the Bass-Serre tree that is fixed by $\pi_1(M_1)$, and fix a basis such that it corresponds to the lattice $\mathcal{O}_v^2$. Then there is a vertex $s_2 \in T$, fixed by $\pi_1(M_2)$, such that $d(s_1, s_2) = n$. Moreover, assume that in this basis s_2 has a representative of the form $t^n\mathcal{O}_v \oplus \mathcal{O}_v$. The first observation is that $\rho_1(\pi_1(M_1)) \subset \mathrm{SL}_2(\mathcal{O}_v)$ because it stabilizes $\mathcal{O}_v^2$. Since ρ_Σ fixes the first edge of the

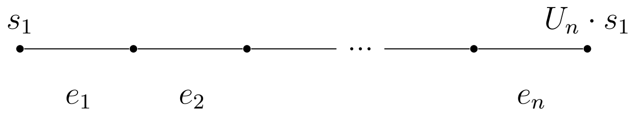
segment $[s_1 s_2]$, in this basis it fixes the lattices $\mathcal{O}_v^2$ and $t\mathcal{O}_v \oplus \mathcal{O}_v$, hence for all $\gamma \in \pi_1(\Sigma)$, $\rho_\Sigma(\gamma) = \left(\begin{smallmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{smallmatrix}\right)$, with $c(\gamma) \in (t)$, hence $\bar{\rho}_\Sigma$ is reducible.
Let $\rho_2' = U_n^{-1}\rho_2 U_n$, then $\rho_2' \cdot s_1 = U_n^{-1}\rho_2 \cdot s_2 = U_n^{-1} \cdot s_2 = s_1$ and we have proved that the representation ρ_2' converges. □

1.3.7 The non-split case

Let S be an incompressible surface associated to an ideal point $v \in \hat{X}$ which is, again, union of n parallel copies $S = S_1 \cup \ldots \cup S_n$, and we assume now that $M \setminus S_i$ is connected. Hence $[S_i] \neq 0 \in H_2(M; \partial M)$, and in particular each component of ∂S_i is a homological longitude. We say that S_i is a *Seifert surface* in M if ∂S_i is connected. Let $V(S_i)$ be a neighborhood of S_i in M, and $E(S_i) = \overline{M \setminus V(S_i)}$. It is a classical fact (see [Oza01, Proposition 2]) that $E(S_i)$ is a handlebody if and only if $\pi_1(E(S_i))$ is free. In this case we say that the surface S_i is *free*. It is the case, for instance, as soon as M is small (does not contain any closed incompressible surfaces), and a necessary and sufficient condition for a knot to contain non-free Seifert surfaces is given in [Oza00].
In the sequel we assume that the Seifert surface S_i is free, say of genus g, and we denote by $H = E(S_i)$ the genus $2g$ handlebody complement of S_i. We assume that $\partial V(S) = S_1 \cup S_n$. We have $M = V(S) \cup_{S_1 \cup S_n} H$, hence the HNN decomposition $\pi_1(M) = \pi_1(H)*_\alpha$, where we fix the basepoint $p \in S_1$, and $\alpha : \pi_1(S_1) \to \pi_1(S_n)$ an isomorphism between those subgroups of $\pi_1(M)$. This means that we have the presentation $\pi_1(M) = \langle \pi_1(H), v \mid v\gamma v^{-1} = \alpha(\gamma), \forall \gamma \in \pi_1(S_1)\rangle$.

Lemma 1.3.16. *One can chose a conjugate of the tautological representation $\rho : \pi_1(M) \to \mathrm{SL}_2(k(X))$ such that the restrictions $\rho_H : \pi_1(H) \to \mathrm{SL}_2(k(X))$ and $\rho_1 : \pi_1(S_1) \to \mathrm{SL}_2(k(X))$ are convergent. Moreover, $\rho(v) = V_n$ with $V_n = \left(\begin{smallmatrix} 0 & t^{n/2} \\ -t^{-n/2} & 0 \end{smallmatrix}\right)$, in particular n is an even integer, and the restriction $\rho_n : \pi_1(S_n) \to \mathrm{SL}_2(k(X))$ is equal to $V_n \rho_1 V_n^{-1}$. Finally, the residual restricted representation $\bar{\rho}_1$ is reducible, and if S_i are Seifert surfaces, one has $\mathrm{Tr}(\bar{\rho}_1(\partial S_i)) = 2$.*

Proof. We fix a vertex s in the Bass-Serre tree T, that corresponds to the lattice $\mathcal{O}_v^2$ and is fixed by $\pi_1(H)$, hence ρ_H is convergent. We denote by e_1 the edge in the tree T incident to s that is fixed by $\pi_1(S_1)$, and the parallel copies of S_1 stabilize a series of edges e_i that form a segment in T, which has $U_n \cdot s_1$ as an end point, as depicted below.



In particular, the representation $\bar{\rho}_1$ is reducible by the same argument that in the proof of Lemma 1.3.15. Now the element $v \in \pi_1(M)$ acts on the tree T in the following way: it sends the vertex s_1 to $U_n \cdot s_1$, but it sends the incident edge e_1 to e_n. Since it acts by isometries, the only possibility is to act as a central rotation with center the mid-point of the segment $[s_1 U_n \cdot s_1]$. But the fundamental group $\pi_1(M)$ acts without inversion, hence n is even, and $\rho(v)$ is of the form $U_{\frac{n}{2}} R_\pi U_{\frac{n}{2}}^{-1} = V_n$, where $R_\pi = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.
The last statement follows from [CCG$^+$94], where it is proven that the eigenvalues of the matrix $\bar{\rho}_1(\partial S_1)$ are roots of unity, of order that divides the number of boundary components of any surface S_i, hence those eigenvalues are equal to 1, and it achieves the proof. □

1.4 Alexander module and character varieties

In this section we prove a theorem due independently to Burde and DeRham ([Bur67, deR67]), that relies the so-called Alexander module of a 3-manifold M, and in particular roots of its Alexander polynomial, with some particular point in the character varieties $X(M)$.

1.4.1 Reducible character in components of irreducible type of the character variety

Definition 1.4.1. Let X be a one dimensional component of irreducible type of the character variety $X(M)$, $v \in \hat{X}$ a finite point in the smooth projective model of X, and $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathcal{O}_v)$ a convergent representative of the tautological representation. We will say the representation ρ is *residually reducible* (resp. *residualy abelian, irreducible, central*) if the residual representation $\bar{\rho} : \pi_1(M) \to \mathrm{SL}_2(k)$ is reducible (resp. abelian, irreducible, central). Recall in addition that a character χ is said central if $\chi(Y_\gamma)^2 = 4$ for all $Y_\gamma \in k[X]$.

Since X is of irreducible type, irreducible characters are dense in X hence for a generic $v \in \hat{X}$, the tautological representation ρ will be residually irreducible. Nevertheless, there may exists a finite set of points where it is residually reducible, they correspond to characters that are intersection points of X with a component of reducible type of $X(M)$. The following proposition uses crucially the hypothesis that the first Betti number of M is 1.

Proposition 1.4.2. *Let $\chi \in X$ a reducible character, $v \in \hat{X}$ the associated valuation, and $\rho : \Gamma \to \mathrm{SL}_2(\mathcal{O}_v)$ a choice of convergent tautological representation. Then the character $\chi_{\bar{\rho}}$ is not central, and in particular the representation ρ is not residually central. Moreover, one can choose the tautological representation ρ such that it is not residually abelian.*

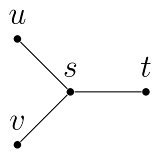
Proof. In [Por97, Lemma 3.9], it is proved that a reducible character $\chi_{\bar{\rho}}$ in a component of irreducible type X of the character variety of a 3 manifold M with $b_1(M) = 1$ cannot be central, because there exists a non abelian representation $\sigma : \Gamma \to \mathrm{SL}_2(k)$ with character $\chi_{\bar{\rho}}$. It proves the first claim. It remains to show that such a non abelian representation is realized as $\bar{\rho}$, for some choice of tautological representation ρ. We give a tree-theoretical argument.

Let T_v be the Bass-Serre tree associated to v, and consider the action of Γ on T_v.

— As the tautological representation is convergent, the subtree T'_v of fixed points is non empty.

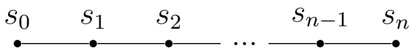
— The tree T'_v is finite, because if not it would contain an half-line, hence from Subsection 1.3.2 the tautological representation would fix a line in $\hat{\mathcal{O}}_v^2$ and it would contradict the irreducibility of ρ.

— The tree T'_v is a segment: assume it contains a vertex of valence at least 3



Then t, u and v represents three distincts points in $k\mathbb{P}^1$, fixed by the residual representation $\bar{\rho}$. Hence $\bar{\rho}(\Gamma) \subset \{\pm \mathrm{Id}\}$ but it is a contradiction with the first part of the proposition.

— We just proved that T' is of the following form:



Fix a basis such that s_i is the lattice $\mathcal{O}_v \oplus t^i \mathcal{O}_v$. Then for any $\gamma \in \Gamma$, $\rho(\gamma) = \begin{pmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_v)$, with $c(\gamma) \in (t^n)$. We conclude by noting that for some $\gamma \in \Gamma$, we have $b(\gamma) \in \mathcal{O}_v^*$; because if not there should be an other fixed point at the left of s_0. Finally, ρ is not residually abelian, and residually reducible as soon as $n > 0$.

□

1.4.2 Alexander module

Here we recall the basic theory of Alexander module. We assume for simplicity that M's first Betti number $b_1(M)$ is equal to 1, despite the theory can be extended to higher cases. We refer to [Rol76, BZH14] for proofs and details on this theory.
We call *abelianization* an epimorphism $\varphi : \pi_1(M) \to \mathbb{Z}$. As $b_1(M) = 1$, there are two possible choices for the homomorphism φ, but nothing we will state here will depend on this choice. Moreover, we abuse of the term abelianization because we omit the possibly non trivial torsion part of the abelianization of the fundamental group.
We call the *infinite cyclic covering* the regular covering $\bar{M}$ with deck transformation's group $\mathbb{Z}$. Let $\mathbb{Z} = \langle t \rangle$, we denote by Λ the group ring $\Lambda = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t^{\pm 1}]$. The action of $\mathbb{Z}$ on $\bar{M}$ turns the homology groups $H_*(\bar{M})$ into Λ-modules, and we call the Λ-module $H_1(\bar{M})$ the *Alexander module.* Notice it is the φ-twisted homology of M with coefficient in Λ.

Proposition 1.4.3. *The Λ-modules $H_i(\bar{M})$ are trivial for $i \geq 2$, $H_0(\bar{M}) \simeq \mathbb{Z}[t^{\pm 1}]/(t-1)$ and the Alexander module is a torsion Λ-module.*

Proof. Since M has a non-empty boundary, it has the homotopy type of a two dimensional cell complex, that can be obtained by collapsing all 3-cells adjacent to the boundary in any cell decomposition of M, hence $H_i(\bar{M}) = 0$ for $i \geq 3$. Moreover, it follows from [Jac80, Chapter V] that any presentation of $\pi_1(M)$ obtained from this cell decomposition has deficiency one, that is $\pi_1(M) = \langle x_1, \ldots, x_n | r_1, \ldots, r_{n-1} \rangle$ and one can assume that $\varphi(x_1)$ is a generator of $\mathbb{Z}$. In fact this cell complex can be chosen to be composed of an unique 0-cell e_0, n loops $e_1^1, \ldots, e_1^n$ based in e_0 corresponding to the generators $x_1, \ldots, x_n$ and $n-1$ disks $e_2^1, \ldots, e_2^{n-1}$ such that the boundary of the disk e_2^i follows the loops $\{e_1^k, k = 1 \ldots n\}$ as indicated by the relator r_i.
Now we consider the complex $C_2(\bar{M}) \to C_1(\bar{M}) \to C_0(\bar{M})$, which is nothing but $\Lambda^{n-1} \to \Lambda^n \to \Lambda$. The first boundary map is given by the $(n \times n-1)$ matrix $A = (\varphi(\partial_{x_i} r_j))_{i,j}$, where $\varphi : \mathbb{Z}[\Gamma] \to \mathbb{Z}[\mathbb{Z}]$ is the natural ring extension of the abelianization homomorphism, and the operators ∂_{x_i} are the Fox derivation operators (see [BZH14] for details). Theorem 9.10 from the given reference states that this matrix has maximal rank, hence $H_2(\tilde{M}) = 0$. The second boundary map is the line matrix $\begin{pmatrix} \varphi(x_1) - 1 & \ldots & \varphi(x_n) - 1 \end{pmatrix}$, and the remaining part of the proposition follows from the obvious fact that $\varphi(x_i) = t$. □

Definition 1.4.4. Let $\Delta_M = \gcd(\{(n-1) \text{ minors of } A\}) \in \mathbb{Z}[t^{\pm 1}]$, where A is the matrix of the proof above. We call this polynomial the *(first) Alexander polynomial* of M. Similarly, we call $\Delta_k(M) = \gcd(\{(n-k) \text{ minors of } A\}) \in \mathbb{Z}[t^{\pm 1}]$ the *k-th Alexander polynomial* of M.

Remark 1.4.5. It follows from the fact that A has maximal rank that none of those polynomials are zero. A proof that they only depend on M and are in fact topological invariants can be found in [BZH14]. Note that they are defined up to a power of t.

The following theorem is due to Burde and DeRham [Bur67, deR67], see [HPSP01] for a more recent treatment and improvement of the result. We give a proof of this well-known fact because we did not find it in those terms in the literature, and we will use some similar techniques along this manuscript.

Theorem 1.4.6. *Let $\chi \in X$ be a reducible character in a component of irreducible type of the character variety. Then there is $\lambda \in k^*$ such that $\Delta(\lambda^2) = 0$ and for all $\gamma \in \pi_1(M)$, $\chi(Y_\gamma) = \lambda^{\varphi(\gamma)} + \lambda^{-\varphi(\gamma)}$.*

Proof. As usual we fix $v \in \hat{X}$ a point in the smooth projective model such that $\nu(v) = \chi$, and $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathcal{O}_v)$ a convergent tautological representation, by hypothesis it is residually reducible, and we can choose it to be not residually abelian, by Proposition 1.4.2. Then the residual representation can be written $\bar{\rho}(\gamma) = \begin{pmatrix} \lambda^{\varphi(\gamma)} & \lambda^{-\varphi(\gamma)}u(\gamma) \\ 0 & \lambda^{-\varphi(\gamma)} \end{pmatrix}$ for some $\lambda \in k^*$. We want to show that λ^2 is a root of the Alexander polynomial Δ_M.

- The first observation is that we know λ to be different from ± 1 from Proposition 1.4.2.
- The group relation $\bar{\rho}(\gamma\delta) = \bar{\rho}(\gamma)\bar{\rho}(\delta)$ implies that the map $u : \Gamma \to k$ satisfies the relation $u(\gamma\delta) = u(\gamma) + \lambda^{2\varphi(\gamma)}u(\delta)$, hence $u \in Z^1(\Gamma, \lambda^2)$ is a cocycle in the λ^2-twisted cohomology of the group with coefficients in the field k.
- Now we prove that the class $[u] \in H^1(\Gamma, \lambda^2)$ is trivial iff the representation $\bar{\rho}$ is abelian. Since $\lambda \neq \pm 1$, $\bar{\rho}$ being residually abelian is equivalent to $\bar{\rho}$ being conjugated to a diagonal representation, that is to the existence of $a, b, c, d \in k$ such that for all $\gamma \in \Gamma$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda^{\varphi(\gamma)} & \lambda^{-\varphi(\gamma)}u(\gamma) \\ 0 & \lambda^{-\varphi(\gamma)} \end{pmatrix} = \begin{pmatrix} \lambda^{\varphi(\gamma)} & 0 \\ 0 & \lambda^{-\varphi(\gamma)} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

It is equivalent to

$$\begin{cases} a\lambda^{-\varphi(\gamma)}u(\gamma) + b\lambda^{-\varphi(\gamma)} = \lambda^{\varphi(\gamma)}b \\ c\lambda(\gamma) = c\lambda^{-1}(\gamma) \\ c\lambda^{-\varphi(\gamma)}u(\gamma) + d\lambda^{-\varphi(\gamma)} = d\lambda^{-\varphi(\gamma)} \end{cases}$$

or $\begin{cases} c = 0 \\ u(\gamma) = \frac{b}{a}(\lambda^{2\varphi(\gamma)} - 1) \end{cases}$, that is u is a coboundary.

— Since $\bar{\rho}$ is not abelian, we have proved that $H^1(\Gamma, \lambda^2) \neq \{0\}$, consequently $H_1(\Gamma, \lambda^2) \neq 0$. But the Universal Coefficients Theorem provides the isomorphism $H_1(\bar{M}) \otimes_{\mathbb{Z}[t,t^{-1}]} \frac{k[t,t^{-1}]}{(t-\lambda^2)} \simeq H_1(M, \lambda^2)$. Take $A(t) \in \mathcal{M}_{n\times(n-1)}(\mathbb{Z}[t^{\pm 1}])$ a presentation matrix for the Alexander module $H_1(\bar{M})$, then the matrix $A(\lambda^2)$ becomes a presentation matrix for $H_1(M, \lambda^2)$. Since the latter is non trivial, we deduce that $\Delta_M(\lambda^2)$, which is the greatest common divisor of the $(n-1)$ minors of the matrix $A(\lambda^2)$, is zero, and the theorem is proved.

□

Example 1.4.7. 1. For the trefoil knot, we recall that the tautological representation is $\rho : a \mapsto \begin{pmatrix} t & 1 \\ -(t^2+1) & -t \end{pmatrix}, b \mapsto \begin{pmatrix} -j & 0 \\ 0 & -j^2 \end{pmatrix}$. It is residually reducible when $t^2 = -1$. The curve $ab^{-1} \in \Gamma$ is a meridian (that is its abelianization is 1), and at the reducible characters $\rho(ab^{-1})$ has eigenvalue $\lambda = \pm ij^2$, thus $\lambda^2 = -j$, which is a root of the Alexander polynomial $\Delta_M(t) = t^2 - t + 1$.

2. For the figure-eight knot, the component of irreducible type and the component of reducible type intersect when the trace of the meridian is $\pm\sqrt{5}$. Thus the square of the eigenvalues are $\frac{3\pm\sqrt{5}}{2}$, the roots of the Alexander polynomial $\Delta_M(t) = t^2 - 3t + 1$.

1.5 The Reidemeister torsion

In this section we give various definitions used for the Redemeister torsion. References are [Mil66], [GKZ94, Appendix A], [Por97, Chapitre 0]. We stress out the fact that we use a convention (namely, how we take the alternating sum in the definition of the determinant of a complex) that corresponds to [GKZ94], but not to [Mil66].

1.5.1 First definition

Given a finite complex C^* of k-vector spaces

$$C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} C^n$$

fix $\{c^i\}_{i=0\dots n}$ and $\{h^i\}_{i=0\dots n}$ families of basis of the C^i's and the H^i's, one can define the *torsion* of the based complex $\operatorname{tor}(C^*, \{c^i\}, \{h^i\})$ to be the alternating product of the determinants of the base change induced by this choices. More precisely, we have the exact sequences

$$0 \to Z^i \to C^i \to B^{i+1} \to 0$$

$$0 \to B^i \to Z^i \to H^i \to 0$$

that define B^i, Z^i and H^i. Pick a system of basis $\{b^i\}$ of the B^i's, first one obtains a basis of Z^i for any i, given by a section $H^i \to Z^i$, and then a section $B^{i+1} \to C^i$ provides a basis of C^i: $b^i \sqcup \bar{h}^i \sqcup \bar{b}^{i+1}$, where the bars denote the image by the chosen sections. Now compare this new basis with c^i, and take the determinant of the matrix which exchange those basis, denoted by $[b^i \sqcup \bar{h}^i \sqcup \bar{b}^{i+1} : c^i]$. One can show that the alternating product of those determinants does not depend of the lifts and of the system $\{b^i\}$ and we define

$$\operatorname{tor}(C^*, \{c^i\}, \{h^i\}) = \prod_i [b^i \sqcup \bar{h}^i \sqcup \bar{b}^{i+1} : c^i]^{(-1)^i} \in k^*/\{\pm 1\}$$

1.5.2 Second definition: the Euler isomorphism

Recall that the determinant of a n-dimensional vector space V is $\det(V) = \Lambda^n V$. Given L a one-dimensional vector space, for convenience of notations we will denote by $L^{\otimes(-1)}$ its dual vector space $\operatorname{Hom}(L, k)$. We define the *determinant* of a complex $\det(C^*) = \bigotimes_i \det(C^i)^{\otimes(-1)^i}$. The cohomology of this complex is naturally graded by the degree, and we have the following theorem.

Theorem 1.5.1. *There is a natural isomorphism*

$$\operatorname{Eu} : \det(C^*) \xrightarrow{\sim} \det(H^*(C^*))$$

Proof. Again, we write the two exact sequences

$$0 \to Z^i \to C^i \to B^{i+1} \to 0$$

$$0 \to B^i \to Z^i \to H^i \to 0$$

Then the proof reduces to the particular case of a short exact sequence:

Lemma 1.5.2. *For an exact sequence of vector spaces*

$$0 \to A \to B \to C \to 0$$

one has an isomorphism

$$\det(A) \otimes \det(C) \simeq \det(B)$$

given, for any choice of basis $\{a_1, ...a_m\}$ of A, $\{c_1, ..., c_n\}$ of C and of a section $C \to B, c_i \mapsto \bar{c}_i$, by

$$(a_1 \wedge ... \wedge a_m) \otimes (c_1 \wedge ... \wedge c_n) \mapsto a_1 \wedge ... \wedge a_m \wedge \bar{c}_1 \wedge ... \wedge \bar{c}_n$$

□

Definition 1.5.3. Given a complex C^*, and a system of basis $\{c^i\}$ of the C^i's, $c^i = \{c^i_1, ..., c^i_{n_i}\}$, then $\bigwedge c^i = c^i_1 \wedge ... \wedge c^i_{n_i}$ is a basis of $\det(C^i)$, and then we denote by $c = \bigotimes_i (\bigwedge c^i)^{\otimes(-1)^i}$ the induced basis of $\det(C^*)$.
Then the torsion of the based complex is defined by

$$\text{tor}(C^*, \{c^i\}) = \text{Eu}(c) \in \det(H^*(C^*))$$

Remark 1.5.4. The two definitions coincide in the following sense:

$$\text{tor}(C^*, \{c^i\}, \{h^i\}) = \text{tor}(C^i, \{c^i\}) \otimes_i (\wedge h^i)^{\otimes(-1)^i}$$

1.5.3 Third definition: torsion of an exact complex (Cayley formula)

If the complex is exact, one has the following alternative description: pick a system of basis $\{c^i\}$ of the C^i's that induces, for each i, a splitting $C^i = \ker d_i \oplus K^i$, where

K^i is a supplementary of $\ker d_i$ in C^i. Then each d_i restricts to an isomorphism $d_{|_{K^i}} : K^i \to \ker d_{i+1}$, and we define

$$\operatorname{tor}(C^*, \{c^i\}) = \prod_i \det(d_{|_{K^i}})^{(-1)^{i+1}}$$

Again, it's defined up to sign since we haven't fixed an order for the basis.

Chapter 2

The adjoint torsion

Let $\bar{X}$ be a component of irreducible type of the augmented character variety $\bar{X}(M)$ and $k(\bar{X})$ its function field. In this chapter we will study the action of the fundamental group $\pi_1(M)$ on the Lie algebra $\mathrm{sl}_2(k(\bar{X}))$, and the Reidemeister torsion of the twisted complex $C^*(M, \mathrm{Ad} \circ \rho)$ for this action. It is what we will call the adjoint torsion. First we will define it as a rational differential form over Y, the smooth projective model of $\bar{X}$, and then we will study its poles and zeros, in particular we will observe that the torsion is regular on the affine part $\bar{X}$ of the Riemann surface Y. We will deduce a "genus formula" that gives a relation between the topology of the augmented character variety and the Euler characteristic of incompressible surfaces in M produced through the Culler-Shalen theory. This chapter contains the results of the article [Ben16].

Notation. The convention that we follow here for the torsion is convenient to express it as a differential form on the character variety. Many authors use the opposite convention, for instance in Section 2.2.3 we use some computations of Jerome Dubois with the other convention. On the other hand, it is the convention taken in [GKZ94, Appendix A].

The results of this chapter are summarized in the following theorems:

Theorem 2.0.1. *Let $\bar{X}$ be a component of irreducible type of the augmented character variety, and let x be a point in $\bar{X}$, with corresponding valuation v. Then the adjoint torsion $\mathrm{tor}(M, \mathrm{Ad} \circ \rho)$ vanishes at v with order given by the length of the torsion part of the $\mathcal{O}_v$-module $H_1(M, \mathrm{Ad} \circ \rho)_v$. In particular $\mathrm{tor}(M, \mathrm{Ad} \circ \rho)$ is regular on $\bar{X}$, and does not vanish if v projects onto a smooth point of $X(M)$.*

The striking point of this theorem, and the more simple to state, is that the torsion defines a regular differential form on $\bar{X}$, which was not expected from the definition.

On the other hand, its zeros are included in the pre-image S of the singular locus of $X(M)$ through the covering map $\bar{X} \to X$, but not every point $x \in S$ will be a zero of the torsion. For instance, the pre-image of an intersection point with normal crossing between two components of irreducible type in $X(M)$ will not, neither will generically be the pre-image of an intersection point between a component of irreducible type and the component of reducible type. We do not know any example of such a zero, and it is related with the fact that we do not know any component of irreducible type with an intrinsic singularity, like a cusp or a self-crossing.

Theorem 2.0.2. *Let $y \in Y$ be an ideal point of the smooth projective model of a component of irreducible type $\bar{X}$ of the augmented character variety, and let v be the corresponding valuation on $k(Y)$. Assume that y detects an incompressible surface $\Sigma \subset M$, such that Σ is connected or is the union of n parallel homeomorphic copies, that the restriction ρ_Σ is not abelian and that the complement of any connected component of Σ in M is the disjoint union of two handlebodies. Then we have the following bound on the order of $\mathrm{tor}(M, \mathrm{Ad} \circ \rho)$ at y:*

$$v(\mathrm{tor}(M, \mathrm{Ad} \circ \rho)) \leq -n\chi(\Sigma) - m$$

where $m = n$ if ρ_Σ is not residually abelian, and else $m = \mathrm{length}(H^1(\Sigma, \mathrm{Ad} \circ \rho_\Sigma)$.

Unless χ is an annulus (see the example of the trefoil knot Section 2.1, which remains true for torus knots in general), there is no reason for the order of $\mathrm{tor}(M, \mathrm{Ad} \circ \rho)$ at y to be negative. In fact, it is not the case for the four non-torus examples we have computed. From this observations we adress the following question:

Question 2.0.3. Is there a hyperbolic knot such that the adjoint torsion form has a pole on the augmented covering of a geometric component?

A somehow related question is the following, which comes naturally after considering the examples at the end of this chapter:

Question 2.0.4. Is the inequality of Theorem 2.0.2 an equality ?

We deduce from Theorem 2.0.2 the following corollary:

Corollary 2.0.5. *Assume that $X(M)$ holds a one-dimensional component X such that every irreducible characters $\chi \in X$ are smooth points, and that for each ideal point y in its smooth projective model of the augmented variety Y one can produce an incompressible surface Σ_y with n_y connected components and which verifies the* *hypothesis of Theorem 2.0.2. Futhermore, assume that the Alexander polynomial of M has simple roots. Then*

$$-\chi(Y) \leq \sum_{y} -n_y \chi(\Sigma_y) - m_y$$

where m_y is defined as in Theorem 2.0.2.

The two first sections of this chapter are devoted to the definition of the adjoint torsion as a differential form on the augmented character variety, and we detail several computations of this form. In the third section we give a proof of Theorem 2.0.1, and some interpretation of those results in terms of torsion in the Kähler differential's module and of Alexander invariants. In the fourth section we begin with an explicit topological computation of the torsion form of the trefoil knot, which has inspired the general proof of Theorem 2.0.2 that follows, and we conclude by checking our results on a series of examples.

2.1 The twisted complex with adjoint action

In this section we study the complex of twisted cohomology of M in the adjoint representation.

2.1.1 Tangent space, differential forms and twisted cohomology

The results of this section can be found in [Mar15]. It's a well-known result [LM85, Wei64] that the tangent space of the character variety at an irreducible character is isomorphic to the first group of twisted cohomology with the adjoint action. Saito's Theorem allows us to prove it in the skein context. This results can serve as a motivation to study the adjoint-twisted complex, which appears to be related with the geometry of the character variety.

Definition 2.1.1. Let $\rho : \Gamma \to \mathrm{SL}_2(k)$ be a representation. The action by conjugation of the group $\mathrm{SL}_2(k)$ on its Lie algebra induces an action of Γ on $\mathrm{sl}_2(k)$, called the *adjoint action*. We define the complex $C^*(M, \mathrm{Ad} \circ \rho)$ of twisted cohomology for this action with coefficients in $\mathrm{sl}_2(k)$.

Theorem 2.1.2. *Let χ be an irreducible character, and $\rho : \pi_1(M) \to \mathrm{SL}_2(k)$ a representation which lifts χ. Then we have an isomorphism $T_\chi X(M) \simeq H^1(M, \mathrm{Ad} \circ \rho)$, where the cohomology coefficient is $\mathrm{sl}_2(k)$ with the adjoint action.*

Sketch of proof, see [Mar15]. Recall that the Zariski tangent space at a point χ is defined as the set of morphism $\phi_\varepsilon : B[\Gamma] \to k[\varepsilon]/(\varepsilon^2)$ such that $\phi_0 = \chi$. Given a cocyle $\varphi \in H^1(M, \mathrm{Ad} \circ \rho)$, one produces a tangent vector at the point χ_ρ by taking the character associated to $\rho_\varepsilon : \Gamma \to \mathrm{SL}_2(k[\varepsilon]/(\varepsilon^2))$, where $\rho_\varepsilon = (\mathrm{Id} + \varepsilon\varphi)\rho$. Then Saito's Theorem ensure this map is invertible. □

Definition 2.1.3. Given a ring A, and a A-algebra B, we define the B-module of A-derivations $\Omega^1_{B/A}$ to be the free B-module generated by the db's, divided by the relations $\{\forall a \in A, da = 0,\ \forall b_1, b_2 \in B, d(b_1 + b_2) = db_1 + db_2$ and $d(b_1 b_2) = b_1 db_2 + b_2 db_1\}$. If X is an irreducible algebraic variety with function field $k(X)$, we denote by $\Omega^1_{k(X)/k}$ the $k(X)$-vector space of *rational differential forms* over X. It is a classical fact (see [Liu06] for instance) that its dimension as a $k(X)$-vector space is the dimension of X as a variety over k.

Recall that $\bar{X}(M)$ is the augmented character variety (see Section 1.1.24) with algebra of functions $\bar{B}[M] = B[M] \otimes_{C[M]} C(\partial M)$. The module of differential $\Omega_{\bar{B}[M]/k}$ is generated by $d(Y_\gamma \otimes 1)$ for $\gamma \in \pi_1(M)$ and by $d(1 \otimes Z_\gamma)$ for $\gamma \in \pi_1(\partial M)$, with $d(Y_\gamma \otimes 1) = d(1 \otimes Z_\gamma) + d(1 \otimes Z_{\gamma^{-1}})$ for any $\gamma \in \pi_1(\partial M)$. The following proposition is adapted, as well as its proof, from [Mar15, Proposition 4.1].

Proposition 2.1.4. *Let $\bar{X}$ be a component of irreducible type, with function ring $k[\bar{X}] \simeq \bar{B}[\Gamma]/\mathfrak{p}$. Let $\rho : \Gamma \to \mathrm{SL}_2(k(\bar{X}))$ the tautological representation, and $H_*(M, \mathrm{Ad} \circ \rho)$ the twisted homology with adjoint action on $\mathrm{sl}_2(k(\bar{X}))$ coefficients. Then there are isomorphisms*

$$\Omega_{\bar{B}[\Gamma]/k} \otimes_{\bar{B}[\Gamma]} k(\bar{X}) \simeq H_1(M, \mathrm{Ad} \circ \rho) \simeq \Omega_{k(\bar{X})/k}$$

Proof. For any $\gamma \in \Gamma$, we denote by $\rho(\gamma)_0$ the trace-free matrix $\rho(\gamma) - \frac{1}{2}\mathrm{Tr}(\rho(\gamma))\,\mathrm{Id}$. Recall that $C_1(M, \mathrm{Ad} \circ \rho)$ is generated by elements of the form $\xi \otimes [\gamma]$, and that $\xi \otimes [\gamma^{-1}] = -\xi \otimes [\gamma]$.
We construct a morphism of $\bar{B}[\Gamma]$-modules

$$\Omega_{\bar{B}[\Gamma]/k} \to H_1(M, \mathrm{Ad} \circ \rho)$$

$$d(Y_\gamma \otimes 1) \mapsto \rho(\gamma)_0 \otimes [\gamma]$$

$$d(1\otimes Z_\gamma)\mapsto \begin{pmatrix} \frac{\lambda}{2} & 0 \\ 0 & -\frac{\lambda}{2} \end{pmatrix}\otimes[\gamma]$$

Notice that it induces

$$d(1\otimes Z_{\gamma^{-1}})\mapsto \begin{pmatrix} \frac{\lambda^{-1}}{2} & 0 \\ 0 & -\frac{\lambda^{-1}}{2} \end{pmatrix}\otimes[\gamma^{-1}] = \begin{pmatrix} -\frac{\lambda^{-1}}{2} & 0 \\ 0 & \frac{\lambda^{-1}}{2} \end{pmatrix}\otimes[\gamma]$$

where λ,λ^{-1} are the eigenvalues of the matrix $\rho(\gamma)$. It induces a $k(\bar{X})$-linear map $\Psi:\Omega_{\bar{B}[\Gamma]/k}\otimes k(\bar{X})\to H_1(M,\mathrm{Ad}\circ\rho)$.
To construct the reciprocal map, we define $\bar{\Lambda}=k(\bar{X})\oplus\varepsilon\Omega_{\bar{B}[M]/k}\otimes k(\bar{X})$, and extend from [Mar15] the map $\varphi:\bar{B}[M]\to\bar{\Lambda}$ by

$$Y_\gamma\otimes 1\mapsto Y_\gamma\otimes 1+\varepsilon d(Y_\gamma\otimes 1)$$

$$1\otimes Z_\gamma\mapsto 1\otimes Z_\gamma+\varepsilon d(1\otimes Z_\gamma)$$

From the Theorem of Saito, we can produce a representation $\rho_\varepsilon:\pi_1(M)\to\mathrm{SL}_2(\bar{\Lambda})$ such that $\chi_{\rho_\varepsilon}=\varphi$, and it follows that the map $\xi\otimes[\gamma]\mapsto\frac{d}{d\varepsilon}\mathrm{Tr}(\xi\rho_\varepsilon(\gamma)\rho(\gamma)^{-1})$ induces an inverse of the map Ψ. We obtain the first isomorphism.
For the second one, first notice that the natural injection $\Omega_{k[\bar{X}]/k}\otimes k(\bar{X})\to\Omega_{k(\bar{X})/k}$ is an isomorphism, then it's a classical fact that the map $\Omega_{\bar{B}[\Gamma]/k}\otimes k[\bar{X}]\to\Omega_{k[\bar{X}]/k}$ is onto, its kernel being the $k[\bar{X}]$-module $\mathfrak{p}/\mathfrak{p}^2$. We conclude the proof by noting that $\mathfrak{p}/\mathfrak{p}^2\otimes_{k[\bar{X}]}k(\bar{X})=0$ because $X(M)$ is supposed to be reduced. □

2.1.2 Computation of the twisted cohomology

In this section we assume that $\bar{X}$ is a one dimensional component of $\bar{X}(M)$, of irreducible type. We denote by H the matrix $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, and by $r^*:H^i(M,\mathrm{Ad}\circ\rho)\to H^i(\partial M,\mathrm{Ad}\circ\rho)$ the morphism induced by the inclusion $\partial M\subset M$.

Proposition 2.1.5. *Let $\rho:\pi_1(M)\to\mathrm{SL}_2(k(\bar{X}))$ be the tautological representation, then $H^0(M,\mathrm{Ad}\circ\rho)=H^i(M,\mathrm{Ad}\circ\rho)=0$ for $i\geq 3$; $H^1(M,\mathrm{Ad}\circ\rho)$ is one dimensional and $H^2(M,\mathrm{Ad}\circ\rho)\simeq k(\bar{X})$, via the homomorphism $\eta\mapsto\mathrm{Tr}((r^*\eta)H)$.*

Proof. Pick a cellular complex with only 0, 1 and 2-cells, that has the same homotopy type than M, then $H^i(M,\mathrm{Ad}\circ\rho)=0,\forall i\geq 3$. By definition $H^0(M,\mathrm{Ad}\circ\rho)$ is the set of $\mathrm{Ad}\circ\rho$-invariants vectors, hence it is trivial because ρ is not abelian (see [Por97, Chapitre 0]. A classical equality is that $\dim\Omega_{k(\bar{X})/k}=\dim\bar{X}$, thus Proposition 2.1.4 together with Universal Coefficients Theorem imply that $\dim H^1(M,\mathrm{Ad}\circ\rho)=1$. The Euler characteristic of M is 0, thus $\dim H^2(M,\mathrm{Ad}\circ\rho)=1$ too.

Let us make the last isomorphism to be explicit: the long exact sequence of the pair $(M, \partial M)$ ends with

$$... \to H^2(M, \mathrm{Ad} \circ \rho) \xrightarrow{r^*} H^2(\partial M, \mathrm{Ad} \circ \rho) \to H^3(M, \partial M, \mathrm{Ad} \circ \rho)$$

Poincaré duality makes the last term vanish. As $\pi_1(\partial M)$ is abelian, $H^0(\partial M, \mathrm{Ad} \circ \rho)$ is not trivial, and so is $H^2(\partial M, \mathrm{Ad} \circ \rho)$; hence r^* is an isomorphism.
Now we use the construction of the augmented variety: up to conjugacy the restriction $\rho_\partial : \pi_1(\partial M) \to \mathrm{SL}_2(k(\bar{X}))$ is of the form $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$, hence its adjoint action on $\mathrm{sl}_2(k(\bar{X}))$ leaves the vector space spanned by H invariant. In other words, $H^0(\partial M, \mathrm{Ad} \circ \rho)$ is generated by H, and the result follows. □

2.2 The adjoint torsion as a differential form

In this section we define the torsion of the adjoint complex as a rational differential form on the augmented character variety. We give an explicit relation with previous work of Joan Porti, Jerome Dubois and others.

2.2.1 The torsion of an $\mathrm{Ad} \circ \rho$-twisted cellular complex

Here we consider the case of the torsion of the cellular complex $C^*(M, \mathrm{Ad} \circ \rho)$ with coefficients in $\mathrm{sl}_2(k(\bar{X}))$ twisted by a representation $\rho : \pi_1(M) \to \mathrm{SL}_2(k(\bar{X}))$. We pick the basis of $\mathrm{sl}_2(k(\bar{X}))$ given by the vectors

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Any cellular decomposition of M provides a $\mathbb{Z}[\pi_1(M)]$-basis of $C_i(\tilde{M})$ for any i, denoted by $\{\tilde{e}_i^1, ...\tilde{e}_i^{n_i}\}$. We will denote by $f^i_{1,E} \in C^i(M, \mathrm{Ad} \circ \rho)$ the map that sends $\tilde{e}_i^1$ on E and extend it in an $\mathrm{Ad} \circ \rho$-equivariant way, and similarly we obtain a basis $f^i = \{f^i_{k,E}, f^i_{k,F}, f^i_{k,H}, k = 1...n_i\}$ which is a basis of $C^i(M, \mathrm{Ad} \circ \rho)$. As in Section 1.5, we denote by $c = \bigotimes_i (\wedge f^i)^{\otimes (-1)^i}$.

Definition 2.2.1. We define the *Reidemeister adjoint torsion of the twisted complex* as

$$\mathrm{tor}(M, \mathrm{Ad} \circ \rho) = \mathrm{Eu}(c) \in \det(H^*(M, \mathrm{Ad} \circ \rho))$$

Remark 2.2.2. It does not depend on a choice of the lifts of the cells e_i^k in the universal cover of M (because the Adjoint map is unimodular) nor of the basis $\{E, F, H\}$ (because the Euler characteristic $\chi(M)$ vanishes) ; neither it depends on the conjugacy class of ρ.

2.2.2 The Reidemeister torsion form

Again we pick $\bar{X}$ to be a component of irreducible type of the augmented character variety $\bar{X}(M)$, and we fix $\rho : \pi_1(M) \to \mathrm{SL}_2(k(\bar{X}))$ a tautological representation. Since $H^0(M, \mathrm{Ad}\circ\rho)$ is trivial and $H^2(M, \mathrm{Ad}\circ\rho)$ is canonically isomorphic to $k(\bar{X})$ from Proposition 2.1.5, the torsion of the adjoint complex is an element of $\det(H^*(M, \mathrm{Ad}\circ\rho)) \simeq \det(H^1(M, \mathrm{Ad}\circ\rho))^*$. But $\det(H^1(M, \mathrm{Ad}\circ\rho))^* \simeq H^1(M, \mathrm{Ad}\circ\rho)^* \simeq H_1(M, \mathrm{Ad}\circ\rho) \simeq \Omega_{k(\bar{X})/k}$, the first isomorphism comes again from Proposition 2.1.5, the second from the Universal Coefficients Theorem and the third from Proposition 2.1.4.

Definition 2.2.3. The *Reidemeister torsion form* is the rational differential form $\mathrm{tor}(M, \mathrm{Ad}\circ\rho) \in \Omega_{k(\bar{X})/k}$.

2.2.3 Examples and computations

We give a relation between the torsion form as we defined it above and the previous work of J.Porti and then J.Dubois in [Por97, Dub06, DHY09]. As Dubois computed explicit formulae in many cases, this will permit us to perform direct computations in several examples.

Proposition 2.2.4 (see also Corollary 4.2 of [Por97])**.** *Given a finite point x in a component of irreducible type $\bar{X}$ of the augmented character variety, and v a valuation corresponding to x, we fix μ a class in $\pi_1(\partial M)$. Then the following holds:*

$$\mathrm{tor}(M, \mathrm{Ad}\circ\rho)(v) = \frac{1}{\tau_\mu(M, \mathrm{Ad}\circ\bar{\rho})} \frac{2}{\sqrt{(\mathrm{Tr}\,\bar{\rho}(\mu))^2 - 4}} dY_\mu$$

where $\tau_\mu(M, \mathrm{Ad}\circ\bar{\rho})$ is the torsion in the sense of Porti-Dubois at the point v, relatively to the curve μ.

Remark 2.2.5. Here we use the convention of [Por97], that corresponds to ours. In [Dub06] for instance, the torsion defined is the inverse of the latter.

Remark 2.2.6. Notice that τ_μ makes sense only when v is not a critical point of Y_μ. If we pull-back this formula on the augmented variety by setting $Y_\mu = Z_\mu + Z_\mu^{-1}$ then the formula becomes rational: $\mathrm{tor}(M, \mathrm{Ad}\circ\rho) = \frac{2}{\tau_\mu} \frac{dZ_\mu}{Z_\mu}$.

Proof. The torsion in the sense of Porti-Dubois is defined pointwise as a function on a smooth open subset U of the character varierty, and it is proved in [Por97] that it is in fact analytic. For $v \in U$, pick any generator P of $H^0(\partial M, \mathrm{Ad} \circ \bar{\rho})$. Then one obtains basis of $H^i(M, \mathrm{Ad} \circ \bar{\rho}), i = 1, 2$, as follows: since we assumed that $\bar{\rho}$ is not a critical point of the trace function Y_μ, the restriction map $H^1(M, \mathrm{Ad} \circ \bar{\rho}) \to H^1(\mu, \mathrm{Ad} \circ \bar{\rho})$ is an isomorphism. The latter is identified to k via $H^1(\mu, \mathrm{Ad} \circ \bar{\rho}) \to k$, $f \mapsto \mathrm{Tr}(Pf(\mu))$, and the composition of those maps is simply the differential dY_μ. Then the generator of $H^1(M, \mathrm{Ad} \circ \bar{\rho})$ is fixed to be $dY_\mu^{-1}(1)$. For the generator of $H^2(M, \mathrm{Ad} \circ \bar{\rho})$, pull-back 1 via the isomorphisms $H^2(M, \mathrm{Ad} \circ \bar{\rho}) \to H^2(\partial M, \mathrm{Ad} \circ \bar{\rho}) \xrightarrow{\phi_P} k$, where $\phi_P(f) = f(P)$. But P appears both in H^1 and in H^2, hence the torsion does not depend on a choice of P. Finally, we have chosen H to be a generator of $H^2(M, \mathrm{Ad} \circ \rho)$, and the isomorphism $H_1(M, \mathrm{Ad} \circ \rho) \to \Omega_{k(Y)/k}$ provides a term $\bar{\rho}_0(\mu)$, we need to normalize by $\sqrt{\frac{\mathrm{Tr}(H^2)}{\mathrm{Tr}(\bar{\rho}_0(\mu)^2)}}$, and the result follows. □

Example 2.2.7 (The trefoil knot)**.** Recall that the tautological representation of the component of irreducible type $X \subset X(M)$ is given by the formulae:

$$\rho(a) = \begin{pmatrix} t & 1 \\ -(t^2+1) & -t \end{pmatrix}, \rho(b) = \begin{pmatrix} -j & 0 \\ 0 & -j^2 \end{pmatrix}$$

In [Dub06], for any boundary curve $\mu, \tau_\mu(\bar{\rho})$ is a constant $1/k$ that does not depend on $\bar{\rho}$. Take μ the meridian ab^{-1}, $Y_\mu = (j - j^2)t$, $Z_\mu = u$, then

$$\mathrm{tor}(M, \mathrm{Ad} \circ \rho) = k\frac{du}{u}$$

It has no zeros, and two poles at zero and infinity, as expected its divisor's degree is -2 because the smooth projective model Y of X is isomorphic to $\mathbb{CP}^1$.

Notation. In the following examples, we will use the notation $f(t) \sim g(t)$ to say that f and g are equivalent around $t = 0$, up to a factor that does not depends on t, that is f and g have the same vanishing order at $t = 0$.

Example 2.2.8 (The figure-eight knot)**.** Here we take μ to be the longitude of M, denote its trace by $Y_\mu = x^4 - 5x^2 + 2$, then $\tau_\mu(x, y) = 5 - 2x^2$ and one obtains $\mathrm{tor}(M, \mathrm{Ad} \circ \rho) = \frac{dZ_\mu}{(5-2x^2)Z_\mu}$. A careful examination shows that it has no poles, and zeros only at infinity: take $x = 1/t$ a local coordinate, $\frac{dZ_\mu}{Z_\mu} = \frac{dY_\mu}{\sqrt{Y_\mu^2 - 4}} \sim \frac{dt}{t}$, hence each of the four ideal points contribute as a zero of order 1. The divisor's degree of the torsion is 4, that confirms the fact that the Riemann surface Y has genus 3.

Example 2.2.9 (The knot 5_2)**.** This example arises from [DHY09]. The fundamental group is isomorphic to $\pi_1(M) = \langle u, v |\ vw = wu\rangle$ where $w = u^{-1}v^{-1}uvu^{-1}v^{-1}$. The component of irreducible type of the character variety is described by the Riley polynomial $\phi(S, U)$. In our setting, with $x = \operatorname{Tr} u = \operatorname{Tr} v$ and $y = \operatorname{Tr} uv$, then $x = S^{\frac{1}{2}} + S^{-\frac{1}{2}}$ and $y = S + S^{-1} - U$, we obtain

$$X = X^{\mathrm{irr}}(M) = \left\{(x, y) \in k^2 |\ -x^2(y-1)(y-2) + y^3 - y^2 - 2y + 1 = 0\right\}$$

This affine curve compactifies with two points at infinity: an ordinary double point corresponding to the two directions $x = \infty, y = 1$ or $y = 2$, and a simple point $x = y = \infty$. Apart from this, the variety is smooth. By the Noether-Plücker formula, its genus is $(d-1)(d-2)/2 - \delta$, with $d = 4$ and $\delta = 1$, hence $g(\hat{X}) = 2$. The extension $\alpha + \alpha^{-1} = x$ gives a $2:1$ map $Y \to \hat{X}$, that ramifies at $x^2 = 4$. The Hurewitz formula implies $\chi(Y) = 2\chi(\hat{X}) - 6 = -10$, hence Y is a curve of genus 6. From [DHY09] again, with μ a longitude, $\tau_\mu = 5x^4(y-2) - x^2(5y^2 + 7y - 31) + 7(y^2 - y - 3)$, and $Y_\mu = (y^3 - 6y^2 + 12y - 8)x^{10} - (3y^4 - 10y^3 - y^2 - 68)x^8 + 3(y^5 - 43y^3 + 48y^2 + 86y - 116)x^6 + (y^6 + 6y^5 - 23y^4 - 28y^3 + 96y^2 + 28y - 105)x^4 + (2y^6 - y^5 - 16y^4 + 6y^3 + 40y^2 - 9y - 34)x^2 + 2$.
As $\operatorname{tor}(M, \operatorname{Ad} \circ \rho) = \frac{dY_\mu}{\tau_\mu \sqrt{Y_\mu^2 - 4}}$, we compute the vanishing order of the torsion at the 3 different ideal points:

1. $x \sim \frac{1}{t}$, $y \sim 1 + t^2$, then $\tau_\mu \sim \frac{1}{t^4}$, $\frac{dY_\mu}{\sqrt{Y_\mu^2 - 4}} \sim \frac{dt}{t}$ and tor $\sim t^3 dt$
2. $x \sim \frac{1}{t}$, $y \sim 2 + 3t$, then $\tau_\mu \sim \frac{1}{t^2}$, $\frac{dY_\mu}{\sqrt{Y_\mu^2 - 4}} \sim \frac{dt}{t}$ and tor $\sim t dt$
3. $x \sim \frac{1}{t(1-2t^2)}$, $y \sim \frac{1}{t^2(1-2t^2)}$, then again $\tau_\mu \sim \frac{1}{t^2}$, $\frac{dY_\mu}{\sqrt{Y_\mu^2 - 4}} \sim \frac{dt}{t}$ and tor $\sim t dt$

Finally, notice that $Y \to \hat{X}$ does not ramify at infinity, hence to each ideal point of $\hat{X}$ correspond 2 ideal points of Y, and the divisor's degree of tor on Y is 10, as expected.

Example 2.2.10 (The knot 6_1)**.** This example arises from [DHY09] too. The fundamental group is $\pi_1(M) = \langle u, v |\ vw = wu\rangle$ where $w = (vu^{-1}v^{-1}u)^2$. The irreducible type part of the character variety is

$$X = \{(x, y) \in \mathbb{C}^2\ | x^4(y-2)^2 - x^2(y+1)(y-2)(2y-3) + (y^3 - 3y - 1)(y-1) = 0\}$$

The two ideal points are non ordinary double points:

1. When $y \to 2$, $x \to \infty$, we have a double point of type "$y^2 - x^6$", its δ-invariant is 3.

2. When $y, x \to \infty$, we have a double point of type "$y^2 - x^8$", its δ-invariant is 4.

Hence $g(\hat{X}) = (d-1)(d-2)/2 - \sum \delta_i = 10 - 3 - 4 = 3$. The covering map $Y \to \hat{X}$ given by $\alpha + \alpha^{-1} = x$ ramifies in eight finite points, thus $\chi(Y) = -16$.
When desingularizing $\hat{X}$ one obtains four ideal points, the same kind of computations as in Example 2.2.9 are shortened as follows:

1. $x \sim \frac{1}{t(1+at^2)}$, $y \sim \frac{2}{1+at^2}$ with a a root of $4a^2+6a+1$ then in both cases $\tau \sim \frac{1}{t^2}$, $\frac{d\lambda}{\lambda} \sim \frac{dt}{t}$ and tor $\sim tdt$
2. $x \sim \frac{1}{t(1-t^2)}$, $y \sim \frac{1}{t^2(1-t^2)}$, then $\tau \sim \frac{1}{t^6}$, $\frac{d\lambda}{\lambda} \sim \frac{dt}{t}$ and tor $\sim t^5 dt$
3. $x \sim \frac{1}{t(1-2t^2+6t^4-25t^6)}$, $y \sim \frac{1}{t^2(1-2t^2+6t^4-25t^6)}$, then $\tau \sim 1$, $\frac{d\lambda}{\lambda} \sim tdt$ and tor $\sim tdt$

Then notice that $Y \to \hat{X}$ is not ramified at infinity, thus the divisor's degree of tor is 16, as expected.

2.3 The torsion at a finite point

Recall that X is a one-dimensional component of the character variety $X(M)$ of a 3-manifold M with boundary a torus, and the same homology as a circle. The curve Y is the smooth projective model of a double covering of X, namely the augmented character variety. Throughout this section $v \in Y$ is a finite point, and $\rho : \Gamma \to \mathrm{SL}_2(\mathcal{O}_v)$ is a convergent tautological representation (see Proposition 1.3.11 and what follows). We prove the following:

Theorem (Theorem 2.0.1)**.** *The differential form* $\mathrm{tor}(M, \mathrm{Ad} \circ \rho)$ *vanishes at* v *with order the length of the torsion part in the* $\mathcal{O}_v$*-module* $H_1(M, \mathrm{Ad} \circ \rho)_v$*. In particular* $\mathrm{tor}(M, \mathrm{Ad} \circ \rho)$ *has no poles nor zeros if* v *projects on a smooth point of* $X(M)$*.*

Definition 2.3.1. Let M be a torsion $\mathcal{O}_v$-module, then $M = \bigoplus_i \mathcal{O}_v/(t_i^{n})$ and we define its length to be equal to $\sum_i n_i$.

2.3.1 Proof of Main Theorem 1

Definition 2.3.2. We will say that a complex of $\mathcal{O}_v$-modules C^* is *rationally exact* if $C^* \otimes k(Y)$ is an exact sequence.

The following theorem is the key argument in the proof of Main Theorem 1. We explain the statement in the simple case of a rationally acyclic complex $\mathcal{O}_v^n \xrightarrow{A} \mathcal{O}_v^m$. Since the complex is generically acyclic, we deduce that $n = m$ and that the morphism A in injective. Now the torsion of the rational complex is $(\det A)^{-1}$, hence its valuation is given by $-v(\det A) = -\operatorname{length}(\mathcal{O}_v^n / \operatorname{im} A)$.

Theorem 2.3.3. *[GKZ94, Appendix A, theorem 30] Let v be a valuation on $k(Y)$, and C^* a rationally exact based complex of free $\mathcal{O}_v$-modules, with basis $\{c^i\}_i$. Then the following holds:*

$$v(\operatorname{tor}(C^* \otimes k(Y), \{c^i\}_i)) = \sum_k (-1)^k \operatorname{length}(H^k(C^*))$$

We will denote by t an uniforming parameter of $\mathcal{O}_v$, that is an element $t \in k(Y)$ with $v(t) = 1$. Hence the one-dimensional vector-space $\Omega_{k(Y)/k}$ is generated by dt. In the sequel we denote by $f \in k(Y)$ the function such that $\operatorname{tor}(M, \operatorname{Ad} \circ \rho) = f dt$ (more precisely, $\operatorname{tor}(M, \operatorname{Ad} \circ \rho) = f dt \otimes H^*$). The strategy is to construct a rationally exact based complex of $\mathcal{O}_v$ modules C^* with $\operatorname{tor}(C^*) = f$, and then use Theorem 2.3.3.

Let us denote by D^* the trivial complex

$$0 \to \Omega^*_{\mathcal{O}_v/k} \xrightarrow{0} H^0(\partial M, \operatorname{Ad} \circ \rho)^*_v$$

Lemma 2.3.4. *The $\mathcal{O}_v$-module $\Omega_{\mathcal{O}_v/k}$ is free of rank one, with dt as a generator.*

Proof. The $\mathcal{O}_v$-module $\Omega_{\mathcal{O}_v/k}$ is free because $\Omega_{\mathcal{O}_v/k} \otimes k \simeq (t)/(t^2) \simeq k$ (it is the cotangent space at v). Its rank is one because [Liu06, Proposition 1.8, Chapter 6] implies that $\Omega_{\mathcal{O}_v/k} \otimes k(Y) \simeq \Omega_{k(Y)/k}$, and it is one-dimensional by [Liu06, Example 1.6 and Lemma 1.13, Chapter 6]. Finally, again [Liu06, Proposition 1.8, Chapter 6] gives $\Omega_{\mathcal{O}_v/k} \simeq \Omega_{k[t]_{(t)}/k} \otimes \mathcal{O}_v$ hence dt is a generator since it generates $\Omega_{k[t]_{(t)}/k}$. □

Let $\alpha : C^1(M, \operatorname{Ad} \circ \rho)_v \to \Omega^*_{\mathcal{O}_v}$ defined by $\alpha(f)(d(Y_\gamma \otimes 1)) = \operatorname{Tr}(f(\gamma)\rho_0(\gamma))$, and $\beta : C^2(M, \operatorname{Ad} \circ \rho)_v \to H^2(M, \operatorname{Ad} \circ \rho)_v \to H^2(\partial M, \operatorname{Ad} \circ \rho)_v \to H^0(\partial M, \operatorname{Ad} \circ \rho)^*_v$ is the composition of the reduction mod $\operatorname{im} d$, the restriction map and the Poincare duality.

Proposition 2.3.5. *The maps α and β induce a morphism of complexes of $\mathcal{O}_v$-modules $\phi : C^*(M, \operatorname{Ad} \circ \rho)_v \to D^*$ that is rationally a quasi-isomorphism.*

Proof. Let us draw the diagram

$$\begin{array}{ccccc} 0 & \xrightarrow{0} & \Omega^*_{\mathcal{O}_v/k} & \xrightarrow{0} & H^0(\partial M, \mathrm{Ad}\circ\rho)^*_v \\ \uparrow 0 & & \uparrow \alpha & & \uparrow \beta \\ C^0(M, \mathrm{Ad}\circ\rho)_v & \xrightarrow{d} & C^1(M, \mathrm{Ad}\circ\rho)_v & \xrightarrow{d} & C^2(M, \mathrm{Ad}\circ\rho)_v \end{array}$$

First we need to show that this diagram commutes. It is clear from the definition that $\beta \circ d = 0$. Now for any $\zeta \in C^0(M, \mathrm{Ad}\circ\rho)_v$ and for any $\gamma \in \Gamma$, we have $\alpha(d\zeta)d(Y_\gamma \otimes 1) = \mathrm{Tr}(d\zeta(\gamma)\rho_0(\gamma))$. As $d\zeta(\gamma) = \rho(\gamma)\zeta\rho^{-1}(\gamma) - \zeta$ and as for any $\xi \in \mathrm{sl}_2(\mathcal{O}_v), \mathrm{Tr}(\xi\rho_0(\gamma)) = \mathrm{Tr}(\xi\rho(\gamma))$, we conclude that $\alpha(d\zeta)d(Y_\gamma \otimes 1) = \mathrm{Tr}[\rho_0(\gamma), \zeta] = 0$ as expected.
Similarly, if $\gamma \in \pi_1(\partial M)$, then $\rho(\gamma) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and we can compute directly that $\alpha(d\zeta)d(1 \otimes Z_\gamma) = \mathrm{Tr}(d\zeta(\gamma)\begin{pmatrix} \frac{\lambda}{2} & 0 \\ 0 & -\frac{\lambda}{2} \end{pmatrix}) = 0$.
Now we prove that the complexes $C^*(M, \mathrm{Ad}\circ\rho)$ and $D^* \otimes k(Y)$ are quasi-isomorphic. We have $H^1(M, \mathrm{Ad}\circ\rho) \simeq H_1(M, \mathrm{Ad}\circ\rho)^* \simeq \Omega^*_{k(Y)/k} \simeq \Omega^*_{\mathcal{O}_v/k} \otimes k(Y)$, the first isomorphism comes from the Universal Coefficient Theorem, the second from Proposition 2.1.4, and the third is a classical fact of algebraic geometry, see [Liu06, Chapter 6]. The very same proof as the end of the argument in Proof of Proposition 2.1.5 permits us to conclude that $H^2(M, \mathrm{Ad}\circ\rho) \simeq H^0(\partial M, \mathrm{Ad}\circ\rho)*$. Since ρ is not abelian, $H^0(M, \mathrm{Ad}\circ\rho) = 0$ and this concludes the proof. □

Definition 2.3.6. The *cone* of the morphism of complexes ϕ is defined as the complex $D^* \oplus C^{*+1}(M, \mathrm{Ad}\circ\rho)_v$:

$$C^0(M, \mathrm{Ad}\circ\rho)_v \xrightarrow{d} C^1(M, \mathrm{Ad}\circ\rho)_v \xrightarrow{d,\alpha} C^2(M, /Ad_\rho)_v \oplus \Omega^*_{\mathcal{O}_v/k} \xrightarrow{\beta} H^0(\partial M, \mathrm{Ad}\circ\rho)^*_v$$

The preceding lemma asserts that the complex $\mathrm{Cone}(\phi)$ is rationally exact. Moreover, it is naturally a based complex, by the natural basis of $C^*(M, \mathrm{Ad}\circ\rho)_v$, the duals of $dt \in \Omega_{\mathcal{O}_v/k}$ and of $H \in H^0(\partial M, \mathrm{Ad}\circ\rho)_v$. Then the torsion of this complex is (see [GKZ94, Appendix A, Proposition 18])

$$\mathrm{tor}(\mathrm{Cone}(\phi) \otimes k(Y)) = \frac{\mathrm{tor}(D^* \otimes k(Y))}{\mathrm{tor}(C^*(M, \mathrm{Ad}\circ\rho))} \tag{2.1}$$

We deduce the following lemma:

Lemma 2.3.7. *Recall that v is a finite point of Y, and that dt is a generator of*

$\Omega_{\mathcal{O}_v/k}$, *we express the torsion form as* $\mathrm{tor}(M, \mathrm{Ad}\circ\rho) = f dt$ *for some* $f \in k(Y)$. *The torsion of the complex* $\mathrm{Cone}(\phi) \otimes k(Y)$ *is* $\frac{1}{f} \in k(Y)$

Proof. By construction $\mathrm{tor}(D^*\otimes k(Y)) = dt \otimes H^*$, and the result follows from fromula (2.1). □

Now we can apply Theorem 2.3.3 to the rationally exact complex $\mathrm{Cone}(\phi)$, and obtain $v(f) = \sum_i (-1)^{i+1} \mathrm{length}(H^i(\mathrm{Cone}(\phi)))$. Now we compute the cohomology of this complex.

The exact sequence $0 \to D^* \to \mathrm{Cone}(\phi) \to C^{*+1}(M, \mathrm{Ad}\circ\rho)_v \to 0$ induces the long exact sequence in cohomology: $0 \to H^0(\mathrm{Cone}(\phi)) \to H^1(M, \mathrm{Ad}\circ\rho)_v \xrightarrow{\alpha} \Omega^*_{\mathcal{O}_v/k} \to H^1(\mathrm{Cone}(\phi)) \to H^2(M, \mathrm{Ad}\circ\rho)_v \to H^0(\partial M, \mathrm{Ad}\circ\rho)^*_v \to H^2(\mathrm{Cone}(\phi)) \to 0$

Lemma 2.3.8. *The morphism* α *is surjective.*

Proof. We construct a section $s : \Omega^*_{\mathcal{O}_v/k} \to H^1(M, \mathrm{Ad}\circ\rho)_v$ as follows. Let $\theta : \Omega_{\mathcal{O}_v/k} \to \mathcal{O}_v$ be a morphism of $\mathcal{O}_v$-modules, equivalently, by the universal property of Ω, θ is a k-derivation $\mathcal{O}_v \to \mathcal{O}_v$. We define $s(\theta) \in H^1(M, \mathrm{Ad}\circ\rho)_v$ by the formula $s(\theta)(\gamma) = \theta(\rho(\gamma))\rho(\gamma)^{-1}$ where by $\theta(\rho(\gamma))$ we mean that we apply θ to each coefficient of $\rho(\gamma)$. We compute directly that $\mathrm{Tr}\, s(\theta(\gamma)) = \theta(\det \rho(\gamma)) = 0$, and that $s(\theta(\gamma\delta)) = s(\theta(\gamma)) + \mathrm{Ad}\circ\rho(g) s(\theta(\delta))$.

Then $\alpha \circ s(\theta)(Y_\gamma) = \mathrm{Tr}(\theta(\rho(\gamma))) = \theta(Y_\gamma)$, and the lemma is proved. □

Lemma 2.3.9. *The morphism* α *is injective, hence* $H^0(\mathrm{Cone}(\phi)) = \{0\}$.

Proof. By the Universal Coefficient Theorem, $H^1(M, \mathrm{Ad}\circ\rho)_v \simeq H_1(M, \mathrm{Ad}\circ\rho)^*_v \simeq \mathcal{O}_v$ because $H_1(M, \mathrm{Ad}\circ\rho) \simeq k(Y)$. The lemma follows. □

Lemma 2.3.10. *Denote by* T *the torsion part of the module* $H_1(M, \mathrm{Ad}\circ\rho)_v$. *The* $\mathcal{O}_v$*-module* $H^1(\mathrm{Cone}(\phi))$ *is isomorphic to the torsion module* T.

Proof. Again by Universal Coefficient Theorem, there is an isomorphism

$$H^2(M, \mathrm{Ad}\circ\rho)_v \simeq H_2(M, \mathrm{Ad}\circ\rho)^*_v \oplus \mathrm{Ext}(H_1(M, \mathrm{Ad}\circ\rho)_v, \mathcal{O}_v) \simeq \mathcal{O}_v \oplus T$$

As α is surjective and $H^0(\partial M, \mathrm{Ad}\circ\rho)^*_v \simeq \mathcal{O}_v$, we deduce that $H^1(\mathrm{Cone}(\phi)) \simeq \ker(H^2(M, \mathrm{Ad}\circ\rho)_v \to H^0(\partial M, \mathrm{Ad}\circ\rho)^*_v) \simeq T$ □

Lemma 2.3.11. *The map* $H^2(M, \mathrm{Ad}\circ\rho)_v \to H^0(\partial M, \mathrm{Ad}\circ\rho)^*_v$ *is surjective, hence* $H^2(\mathrm{Cone}(\phi)) = \{0\}$.

Proof. Under the isomorphism $H^2(\partial M, \mathrm{Ad}\circ\rho)_v \simeq H^0(\partial M, \mathrm{Ad}\circ\rho)_v^*$, this map is just the last one of the long exact sequence of the pair $(M, \partial M)$. □

Proof of Main Theorem 1. Now we just have to fit together the arguments: write $\mathrm{tor}(M, \mathrm{Ad}\circ\rho) = f dt$, the vanishing order of $\mathrm{tor}(M, \mathrm{Ad}\circ\rho)$ at v is given by $v(f) = -v(\mathrm{tor}(\mathrm{Cone}(\phi) \otimes k(Y))) = \mathrm{length}(H^1(\mathrm{Cone}(\phi))) = \mathrm{length}(T)$. □

2.3.2 Interpretation of the Theorem: singularities and Alexander module

The aim of this section is to provide a geometric signification to the length of the module $H_1(M, \mathrm{Ad}\circ\rho)_v$ that appears in the statement of the Main Theorem 1.
Notice that if v corresponds to an irreducible character, $H_1(M, \mathrm{Ad}\circ\rho)_v \simeq \Omega_{\bar{B}[\Gamma]/k} \otimes \mathcal{O}_v$. If $\nu(v)$ is a smooth point, the latter is isomorphic to the localization of the module of differentials at v: $H_1(M, \mathrm{Ad}\circ\rho)_v \simeq \Omega_{\mathcal{O}_v/k}$, and this is simply $\mathcal{O}_v$. Hence we are interessed by the cases when $\nu(v)$ is not a smooth point of Y, or does correspond to a reducible character. This case is a singular case too: here v corresponds in $X(M)$ to an intersection point of a component of irreducible type and the reducible component. We treat both cases.

Singularities at irreducible characters

One has the following exact sequence $0 \to T \to \Omega_{\bar{B}[\Gamma]/k} \otimes \mathcal{O}_v \to \Omega_{\mathcal{O}_v/k}$ provided by the normalization map ν. Thus $\mathrm{length}(T)$ is an invariant of the branch of v at the singularity $\nu(v)$. We do not know any general formula, but we are able to compute it directly if a curve equation is given. Notice that this general question relies on the (still open) following problem: given a point of a curve $x \in X$, is it true that x is smooth iff $\Omega_{\mathcal{O}_x/k} \simeq \mathcal{O}_x$? See [Ber94] for a survey of the topic. We treat some examples of plane singularities.
Assume $x = (0, 0) \in k^2$, and C is the curve defined by the polynomial $X^p - Y^q$, with $p < q$. The singular point x has multiplicity p ; pick $\tilde{x}$ a pre-image of x by the normalization $\nu : \tilde{C} \to C$, and denote its discrete valuation ring by $\mathcal{O}$. Denote by $n = \gcd(p, q), p' = \frac{p}{n}, q' = \frac{q}{n}$. The normalization $\nu : \mathbb{A}^1_k \to C$ is given by

$$k[X, Y]/(X^p - Y^q) \to k[S]$$

$$X \mapsto S^{q'}, Y \mapsto S^{p'}$$

We compute $\Omega_{\mathcal{O}_x/k} = \mathcal{O}_x dX \oplus \mathcal{O}_x dY/(pX^{p-1}dX - qY^{q-1}dY)$, thus $\Omega_{\mathcal{O}_x/k} \otimes \mathcal{O} = \mathcal{O}dX \oplus \mathcal{O}dY/(pS^{q'(p-1)}dX - qS^{p'(q-1)}dY)$. The morphism $\Omega_{\mathcal{O}_x/k} \otimes \mathcal{O} \to \Omega_{\mathcal{O}/k}$ sends dX to $q'S^{q'-1}dS$ and dY to $p'S^{p'-1}dS$. The kernel of this morphism is generated by $p'dX - q'S^{q'-p'}dY \in \Omega_{\mathcal{O}_x/k} \otimes \mathcal{O}$, and its annihilator is $(nS^{q'(p-1)})$. Hence $T \simeq \mathcal{O}/(S^{q'(p-1)})$ and $\mathrm{length}(T) = q'(p-1)$.

Singularities at reducible characters

In this section we focus on $v \in Y$ that corresponds to a reducible character. It is precisely the case when it corresponds to an intersection point of the component of irreducible type corresponding to Y with the reducible component. By Section 1.3.11, we can pick the tautological representation $\rho : \Gamma \to \mathrm{SL}_2(\mathcal{O}_v)$ to be not residually abelian.
We need to study the $\mathcal{O}_v$-module $H_1(M, \mathrm{Ad} \circ \rho)_v$, but here we do not dispose of an interpretation in terms of the cotangent space. The strategy is the following: recall that we can write $H_1(M, \mathrm{Ad} \circ \rho)_v = \mathcal{O}_v \oplus \bigoplus_{i=1}^{l} \mathcal{O}_v/(t^{n_i})$, so we need to compute $\sum_i n_i$. First we consider $H_1(M, \mathrm{Ad} \circ \rho)_v \otimes \mathcal{O}_v/(t) \simeq H_1(M, \mathrm{Ad} \circ \bar{\rho})$, it is a k-vector space whose dimension will be the integer $l+1$. Then we prove that under some hypothesis, all of the n_i's are equal to 1.
Up to conjugacy, we know that we can fix the tautological representation $\rho : \Gamma \to \mathrm{SL}_2(\mathcal{O}_v)$ such that $\forall \gamma \in \Gamma, \bar{\rho}(\gamma) = \begin{pmatrix} \lambda^{\varphi(\gamma)} & \lambda^{-\varphi(\gamma)}u(\gamma) \\ 0 & \lambda^{-\varphi(\gamma)} \end{pmatrix}$ for some $\lambda \in k^*$ such that $\Delta_M(\lambda^2) = 0$, see Section 1.4.2.

Lemma 2.3.12. *The $\mathcal{O}_v$-module $H_0(M, \mathrm{Ad} \circ \rho)_v$ is trivial.*

Proof. Using twice the Universal Coefficient Theorem, we have

$$H_0(M, \mathrm{Ad} \circ \rho)_v \otimes_{\mathcal{O}_v} \mathcal{O}_v/(t) \simeq H_0(M, \mathrm{Ad} \circ \bar{\rho}) \simeq H^0(M, \mathrm{Ad} \circ \bar{\rho})^*$$

and the last term is trivial since $\bar{\rho}$ is not abelian. □

This section aims to relate the order of the torsion at such a finite point v where the tautological representation is residually reducible, with the order of λ^2 as a root of the Alexander polynomial. We will denote by $C^*(M, k_{\lambda^{\pm 2}})$ the complex of group cohomology of Γ with coefficients in k twisted by the action of λ^2, resp. λ^{-2}. As we proved in Section 1.4.2, the map $u : \Gamma \to k$ is a non trivial cocycle in the first cohomology group $H^1(M, \lambda^2)$.

Let fix some notations: the adjoint action of $\bar{\rho}$ on $\mathrm{sl}_2(k)$ has the following matrix in the basis $\{E, H, F\}$:

$$\operatorname{Ad}\circ\bar{\rho} = \begin{pmatrix} \lambda^2 & -2u & -\lambda^{-2}u^2 \\ 0 & 1 & \lambda^{-2}u \\ 0 & 0 & \lambda^{-2} \end{pmatrix} \tag{2.2}$$

The Γ-module $\mathrm{sl}_2(k)$ will be denoted by $\operatorname{Ad}\circ\bar{\rho}$, and K will be the submodule $\mathrm{span}_k\langle E, H\rangle$ with the induced Γ-action given by $\left(\begin{smallmatrix} \lambda^2 & -2u \\ 0 & 1 \end{smallmatrix}\right)$. We denote by $k_{\lambda^{-2}}$ the submodule $\mathrm{span}_k\langle F\rangle$. From the matrix (2.2) we obtain the short exact sequence of Γ-modules:

$$0 \to K \to \operatorname{Ad}\circ\bar{\rho} \to k_{\lambda^{-2}} \to 0$$

that gives rise to the long exact sequence:

$$\begin{aligned} 0 \to H^1(M, K) \to H^1(M, \operatorname{Ad}\circ\bar{\rho}) \to H^1(M, \lambda^{-2}) \\ \to H^2(M, K) \to H^2(M, \operatorname{Ad}\circ\bar{\rho}) \to H^2(M, \lambda^{-2}) \to 0 \end{aligned} \tag{2.3}$$

Notice that $H^0(M, \lambda^2)$ is trivial as soon as there exists $\gamma \in \Gamma$ such that $\lambda^2(\gamma) \neq 1$, i.e. as soon as $\bar{\rho}$ is not central as a character.
Similarly, there is the short exact sequence of Γ-modules:

$$0 \to k_{\lambda^2} \to K \to k \to 0$$

where k_{λ^2} denotes the submodule $\mathrm{span}_k\langle E\rangle$ and k denotes the submodule $\mathrm{span}_k\langle H\rangle$, hence we obtain the following exact sequence

$$\begin{aligned} 0 \to H^0(M, k) \to H^1(M, \lambda^2) \to H^1(M, K) \\ \to H^1(M, k) \xrightarrow{\delta} H^2(M, \lambda^2) \to H^2(M, K) \to 0 \end{aligned} \tag{2.4}$$

Definition 2.3.13. We define the *cup-bracket*:

$$[.\cup.] : H^1(\Gamma, \operatorname{Ad}\circ\bar{\rho}) \times H^1(\Gamma, \operatorname{Ad}\circ\bar{\rho}) \to H^2(\Gamma, \operatorname{Ad}\circ\bar{\rho})$$

$$(\zeta_1, \zeta_2) \mapsto ([\zeta_1 \cup \zeta_2] : (\gamma, \delta) \mapsto [\zeta_1(\gamma), \operatorname{Ad}\circ\bar{\rho}(\gamma)\zeta_2(\delta)])$$

Definition 2.3.14. We define the first order deformation of $\bar{\rho}$ as the map $\xi \in C^1(M, \operatorname{Ad}\circ\bar{\rho})$ such that in $\mathrm{SL}_2(\mathcal{O}_v/(t^2))$, the mod (t^2)-reduced tautological representation is given by $\rho' = (\operatorname{Id} + t\xi)\bar{\rho}$.

The following lemma recalls the fact of deformation theory (see [Gol84, section 1.4] for instance) that ξ is a cocycle in $H^1(M, \mathrm{Ad} \circ \bar{\rho})$ and that $[\xi \cup \xi] = 0$ in this context, in other words, ξ lies in the kernel of the morphism $B : H^1(M, \mathrm{Ad} \circ \bar{\rho}) \to H^2(M, \mathrm{Ad} \circ \bar{\rho})$ that sends a cocycle ζ on the cup-bracket $[\zeta \cup \xi]$.

Lemma 2.3.15. *The first order deformation ξ of ρ is a cocycle, and $B(\xi) = 0 \in H^2(M, \mathrm{Ad} \circ \bar{\rho})$.*

Proof. The first claims follows from the fact that, for all $\gamma, \delta \in G$, $\rho'(\gamma\delta) = \rho'(\gamma)\rho'(\delta) \mod(t^2)$. Moreover, let $\rho'' = (\mathrm{Id} + t\xi + t^2\eta)\bar{\rho}$ be the $\mathrm{mod}(t^3)$ representation $: \Gamma \to \mathrm{SL}_2(\mathcal{O}_v/(t^3))$. Again, ρ'' is a group homomorphism, hence a direct computation gives $\eta(\gamma\delta) = \eta(\gamma) + \mathrm{Ad} \circ \bar{\rho}(\gamma)\eta(\delta) + \xi(\gamma) \mathrm{Ad} \circ \bar{\rho}(\gamma)\xi(\delta)$. This equality can be written as $d\eta(\gamma, \delta) = \xi \cup \xi(\gamma, \delta) \in H^2(\Gamma, \mathrm{Ad} \circ \bar{\rho} \otimes \mathrm{Ad} \circ \bar{\rho})$, hence $\xi \cup \xi = 0$. On the other hand,

$$\begin{aligned}
d\xi^2(\gamma, \delta) &= \xi^2(\gamma\delta) - \xi^2(\gamma) - \mathrm{Ad} \circ \bar{\rho}(\gamma)\xi^2(\delta) \\
&= (\xi(\gamma) + \mathrm{Ad} \circ \bar{\rho}(\gamma)\xi(\delta))^2 - \xi^2(\gamma) - \mathrm{Ad} \circ \bar{\rho}(\gamma)\xi^2(\delta) \\
&= \xi^2(\gamma) + \xi(\gamma) \mathrm{Ad} \circ \bar{\rho}(\gamma)\xi(\delta) + \mathrm{Ad} \circ \bar{\rho}(\gamma)(\xi(\delta))\xi(\gamma) \\
&+ (\mathrm{Ad} \circ \bar{\rho}(\gamma)\xi(\delta))^2 - \xi^2(\gamma) - (\mathrm{Ad} \circ \bar{\rho}(\gamma)\xi(\delta))^2 \\
&= \xi(\gamma) \mathrm{Ad} \circ \bar{\rho}(\gamma)\xi(\delta) + \mathrm{Ad} \circ \bar{\rho}(\gamma)(\xi(\delta))\xi(\gamma) \\
&= -[\xi \cup \xi](\gamma, \delta) + 2\xi \cup \xi(\gamma, \delta)
\end{aligned}$$

hence $[\xi \cup \xi] = d(2\eta - \xi^2) = 0 \in H^2(\Gamma, \mathrm{Ad} \circ \bar{\rho})$. □

The theorem is the following:

Theorem 2.3.16. *Let $v \in Y$ a finite point, assume that the tautological representation $\rho : \Gamma \to \mathrm{SL}_2(\mathcal{O}_v)$ is residually reducible with residual representation $\bar{\rho}(\gamma) = \begin{pmatrix} \lambda^{\varphi(\gamma)} & \lambda^{-\varphi(\gamma)}u(\gamma) \\ 0 & \lambda^{-\varphi(\gamma)} \end{pmatrix}$ and denote by $r \geq 1$ the order of λ^2 as a root of the Alexander poynomial Δ_M. Assume moreover that the morphism B has maximal rank, i.e. $\ker B = k\langle\xi\rangle$. Then the torsion form's vanishing order at v is bounded by $2r - 2$.*

Remark 2.3.17. In the generic case when λ^2 is a simple root, then the technical assumption is automatically satisfied and the theorem always holds: thus in this case $\mathrm{tor}(M, \mathrm{Ad} \circ \rho)$ does not vanish at v.

Proof. As λ^2 is a root of order r of the Alexander polynomial, the Alexander module $H_1(M, k[t^{\pm 1}]_\varphi)$ has $(t - \lambda^2)$-length equal to r, in other words, its $(t - \lambda^2)$-torsion is of the form $\bigoplus_{i=1}^{l} \frac{k[t^{\pm 1}]}{(t-\lambda^2)^{n_i}}$ with $\sum_{i=1}^{l} n_i = r$. In particular $1 \leq l \leq r$ and from the Universal Coefficients Theorem, one obtains that $\dim H^1(M, \lambda^2) = l \in \{1, ...r\}$. If $l = r$, then we use Proposition 2.3.19 (postponed at the end of this section) to conclude that the map δ in the sequence (2.4) is injective, hence $\dim H^1(M, K) = \dim H^1(M, \lambda^2) - 1 = r - 1$. Otherwise $l \leq r - 1$, and as $\dim(H^1(M, k)) = 1$, again by sequence (2.4) the inequality $\dim(H^1(M, K)) \leq r - 1$ holds.

Now we consider the sequence (2.3). As the Alexander polynomial is symmetric, we know that $\dim(H^1(M, \lambda^{-2})) = \dim(H^1(M, \lambda^2)) \leq r$. A careful observation of this exact sequence is now enough to prove that $\dim H^1(M, \mathrm{Ad} \circ \bar{\rho}) \leq 2r - 1$. We denote this dimension by m.

Consider the sequence of Γ-modules

$$0 \to \mathrm{sl}_2(\mathcal{O}_v/(t)) \xrightarrow{.t} \mathrm{sl}_2(\mathcal{O}_v/(t^2)) \xrightarrow{\bmod t} \mathrm{sl}_2(\mathcal{O}_v/(t)) \to 0$$

From this we get

$$\begin{aligned} 0 \to H^1(M, \mathrm{Ad} \circ \bar{\rho}) \to H^1(M, \mathrm{Ad} \circ \rho') \xrightarrow{p} H^1(M, \mathrm{Ad} \circ \bar{\rho}) \\ \xrightarrow{B} H^2(M, \mathrm{Ad} \circ \bar{\rho}) \to H^2(M, \mathrm{Ad} \circ \rho') \to H^2(M, \mathrm{Ad} \circ \bar{\rho}) \to 0 \quad (2.5) \end{aligned}$$

where $\mathrm{Ad} \circ \rho'$ denotes the homology of the complex with coefficients in $\mathrm{sl}_2(\mathcal{O}_v/(t^2))$. By Lemma 2.3.18 below, the connection operator is nothing but the map B we defined above.

We write $H_1(M, \mathrm{Ad} \circ \rho)_v = \mathcal{O}_v \oplus \bigoplus_{i=1}^{m-1} \mathcal{O}_v/(t^{n_i})$, hence we have $H^1(M, \mathrm{Ad} \circ \bar{\rho}) = \mathcal{O}_v/(t) \oplus \bigoplus_i \mathcal{O}_v/(t)$. Moreover we have $H_1(M, \mathrm{Ad} \circ \rho') \simeq \mathcal{O}_v/(t^2) \oplus \bigoplus_i \mathcal{O}_v/(t^{\min(n_i,2)})$, and by the Universal Coefficients Theorem,

$$H^1(M, \mathrm{Ad} \circ \rho') \simeq \mathrm{Hom}(H_1(M, \mathrm{Ad} \circ \rho'), \mathcal{O}_v/(t^2)) \simeq H_1(M, \mathrm{Ad} \circ \rho')$$

The first terms of equation (2.5) become

$$0 \to \mathcal{O}_v/(t) \oplus \bigoplus_i \mathcal{O}_v/(t) \xrightarrow{i} \mathcal{O}_v/(t^2) \oplus \bigoplus_i \mathcal{O}_v/(t^{\min(n_i,2)}) \xrightarrow{p} \mathcal{O}_v/(t) \oplus \bigoplus_i \mathcal{O}_v/(t) \xrightarrow{B} ...$$

The image of p is the kernel of B, thus it is a copy of $\mathcal{O}_v/(t)$ generated by ξ. Hence the image of i is $\mathcal{O}_v/(t) \oplus \bigoplus_i \mathcal{O}_v/(t^{\min(n_i,2)})$, and a simple count of dimensions proves that $n_i = 1$ for all i. In conclusion $H_1(M, \mathrm{Ad} \circ \rho)_v \simeq \mathcal{O}_v \oplus (\mathcal{O}_v/(t))^{m-1}$ and the torsion vanishes at order $m-1 \leq 2r-2$. □

Lemma 2.3.18. *The connexion operator in the sequence (2.5) is the map* $B : H^1(M, \mathrm{Ad} \circ \bar{\rho}) \to H^2(M, \mathrm{Ad} \circ \bar{\rho})$, $\zeta \mapsto [\xi \cup \zeta]$.

Proof. We draw the diagram

$$\begin{array}{ccccccccc} 0 \to & C^1(M, \mathrm{Ad} \circ \rho) & \longrightarrow & C^1(M, \mathrm{Ad} \circ \rho') & \longrightarrow & C^1(M, \mathrm{Ad} \circ \rho) & \to 0 \\ & \downarrow d & & \downarrow d & & \downarrow d & \\ 0 \to & C^2(M, \mathrm{Ad} \circ \rho) & \longrightarrow & C^2(M, \mathrm{Ad} \circ \rho') & \longrightarrow & C^2(M, \mathrm{Ad} \circ \rho) & \to 0 \end{array}$$

with exact rows. We pick $\zeta \in Z^1(M, \mathrm{Ad} \circ \rho)$, and we lift it to $\tilde{\zeta} \in C^1(M, \mathrm{Ad} \circ \rho')$. For $\gamma, \delta \in \Gamma$, we compute $d\tilde{\zeta}(\gamma, \delta) = \tilde{\zeta}(\gamma) + \mathrm{Ad} \circ \bar{\rho}(\gamma)\tilde{\zeta}(\delta) + t(\xi(\gamma) \mathrm{Ad} \circ \bar{\rho}(\gamma)\tilde{\zeta}(\delta) - \mathrm{Ad} \circ \bar{\rho}(\gamma)\tilde{\zeta}(\delta)\xi(\gamma)) = d\zeta(\tilde{\gamma}, \delta) + t[\xi \cup \zeta](\gamma, \delta)$. As $d\zeta = 0$, the result follows. □

Proposition 2.3.19. *Recall that r is the order of λ^2 as a root of the Alexander polynomial. If $\dim H^1(M, \lambda^2) = r$, then the morphism $\delta : H^1(M, k) \to H^2(M, \lambda^2)$ is injective.*

The proof consists of the following succession of lemmas.

Lemma 2.3.20. *The morphism $\delta : H^1(M, k) \to H^2(M, \lambda^2)$ is the map $\varphi \mapsto 2u \cup \varphi$.*

Proof. Again, we draw the diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & C^1(M, \lambda^2) & \longrightarrow & C^1(M, K) & \longrightarrow & C^1(M, k) & \longrightarrow 0 \\ & \downarrow d & & \downarrow d & & \downarrow d & \\ 0 \longrightarrow & C^2(M, \lambda^2) & \longrightarrow & C^2(M, K) & \longrightarrow & C^2(M, k) & \longrightarrow 0 \end{array}$$

with exact rows.

Pick $\varphi \in C^1(M, k)$ such that $d\varphi = 0$. Take $\tilde{\varphi} \in C^1(M, K)$ defined by $\tilde{\varphi}(\gamma) = \varphi(\gamma)H$. Then compute $d\tilde{\varphi}(\gamma, \delta) = \tilde{\varphi}(\gamma\delta) - \tilde{\varphi}(\gamma) - \gamma.\tilde{\varphi}(\delta)$ Observe that by the adjoint matrix (2.2) we have $\gamma.\tilde{\varphi}(\delta) = \mathrm{Ad} \circ \bar{\rho}(\gamma)_{|K}\tilde{\varphi}(\delta) = \tilde{\varphi}(\delta) - 2u(\gamma)\varphi(\delta)E$. As $\varphi \in Z^1(M, k)$, we get that $\tilde{\varphi}(\gamma\delta) = \tilde{\varphi}(\gamma) + \tilde{\varphi}(\delta)$, thus

$$d\tilde{\varphi}(\gamma, \delta) = -2u(\gamma)\varphi(\delta)E$$

The conclusion follows. □

The multiplication by $t-\lambda^2$ induces the short exact sequence of Γ-modules:

$$0 \to k[t^{\pm 1}]_\varphi \xrightarrow{t-\lambda^2} k[t^{\pm 1}]_\varphi \to k_{\lambda^2} \to 0 \tag{2.6}$$

Lemma 2.3.21. *The sequence above induces the following long exact sequence:*

$$0 \to H^1(M,\lambda^2) \to H_1(M,k[t^{\pm 1}]_\varphi) \to H_1(M,k[t^{\pm 1}]_\varphi) \to H^2(M,\lambda^2) \to 0 \tag{2.7}$$

Proof. Let's write the long exact sequence in homology induced by (2.6)

$$... \to H^1(M,\lambda^2) \to H^2(M,k[t^{\pm 1}]_\varphi) \to H^2(M,k[t^{\pm 1}]_\varphi) \to H^2(M,\lambda^2) \to ...$$

First, let's prove that the first and the last arrows of this sequence are 0's. We remark that the missing term on the right is $H^3(k[t^{\pm 1}]) \simeq 0$. Then as the missing term on the left is $H^1(k[t^{\pm 1}]_\varphi) \simeq \mathrm{Ext}(H_0(M,k[t^{\pm 1}]_\varphi),k[t^{\pm 1}]) \simeq k[t^{\pm 1}]/(t-1)$ by the universal coefficients theorem, we obtain that the map induced by multiplication by $t-\lambda^2$

$$H^1(M,k[t^{\pm 1}]_\varphi) \xrightarrow{.(t-\lambda^2)} H^1(M,k[t^{\pm 1}]_\varphi)$$

is surjective, thus the first map on the left is $H^1(M,k[t^{\pm 1}]_\varphi) \xrightarrow{0} H^1(M,\lambda^2)$.
Now, we have the universal coefficients theorem that gives the following isomorphism $H^2(M,k[t^{\pm 1}]_\varphi) \simeq \mathrm{Ext}(H_1(M,k[t^{\pm 1}]_\varphi),k[t^{\pm 1}])$. Since $H_1(M,k[t^{\pm 1}]_\varphi)$ is a torsion module, the latter is $H_1(M,k[t^{\pm 1}]_\varphi)$ itself, that ends the proof of the lemma. □

We denote by $\theta : H^1(M,\lambda^2) \to H^2(M,\lambda^2)$ the composition of the first and the third map in the sequence (2.7).

Lemma 2.3.22. $\theta(z) = \lambda^{-2}\varphi_\cup z$, *in particular* $2\lambda^2\theta(u) = \delta(\varphi)$.

Proof. Again, consider the sequence:

$$0 \to H^1(M,\lambda^2) \to H^2(M,k[t^{\pm 1}]_\varphi) \to H^2(M,k[t^{\pm 1}]_\varphi) \to H^2(M,\lambda^2) \to 0$$

Then the same kind of computation as in the proof of Lemmas 2.3.18 and 2.3.20 gives the result. We sketch the proof: given $z \in H^1(M,\lambda^2)$, first lift it as $\tilde{z}(t) \in C^1(M,k[t^{\pm 1}]_\varphi)$. Denote by $D : C^1(M,k[t^{\pm 1}]_\varphi) \to C^2(M,k[t^{\pm 1}]_\varphi)$ the boundary map ; then send $\tilde{z}$ on $\frac{D\tilde{z}(t)}{t-\lambda^2} \in C^2(M,k[t^{\pm 1}]_\varphi)$, and then take the evaluation at $t=\lambda^2$. The result is precisely $\frac{d}{dt}\Big|_{t=\lambda^2} D\tilde{z}(t) = \lambda^{-2}\varphi_\cup z \in H^2(M,\lambda^2)$ □

Lemma 2.3.23. *If* $\dim(H^1(M,\lambda^2)) = r$, *then the map* θ *is an isomorphism.*

Proof. Consider the endomorphism A of $H_1(M, k[t^{\pm 1}]_\varphi)$ induced by the multiplication by $(t-\lambda^2)$. Its kernel (and cokernel) is exactly $\bigoplus_{i=1}^{r} \frac{k[t^{\pm 1}]}{(t-\lambda^2)}$ because teh dimension of the vector space $(H^1(M, \lambda^2))$ is equal to r, hence θ is an isomorphism $\ker A \to \operatorname{coker} A$. □

Proof of Proposition 2.3.19. The map θ is an isomorphism, in particular $\theta(u) \neq 0$, hence $\delta(\varphi) \neq 0$ and δ is injective. □

2.4 The torsion at an ideal point

In this section we consider $v \in Y$ an ideal point. We denote by t an uniforming parameter of the valuation ring $\mathcal{O}_v$.
We mentioned in Section 1.3 the construction of Marc Culler and Peter Shalen: such an ideal point induces an action of $\pi_1(M)$ on a simplicial tree T_v. In [Sha02], Shalen explains how one can produce a so-called *dual surface* Σ to the action of $\pi_1(M)$ on T_v. Moreover, this surface can be chosen to be *incompressible* in the manifold M, that is such that the inclusion map $\pi_1(\Sigma) \to \pi_1(M)$ is injective.

Definition 2.4.1. We say that $\Sigma \subset M$ is *free* is its complement $M \setminus \Sigma$ has a free fundamental group.

This section is devoted to the proof of the following theorem:

Theorem (Theorem 2.0.2)**.** *Let M be a 3-manifold with boundary a torus and rational homology of a circle ; and let $v \in Y$ an ideal point in the smooth projective model of a one dimensional component of irreducible type of the augmented character variety. Assume that an incompressible surface Σ associated to v is connected or is the union of n parallel homeomorphic copies, is free, is not the Seifert surface, and assume that the restricted tautological representation $\rho_\Sigma : \pi_1(\Sigma) \to \mathrm{SL}_2(\mathcal{O}_v)$ is not abelian. Then the following inequality holds:*

$$v(\operatorname{tor}(M, \operatorname{Ad} \circ \rho)) \leq -n\chi(\Sigma) - m$$

where $m = n$ if ρ_Σ is not residually abelian, and m is the length of the torsion module $H^1(\Sigma, \operatorname{Ad} \circ \rho_\Sigma)_v$ if ρ_Σ is residually abelian.

Remark 2.4.2. 1. Notice that since $H_2(M, \partial M) = \mathbb{Z}$ generated by the incompressible Seifert surface, any incompressible non-Seifert surface with boundary is homologically trivial, hence is separating.

2. Moreover, it is a consequence from [Prz83, Proposition 4.3] that as soon as M is irreducible and small, any incompressible surface Σ is free.

3. Finally, the hypothesis that the irreducible tautological representation $\rho : \pi_1(M) \to \mathrm{SL}_2(k(Y))$ remains irreducible (or at least does not become abelian) is generic, in the sense that it is Zariski-open in the character variety of Σ. Moreover, if X contains the character of a faithful representation, then the tautological representation $\rho : \pi_1(M) \to \mathrm{SL}_2(k(X))$ is faithful, thus ρ_Σ is abelian iff $\pi_1(\Sigma)$ is abelian, that is iff Σ is an annulus. On the other hand, ideal points in character varieties of torus knots produce incompressible annuli in their complement. Since we can compute "by hand" the torsion in those cases (see below Section 2.4.1), we do not attempt to treat the case when ρ_Σ is abelian. Finally, we observe that the conclusion of Theorem 2.0.2 remains true (even the equality of Question 2.0.4).

2.4.1 An example: a direct computation of the torsion of the trefoil knot using its incompressible non-Seifert surface

The trefoil complement M carries an unique incompressible, non Seifert surface, which is an annulus. It is depicted on Figure 2.1.

The complement of this incompressible annulus A is the disjoint union of two solid torii, M_1 is the "interior" of T, and M_2 its "exterior".

Recall the following presentation of the fundamental group $\pi_1(M) = \langle a, b |\ a^2 = b^3 \rangle$, where a is the generator of the infinite cyclic fundamental group of M_1, b is the generator of the infinite cyclic fundamental group of M_2 and $a^2 = b^3 = u$ is the generator of the fundamental group of A.

We recall the description of the component of irreducible type X of the character variety $X(M)$. Notice that the subgroup $\langle u \rangle$ is the center of $\pi_1(M)$, hence any irreducible representation must send it onto $\{\pm \mathrm{Id}\}$. Moreover, if $\rho(u) = \mathrm{Id}$, then $\rho(a) = \pm \mathrm{Id}$ what would imply that ρ is reducible. Hence we fix $\rho(u) = -\mathrm{Id}$, and we can choose, up to conjugacy, $\rho(b) = \begin{pmatrix} -j & 0 \\ 0 & -j^2 \end{pmatrix}$, where j is a primitive third root of 1. If the upper-right entry of $\rho(a)$ was 0, then again ρ would be reducible, hence we can fix, up to conjugacy that preserves the diagonal matrix $\rho(b)$, this entry to be 1. Now observe that the Cayley-Hamilton identity $\rho(a)^2 - \mathrm{Tr}(\rho(a))\rho(a) + \mathrm{Id} = 0$ implies $\mathrm{Tr}(\rho(a)) = 0$, and the identity $\det(\rho(a)) = 1$ leads us to the following expression of

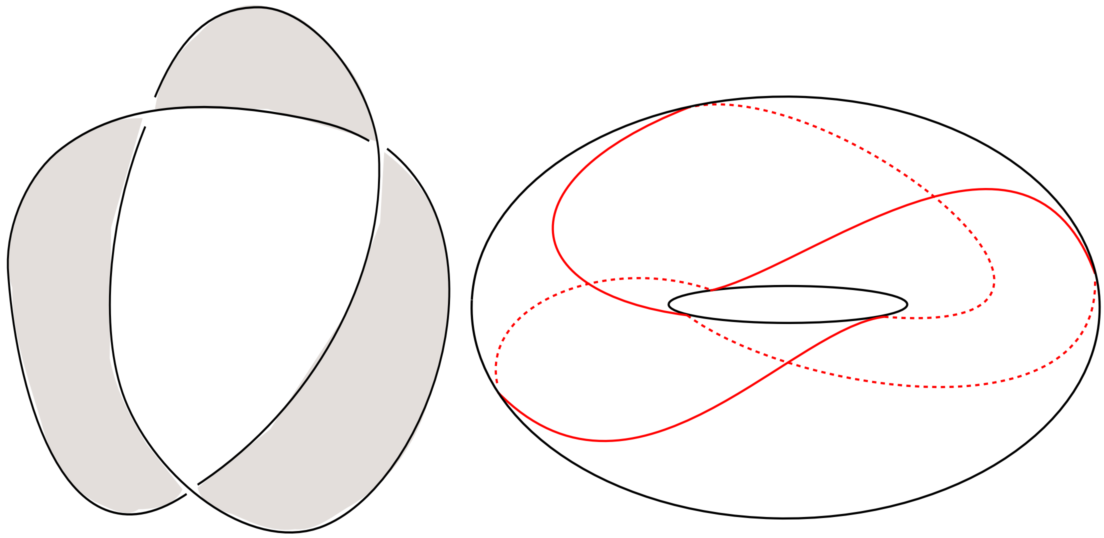


Figure 2.1 – On the left, the classical diagram of a trefoil knot, with an incompressible Möbius band in its complement. On the right, the red curve is an embedding of the trefoil knot in a torus T. The complement of a tubular neighborhood of this curve in this torus is an incompressible annulus, which is the orientation covering of the Möbius band on the left.

$\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C}(t))$:

$$\rho(a) = \begin{pmatrix} t & 1 \\ -(t^2+1) & -t \end{pmatrix}, \rho(b) = \begin{pmatrix} -j & 0 \\ 0 & -j^2 \end{pmatrix}$$

In particular X is a rational curve, parametrized by t.

The aim of the present section is to compute "by hand" the order of the Reidemeister torsion form when t goes to infinity, that is the asymptotic of the torsion of the complex $C^0(M, \mathrm{Ad} \circ \rho) \to C^1(M, \mathrm{Ad} \circ \rho) \to C^2(M, \mathrm{Ad} \circ \rho)$.

The splitting $M = M_1 \cup_A M_2$ provides the following exact sequence of complexes

$$0 \to C^*(M, \mathrm{Ad} \circ \rho) \to C^*(M_1, \mathrm{Ad} \circ \rho)) \oplus C^*(M_2, \mathrm{Ad} \circ \rho) \to C^*(A, \mathrm{Ad} \circ \rho) \to 0$$

that induces the long exact Mayer-Vietoris sequence $\mathcal{H}$:

$$0 \to H^0(M, \mathrm{Ad} \circ \rho) \to H^0(M_1, \mathrm{Ad} \circ \rho) \oplus H^0(M_2, \mathrm{Ad} \circ \rho) \to \dots$$

$$\to H^1(A, \mathrm{Ad} \circ \rho) \to H^2(M, \mathrm{Ad} \circ \rho) \to 0$$

The sequence ends there because M_i and A are homotopic to a circle, hence they do not have homology in degree greater than one. Since $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C}(t))$ is irreducible, $H^0(M, \mathrm{Ad} \circ \rho) = \{\zeta \in \mathrm{sl}_2(\mathbb{C}(t)) | \ \rho(\gamma)\zeta\rho(\gamma)^{-1} = \zeta, \forall \gamma \in \pi_1(M)\} = \{0\}$, $H^1(M, \mathrm{Ad} \circ \rho) \simeq \mathbb{C}(t)$ because X is one dimensional, and so is $H^2(M, \mathrm{Ad} \circ \rho)$.
This sequence allows us to use the following formula (see [Mil66])

$$\mathrm{tor}(M, \mathrm{Ad} \circ \rho) = \frac{\mathrm{tor}(M_1)\,\mathrm{tor}(M_2)}{\mathrm{tor}(A)}\,\mathrm{tor}(\mathcal{H})$$

where we need to specify some choices of basis of the involved complexes.
The representation restricted to M_1 is generated by $\rho(a)$, which is conjugated to $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$. The twisted complex is $C^0(M_1, \mathrm{Ad} \circ \rho) \simeq \mathrm{sl}_2(\mathbb{C}(t)) \xrightarrow{\mathrm{Ad}_{\rho(a)} - \mathrm{Id}} C^1(M, \mathrm{Ad} \circ \rho) \simeq \mathrm{sl}_2(\mathbb{C}(t))$. The matrix $\mathrm{Ad} \circ \rho(a) - \mathrm{Id} = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{smallmatrix}\right)$ has rank 2, hence $H^0(M_1, \mathrm{Ad} \circ \rho)$ and $H^1(M_1, \mathrm{Ad} \circ \rho)$ are one-dimensional $\mathbb{C}(t)$ vector-spaces. The matrix $\rho(a)$ provides a choice of generators for both $H^0(M_1, \mathrm{Ad} \circ \rho)$ and $H^1(M_1, \mathrm{Ad} \circ \rho)$. With this choice of basis, the torsion $\mathrm{tor}(M_1) \in \mathbb{C}^*$ is a constant independent of t.
The same arguments show that $H^0(M_2, \mathrm{Ad} \circ \rho) \simeq H^1(M_2, \mathrm{Ad} \circ \rho) \simeq \mathbb{C}(t)$, with common generator given by the matrix $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$; again the torsion $\mathrm{tor}(M_2)$ is a constant.
Finally, the map $\mathrm{Ad} \circ \rho(u) - \mathrm{Id}$ is trivial, hence $H^0(A, \mathrm{Ad} \circ \rho) \simeq H^1(A, \mathrm{Ad} \circ \rho) \simeq \mathrm{sl}_2(\mathbb{C}(t))$, we choose the classical basis $\{H, E, F\}$. In this basis $\mathrm{tor}(A) = 1$.
By a simple count of dimension, one sees that the sequence $\mathcal{H}$ splits into two subsequences

$$0 \to H^0(M_1, \mathrm{Ad}_\rho) \oplus H^0(M_2, \mathrm{Ad}_\rho) \xrightarrow{\alpha_1} H^0(A, \mathrm{Ad}_\rho) \xrightarrow{\delta_1} H^1(M, \mathrm{Ad}_\rho) \to 0 \qquad (2.8)$$

$$0 \to H^1(M_1, \mathrm{Ad}_\rho) \oplus H^1(M_2, \mathrm{Ad}_\rho) \xrightarrow{\alpha_2} H^1(A, \mathrm{Ad}_\rho) \xrightarrow{\delta_2} H^2(M, \mathrm{Ad}_\rho) \to 0 \qquad (2.9)$$

As we identify $H^1(M, \mathrm{Ad}_\rho)$ to the Zariski tangent space, we pick as a basis the vector $\partial t = \frac{d}{dt}(\rho)\rho^{-1}$ that sends a to the matrix $\left(\begin{smallmatrix} -t & -1 \\ t^2-1 & t \end{smallmatrix}\right)$ and b to 0. On the other hand,

$\alpha_1(\zeta_1, \zeta_2) = \zeta_1 - \zeta_2$ and we compute the map δ_1 by a "diagram chasing":

$$\begin{array}{ccccccccc}
0 \longrightarrow & C^0(M, \mathrm{Ad}_\rho) & \xrightarrow{i_0} & C^0(M_1, \mathrm{Ad}_\rho) \oplus C^0(M_2, \mathrm{Ad}_\rho) & \xrightarrow{j_0} & C^0(A, \mathrm{Ad}_\rho) & \longrightarrow 0 \\
& \downarrow d & & \downarrow d & & \downarrow d & \\
0 \longrightarrow & C^1(M, \mathrm{Ad}_\rho) & \xrightarrow{i_1} & C^1(M_1 \mathrm{Ad}_\rho) \oplus C^1(M_2, \mathrm{Ad}_\rho) & \xrightarrow{j_1} & C^1(A, \mathrm{Ad}_\rho) & \longrightarrow 0
\end{array}$$

We need to represent the system $(A, M_1 \cup M_2, M)$ as a CW-complex to describe explicitly the maps involved in the previous diagram:

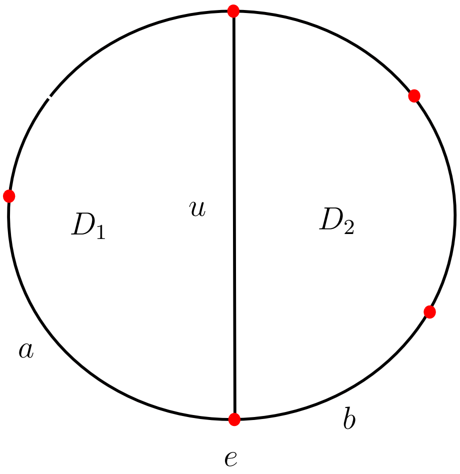


Figure 2.2 – This represents a cellular decompositon of M with one 0-cell e, three 1-cells a, b and u corresponding to generators of $\pi_1(M_1), \pi_1(M_2)$ and $\pi_1(A)$ respectively and two 2-cells D_1 and D_2 representing the relations $a^2 = u = b^3$. The 1-cells are oriented from the bottom to the top, and 2-cells clockwise.

Then $C^0(M, \mathrm{Ad}_\rho) \simeq C^0(M_i, \mathrm{Ad}_\rho) \simeq C^0(A, \mathrm{Ad}_\rho) \simeq \mathrm{sl}_2(\mathbb{C}(t))$ generated by the unique 0-cell e ; $C^1(M, \mathrm{Ad}_\rho) \simeq \mathrm{sl}_2(\mathbb{C}(t)) \cdot a \oplus \mathrm{sl}_2(\mathbb{C}(t)) \cdot b \oplus \mathrm{sl}_2(\mathbb{C}(t)) \cdot z$, $C^1(M_i, \mathrm{Ad}_\rho) \simeq \mathrm{sl}_2(\mathbb{C}(t))^2$ generated by a and z (resp. by b and z), and $C^1(A, \mathrm{Ad}_\rho) \simeq \mathrm{sl}_2(\mathbb{C}(t)) \cdot z$. Finally, $i_1(\zeta_a, \zeta_b, \zeta_z) = (\zeta_a, \zeta_z) \oplus (\zeta_b, \zeta_z)$ and $j_1((\zeta_a, \zeta_z) \oplus (\xi_b, \xi_z)) = \zeta_z - \xi_z$. The degree zero maps are defined as usual.
Now we are ready to compute δ_1: pick $\zeta \in \mathrm{sl}_2(\mathbb{C}(t)) = C^0(A, \mathrm{Ad}_\rho)$ and lift it to $\zeta \oplus 0 \in C^0(M_1, \mathrm{Ad}_\rho) \oplus C^0(M_2, \mathrm{Ad}_\rho)$. Then $d(\zeta \oplus 0) = (\mathrm{Ad}_{\rho(a)} \cdot \zeta - \zeta, 0) \oplus (0, 0) \in C^1(M_1, \mathrm{Ad}_\rho) \oplus C^1(M_2, \mathrm{Ad}_\rho)$, and thus one obtains $\delta_1(\zeta) = [(\mathrm{Ad}_{\rho(a)} \cdot \zeta - \zeta, 0, 0)] \in H^1(M, \mathrm{Ad}_\rho)$.
Finally, the torsion of the sequence (2.8) is the determinant of the basis $\{\alpha_1(\rho(a) \oplus H), \delta_1^{-1}(\partial t)\}$ in $\{H, E, F\}$. A direct computation shows that $\mathrm{Ad}_{\rho(a)} F - F = \partial t$,

hence we compute the determinant of the basis $\{\rho(a), H, F\}$ in $\{H, E, F\}$, hence $\begin{vmatrix} t & 1 & 0 \\ 1 & 0 & 0 \\ -(t^2+1) & 0 & 1 \end{vmatrix} = -1$. As an element of $\det(H^0(M_1, \mathrm{Ad}_\rho) \oplus H^0(M_2, \mathrm{Ad}_\rho))^* \otimes \det(H^0(A, \mathrm{Ad}_\rho)) \otimes \det(H^1(M, \mathrm{Ad}_\rho))^*$, then $\mathrm{tor}(2.8) = -dt \frac{H \wedge E \wedge F}{\rho(a) \wedge H}$, where dt is the dual basis of ∂t.

Now consider the second part of the sequence

$$0 \to H^1(M_1, \mathrm{Ad}_\rho) \oplus H^1(M_2, \mathrm{Ad}_\rho) \xrightarrow{\alpha_2} H^1(A, \mathrm{Ad}_\rho) \xrightarrow{\delta_2} H^2(M, \mathrm{Ad}_\rho) \to 0$$

First, we compute δ_2 from the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^1(M, \mathrm{Ad}_\rho) & \xrightarrow{i_1} & C^1(M_1, \mathrm{Ad}_\rho) \oplus C^1(M_2, \mathrm{Ad}_\rho) & \xrightarrow{j_1} & C^1(A, \mathrm{Ad}_\rho) & \longrightarrow & 0 \\ & & \downarrow d & & \downarrow d & & \downarrow d & & \\ 0 & \longrightarrow & C^2(M, \mathrm{Ad}_\rho) & \xrightarrow{i_2} & C^2(M_1 \mathrm{Ad}_\rho) \oplus C^2(M_2, \mathrm{Ad}_\rho) & \xrightarrow{j_2} & C^2(A, \mathrm{Ad}_\rho) & \longrightarrow & 0 \end{array}$$

Notice that $C^2(M, \mathrm{Ad}_\rho) \simeq \mathrm{sl}_2(\mathbb{C}(t)) \cdot D_1 \oplus \mathrm{sl}_2(\mathbb{C}(t)) \cdot D_2$, and that this direct sum corresponds to $C^2(M_1, \mathrm{Ad}_\rho) \oplus C^2(M_2, \mathrm{Ad}_\rho)$.

We pick $\zeta \in C^1(A, \mathrm{Ad}_\rho) \simeq \mathrm{sl}_2(\mathbb{C}(t))$ and we lift it to $(0, \zeta) \oplus (0, 0) \in C^1(M_1, \mathrm{Ad}_\rho) \oplus C^1(M_2, \mathrm{Ad}_\rho)$. Then we compute the coboundary map

$$\begin{aligned} d : & C^1(M_1, \mathrm{Ad}_\rho) \oplus C^1(M_2, \mathrm{Ad}_\rho) \to C^2(M_1, \mathrm{Ad}_\rho) \oplus C^2(M_2, \mathrm{Ad}_\rho) \\ & (\eta_1, \xi_1) \oplus (\eta_2, \xi_2) \mapsto (\eta_1 + \mathrm{Ad}_{\rho(a)} \cdot \eta_1 - \xi_1) \oplus (\eta_2 + \mathrm{Ad}_{\rho(b)} \cdot \eta_2 + \mathrm{Ad}_{\rho(b)^2} \cdot \eta_2 - \xi_2) \end{aligned}$$

Hence $d((0, \zeta) \oplus (0, 0)) = -\zeta \oplus 0$ and $\delta_2(\zeta) = [(-\zeta, 0)] \in H^2(M, \mathrm{Ad}_\rho)$.

Now we need to find a basis of $H^2(M, \mathrm{Ad}_\rho) \simeq H^2(\partial M, \mathrm{Ad}_\rho) \simeq H^0(\partial M, \mathrm{Ad}_\rho)^*$. The vector space $H^0(\partial M, \mathrm{Ad}_\rho)$ is generated by any trace-free matrix commuting with $\pi_1(M)$; pick $ab^{-1} = x$ as a meridian, that is a curve that encircles the knot once. Then the trace-free matrix $\rho(ab^{-1})_0$ commutes with $\rho(x)$, and thus commutes to every element of $\pi_1(\partial M)$, we fix it as a basis of $H^0(\partial M, \mathrm{Ad}_\rho)$. To deduce a basis of $H^2(M, \mathrm{Ad}_\rho)$, we need to compute the map $H^2(M, \mathrm{Ad}_\rho) \to H^2(\partial M, \mathrm{Ad}_\rho)$, using the following cellular decomposition of ∂M. We shall furnish some explanations on how we produce such a cellular decomposition: the first point is to recall that the trefoil can be drawn lying on a 2-torus embedded in the 3-sphere in the usual way, as in picture 2.1. Another way to draw it, more schematically, is the following depicted in Figure 2.4.

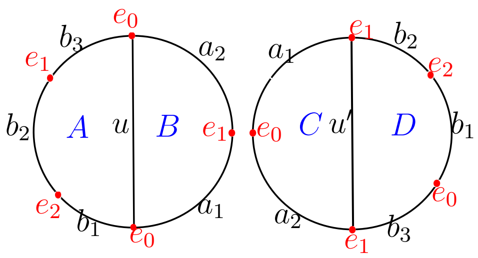


Figure 2.3 – Here is depicted a cellular decomposition of the boundary of M. The e's denote 0-cells, a's,b's and u's are 1-cells and 2-cells are denoted using capital letters. Cells denoted identically are identified.

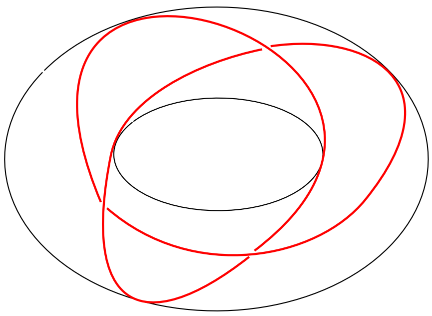


Figure 2.4 – An other picture of the trefoil in a 2-torus

The torus that holds the knot splits the 3-sphere into two solid tori T_1 and T_2. A slice of each of those tori is depicted on the figure 2.5, with red points corresponding to the knot crossing the boundary of the disk. "Flowing" those points toward the center retracts the slice of the knot complement on a graph, as shown here. Now we can define an equivariant cellular map $\phi : \partial\tilde{M} \to \tilde{M}$ between lifts of those cellular decompositions to universal covers: $\phi(\tilde{e}_0) = \tilde{e}, \phi(\tilde{e}_1) = a \cdot \tilde{e}, \phi(\tilde{e}_2) = b \cdot \tilde{e}, \phi(\tilde{a}_1) = \tilde{a}, \phi(\tilde{a}_2) = a \cdot \tilde{a}, \phi(\tilde{b}_1) = \tilde{b}, \phi(\tilde{b}_2) = b \cdot \tilde{b}, \phi(\tilde{b}_3) = b^2 \cdot \tilde{b}, \phi(\tilde{A}) = \tilde{D}_2, \phi(\tilde{B}) = \tilde{D}_1, \phi(\tilde{C}) = a \cdot \tilde{D}_1, \phi(\tilde{D}) = b^2 \cdot \tilde{D}_2$.

Then

$$
\begin{aligned}
\phi^* : H^2(M, \mathrm{Ad}_\rho) &\to H^2(\partial M, \mathrm{Ad}_\rho) \\
(\zeta_{D_1}, \zeta_{D_2}) &\mapsto (\zeta_{D_2}, \zeta_{D_1}, \mathrm{Ad}_{\rho(a)} \cdot \zeta_{D_1}, \mathrm{Ad}_{\rho(b^2)} \cdot \zeta_{D_2})
\end{aligned}
$$

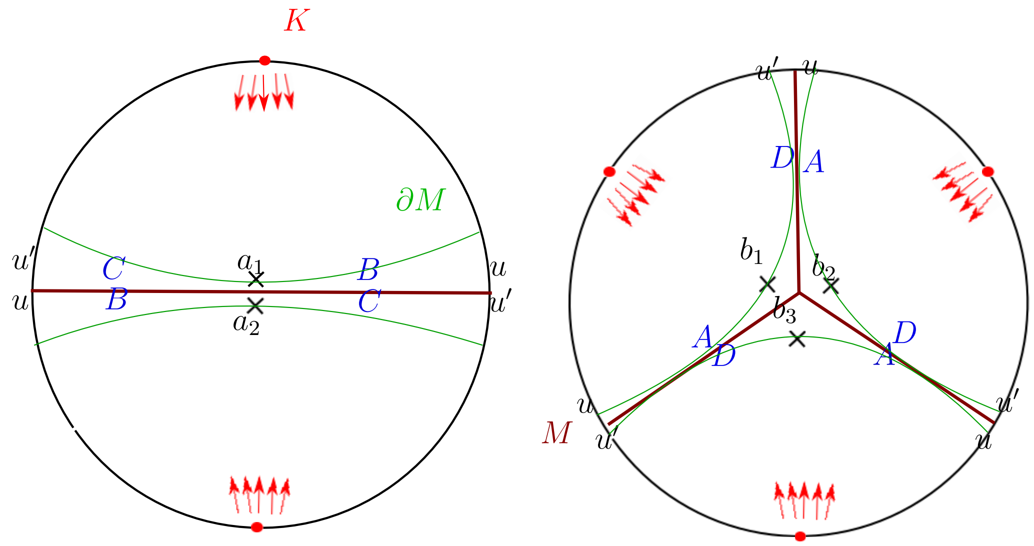


Figure 2.5 – The complement M of the knot K retracts on the union of the red radial segments. The boundary ∂M, in green on the figure, can be seen as foliating the disks A, B, C and D of figure 2.3 by half-segments from a_i or b_i to u or u'.

Now we claim that $(F, 0) \in H^2(M, \mathrm{Ad}_\rho)$ is dual to the choice of basis $\rho(ab^{-1})_0 \in H^0(\partial M)$, we the evaluation against the vector $[A + B + C + D] \otimes \rho(ab^{-1}) \in H_2(\partial M, \mathrm{Ad}_\rho)$:

$$\begin{aligned}\mathrm{Tr}(\phi^*(F,0)((A+B+C+D)\otimes\rho(ab^{-1})_0)) &= \mathrm{Tr}((F + \mathrm{Ad}_{\rho(a)}\cdot F)\rho(ab^{-1})_0) \\ &= \mathrm{Tr}\begin{pmatrix} -t & -1 \\ t^2+1 & t \end{pmatrix}\begin{pmatrix} t/2 & -j \\ (t^2+1)j^2 & -t/2 \end{pmatrix} \\ &= 1\end{aligned}$$

Again, the torsion is the determinant of the basis $\{\rho(a), H, -F\}$ in the standard basis $\{H, E, F\}$, which is equal to 1. Seen as an element of the vector space $\det(H^1(M_1, \mathrm{Ad}_\rho) \otimes H^1(M_2, \mathrm{Ad}_\rho)) \otimes \det(H^1(A, \mathrm{Ad}_\rho))^* \otimes \det(H^2(M, \mathrm{Ad}_\rho))$, one conclude that

$$\mathrm{tor}(2.9) = \frac{H \wedge E \wedge F}{\rho(a) \wedge H \otimes \rho(ab^{-1})_0}$$

We normalize $\rho(ab^{-1})_0$ in order to have consistent choice with the previous subsection: $|\rho(ab^{-1})_0| = \sqrt{\mathrm{Tr}(\rho(ab^{-1})_0^2)} = \sqrt{t^2-4}$, and finally

$$\mathrm{tor}(M, \mathrm{Ad} \circ \rho) = \frac{dt}{\sqrt{t^2-4}}$$

It is not rational, but on the double cover of the character variety given by $u + u^{-1} = t$, one obtains

$$\mathrm{tor}(M, \mathrm{Ad} \circ \rho) = \frac{du}{u}$$

2.4.2 Proof of theorem 2.0.2, the case when ρ_Σ is not residually abelian

In this section we prove Theorem 2.0.2 in the case where ρ_Σ is not residually abelian. This case is the generic case, nevertheless computational evidences suggest that there are many examples where it is not the case, hence we will treat the residually abelian case in the next section.

Recall from Section 1.3.6 that there the splitting $M = M_1 \cup_\Sigma M_2$ provides representations $\rho_i : \pi_1(M_i) \to \mathrm{SL}_2(k(Y))$ with ρ_1 convergent, and $\rho_2 = U_n \rho_2' U_n^{-1}$, where the matrix U_n is $\left(\begin{smallmatrix} t^n & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

it induces the following Mayer-Vietoris exact sequence of $k(Y)$-vector spaces $\mathcal{H}$:

$$0 \to H^1(M, \mathrm{Ad} \circ \rho) \xrightarrow{(i_1^*, i_2^*)} H^1(M_1, \mathrm{Ad} \circ \rho_1) \oplus H^1(M_2, \mathrm{Ad} \circ \rho_2)$$

$$\xrightarrow{j_1^* - j_2^*} H^1(\Sigma, \mathrm{Ad} \circ \rho_\Sigma) \xrightarrow{\delta} H^2(M, \mathrm{Ad} \circ \rho) \to 0$$

In this sequence the missing term on the left $H^0(\Sigma, \mathrm{Ad} \circ \rho)$ is trivial because ρ_Σ is not abelian. Moreover, we can consider the complexes: $C^*(\Sigma, \mathrm{Ad} \circ \rho_\Sigma)$ and $C^*(M_i, \mathrm{Ad} \circ \rho_i)$, $i = 1, 2$, with their natural geometric bases c_Σ, c_1, c_2, and with a choice of bases h_Σ, h_1, h_2 of their homology groups. That allows us to define their torsion $\mathrm{tor}(\Sigma, h_\Sigma)$, $\mathrm{tor}(M_i, h_i) \in k(Y)$. We pick any basis of $H^1(M, \mathrm{Ad} \circ \rho)$, and again H as a basis of $H^2(M, \mathrm{Ad} \circ \rho)$. Then we have the following theorem of Milnor [Mil66]:

Theorem 2.4.3.

$$\mathrm{tor}(M, \mathrm{Ad} \circ \rho) = \frac{\mathrm{tor}(M_1)\,\mathrm{tor}(M_2)}{\mathrm{tor}(\Sigma)} \mathrm{tor}(\mathcal{H}) \in \Omega_{k(Y)/k}$$

This equality (and the left-hand side term) does not depend on the choice of bases we made.

Proposition 2.4.4. *The terms* $\mathrm{tor}(M_1), \mathrm{tor}(M_2)$ *and* $\mathrm{tor}(\Sigma)$ *lie in* $\mathcal{O}_v^*$

Proof. Those factors are torsion of based complex of $k(Y)$-vector spaces with based homology, hence they lie in $k(Y)^*$ by definition. Since the representations ρ_1, ρ_2' and ρ_Σ are convergent, one can define the complexes of $\mathcal{O}_v$-modules $C^*(\Sigma, \mathrm{Ad} \circ \rho_\Sigma)_v$, $C^*(M_1, \mathrm{Ad} \circ \rho_1)_v$ and $C^*(M_2, \mathrm{Ad} \circ \rho_2')_v$ with their homology groups. Moreover, we could have chosen the bases $c_\Sigma, c_1, c_2, h_\Sigma, h_1, h_2$ of the paragraph above to generate those terms as $\mathcal{O}_v$-modules, because $C^*(\Sigma, \mathrm{Ad} \circ \rho_\Sigma)_v, \ldots, H^*(M_i, \mathrm{Ad} \circ \rho_i)_v$ are free $\mathcal{O}_v$-modules (see Remark 2.4.5), and those choices do not affect the computation of the torsion. To be precise, we assume that we have chosen for instance a basis h_2 of the free $\mathcal{O}_v$-module $H^1(M_2, \mathrm{Ad} \circ \rho_2')_v$ that spans $H^1(M_2, \mathrm{Ad} \circ \rho_2')$ as a $k(Y)$-vector space, and that it is mapped on a basis through the isomorphism of $k(Y)$-vector spaces $H^1(M_2, \mathrm{Ad} \circ \rho_2') \to H^1(M_2, \mathrm{Ad} \circ \rho_2)$. Finally, the map $H^1(M_1, \mathrm{Ad} \circ \rho_1)_v \to H^1(\Sigma, \mathrm{Ad} \circ \rho_\Sigma)_v$ identifies the basis h_1 to a sub-basis of h_Σ.
Now we prove that the torsions of this complexes lie in $\mathcal{O}_v^*$: let us perform the computation for, say, M_1. The complex is $C^0(M_1, \mathrm{Ad} \circ \rho_1)_v \xrightarrow{A} C^1(M_1, \mathrm{Ad} \circ \rho_1)_v$. Since $H^0(M_1, \mathrm{Ad} \circ \rho_1)_v$ is trivial, the matrix A is the matrix of an injective $\mathcal{O}_v$-linear morphism. Moreover, $H^1(M, \mathrm{Ad} \circ \rho_1)_v$ is free, hence the determinant of the restriction $\bar{A} : C^0(M_1, \mathrm{Ad} \circ \rho_1)_v \to \mathrm{im}(A)$ is an invertible: $\det \bar{A} \in \mathcal{O}_v^*$ as claimed. □

Since $\mathrm{tor}(M_1), \mathrm{tor}(M_2)$ and $\mathrm{tor}(\Sigma)$ take values in $\mathcal{O}_v^*$, the valuation of $\mathrm{tor}(M, \mathrm{Ad} \circ \rho)$ is determined by the torsion of the exact sequence $\mathcal{H}$.

Remark 2.4.5.

1. As we wish to compute the valuation of the torsion, it would be better to study a complex of $\mathcal{O}_v$-modules, having in mind Theorem 2.3.3. That will be the first step of the proof.
2. Notice that we have isomorphisms: $k(Y) \xrightarrow{\sim} H^1(M, \mathrm{Ad} \circ \rho), 1 \mapsto (\frac{d}{dt}\rho)\rho^{-1}$ and $H^2(M, \mathrm{Ad} \circ \rho) \xrightarrow{\sim} k(Y), \lambda \mapsto \mathrm{Tr}(\lambda(\partial M)H)$.
3. Since ρ_Σ is residually non abelian and so are ρ_1, ρ_2', the only cohomology groups that are non trivial as $\mathcal{O}_v$-modules for them are $H^1(M_1, \mathrm{Ad} \circ \rho_1)_v$, $H^1(M_2, \mathrm{Ad} \circ \rho_2')_v$ and $H^1(\Sigma, \mathrm{Ad} \circ \rho_\Sigma)_v$. Moreover it comes from Universal Coefficients Theorem that they are free modules because the following k-vector spaces $H^0(M_i, \mathrm{Ad} \circ \bar{\rho}_i)$ and $H^0(\Sigma, \mathrm{Ad} \circ \bar{\rho}_\Sigma)$ are trivial.

Lemma 2.4.6. *The sequence of* $k(Y)$*-vector spaces*

$$0 \to k(Y) \xrightarrow{d_1} H^1(M_1, \mathrm{Ad} \circ \rho_1) \oplus H^1(M_2, \mathrm{Ad} \circ \rho_2') \xrightarrow{d_2} H^1(\Sigma, \mathrm{Ad} \circ \rho_\Sigma) \xrightarrow{\delta} k(Y) \to 0$$

is exact, where the morphisms are given by $d_1 : 1 \mapsto \left((\frac{d}{dt}\rho_1)\rho_1^{-1}, (\frac{d}{dt}\rho_2')\rho_2'^{-1}\right)$, $d_2 :$ $(\zeta_1, \zeta_2) \mapsto \zeta_1 - U_n\zeta_2U_n^{-1} = \zeta_1 - \begin{pmatrix} x_2 & t^n y_2 \\ t^{-n}z_2 & -x_2 \end{pmatrix}$ *and* $\delta : \lambda \mapsto \mathrm{Tr}(\lambda([\partial\Sigma])H)$.

Proof. We just used the isomophism $H^1(M_2, \mathrm{Ad}\circ\rho_2') \xrightarrow{\sim} H^1(M_2, \mathrm{Ad}\circ\rho_2)$ given by $\zeta_2 \mapsto U_n\zeta_2U_n^{-1}$ and rewrite the sequence. Then we compute the morphisms d_1, d_2 and δ:

$$\begin{aligned} U_n^{-1}(\frac{d}{dt}\rho_2)\rho_2^{-1}U_n &= U_n^{-1}\frac{d}{dt}(U_n\rho_2'U_n^{-1})(U_n\rho_2'^{-1}U_n^{-1})U_n \\ &= (\frac{d}{dt}\rho_2' + U_n^{-1}\frac{d}{dt}U_n\rho_2' + \rho_2'\frac{dU_n^{-1}}{dt}U_n)\rho_2'^{-1} \\ &= (\frac{d}{dt}\rho_2')\rho_2'^{-1} + \frac{1}{t^n}(\rho_2'N\rho_2'^{-1} - N) \end{aligned}$$

where the matrix N is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Observe then that the last term on the right is the boundary of $N_0 = N - \frac{1}{2}\,\mathrm{Id}$, and the first assertion follows. The second is clear.
For the third point, the naturality of the Mayer-Vietoris sequence and the exact sequence of a pair provide the following diagram.

$$\begin{array}{ccccc} H^1(\Sigma, \mathrm{Ad}\circ\rho) & \xrightarrow{\delta} & H^2(M, \mathrm{Ad}\circ\rho) & & \\ \downarrow & & \downarrow & & \\ H^1(\partial\Sigma, \mathrm{Ad}\circ\rho) & \longrightarrow & H^2(\partial M, \mathrm{Ad}\circ\rho) & \xrightarrow{\sim} & H^0(\partial M, \mathrm{Ad}\circ\rho)^* \end{array}$$

As the second vertical arrow is an isomorphism, it's enough to compute the composition $H^1(\Sigma, \mathrm{Ad}\circ\rho) \to H^1(\partial\Sigma, \mathrm{Ad}\circ\rho) \to H^2(\partial M, \mathrm{Ad}\circ\rho) \to k(Y)$, which is simply $\lambda \mapsto \mathrm{Tr}(\lambda([\partial\Sigma])H)$. □

Now each term of the sequence can be thought as an $\mathcal{O}_v$-module tensorized by $k(Y)$, but the map d_2 does not restrict to a morphism of $\mathcal{O}_v$-module. Hence in the sequel we will consider the exact sequence $\mathcal{H}_t$:

$$0 \to k(Y) \xrightarrow{d_1} H^1(M_1, \mathrm{Ad}\circ\rho_1) \oplus H^1(M_2, \mathrm{Ad}\circ\rho_2') \xrightarrow{t^n.d_2} H^1(\Sigma, \mathrm{Ad}\circ\rho_\Sigma) \xrightarrow{\delta} k(Y) \to 0$$

where we just have multiplied d_2 by t^n. We will denote by D_2 this new map, which restricts to morphism of $\mathcal{O}_v$-modules $H^1(M_1, \mathrm{Ad}\circ\rho_1)_v \oplus H^1(M_2, \mathrm{Ad}\circ\rho_2')_v \xrightarrow{D_2} H^1(\Sigma, \mathrm{Ad}\circ\rho_\Sigma)_v$.

From now on we suppose that the choices of bases we made h_Σ, h_1 and h_2 gave splittings $H^1(M_1, \mathrm{Ad}\circ\rho_1) \oplus H^1(M_2, \mathrm{Ad}\circ\rho_2') = \ker d_2 \oplus E_1$, and $H^1(\Sigma, \mathrm{Ad}\circ\rho_\Sigma) = d_2(E_1) \oplus E_2$. Let Δ_2 be the restricted map $D_{2|_{E_1}} : E_1 \to d_2(E_1)$.

Lemma 2.4.7.

$$\mathrm{tor}(\mathcal{H}_t) = \frac{1}{\det \Delta_2} c, \textit{ with } c \in \mathcal{O}_v^*$$

Proof. Considering the definition of the torsion of Section 1.5.3, the following equality holds:

$$\mathrm{tor}(\mathcal{H}_t) = \frac{\det(d_1 : k(Y) \to d_1(k(Y))) \det(\delta : E_2 \to k(Y))}{\det \Delta_2}$$

Then we conclude the proof noting that the numerator lies in $\mathcal{O}_v^*$. □

Hence we are now reduced to compute $v(\det(\Delta_2))$.
To do this, the idea is the following: recall that the completion of the valuation ring $\mathcal{O}_v$ is isomorphic to $k[[t]]$, the ring of formal series. Consider a matrix $A \in \mathcal{M}_n(\mathcal{O}_v)$ as a formal series $A = \sum t^i A_i$, with $A_i \in \mathcal{M}_n(k)$, the problem is to compute the valuation of its determinant. If $\det A_0 \neq 0$, then A is invertible, $\det A \in \mathcal{O}_v^*$ and $v(\det A) = 0$. If not, we have $k^n \xrightarrow{A_0} k^n$ which is not invertible and define $H^0(A_0) = \ker A_0$, $H^1(A_0) = \mathrm{coker}\, A_0$, hence $H^0(A_0) \simeq H^1(A_0) \neq 0$. Pick $P, Q \in \mathrm{GL}_n(k)$ such that $PA_0Q = \left(\begin{smallmatrix} 0 & 0 \\ 0 & I_{n-r_0} \end{smallmatrix}\right)$ is diagonal, where $r_0 = \dim \ker A_0$, and I_{n-r_0} is the $(n - r_0)$ identity matrix. Then to compute $\det A$, it's enough to compute the determinant of the $r_0 \times r_0$ first block of $A_1 + tA_2$.... More precisely $\det A = t^{r_0} \det A_1' + o(t^{r_0})$, where A_1' is the restriction of $\sum t^i A_{i+1}$ to $H^0(A_0) \otimes k[[t]]$, followed by the projection $k[[t]]^n \to H^1(A_0) \otimes k[[t]]$.
One proceeds by induction, the argument is formalized in the following lemma:

Lemma 2.4.8. *Let $A : \mathcal{O}_v^n \to \mathcal{O}_v^n$ a morphism such that $\det A \neq 0$. Working in the completion $\hat{\mathcal{O}}_v$ if necessary, we define $A_{\geq 0} = A, A_{\geq i+1} = \frac{d}{dt} A_{\geq i}$ restricted to $H^0(A_{\geq i}(0)) \otimes k[[t]]$ followed by the projection $k[[t]]^{n - \sum_{k=0}^{i} r_k} \to H^1(A_{\geq i}(0)) \otimes k[[t]]$, and $r_i = \dim \ker A_{\geq i}(0)$.*
Then $\det(A) = t^{\sum r_i} c$, with $c \in \mathcal{O}_v^*$.

Proof. Define the sequence $(r_n)_n$ as in the lemma. As $\det A \neq 0$, there is an i_0 such that $r_{i_0} = 0$. Take $0 < i \leq i_0$, after fixing appropriated bases of $\ker A_{\geq i-1}(0)$, one write $A_{\geq i}(0)$ as a diagonal matrix, with r_{i-1} zeros on the diagonal, and 1's after. Then the classical formula for the determinant tells us that $\det A_{\geq i} = t^{r_i} \det A_{\geq i+1} + o(t^{r_i})$, and the result follows by induction. □

We will apply this lemma to the morphism Δ_2. Recall that for $\rho_1, \rho_2', \rho_\Sigma : \Gamma \to \mathrm{SL}_2(\mathcal{O}_v)$, we have the so-called residual representations $\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_\Sigma : \Gamma \to \mathrm{SL}_2(k)$ taking values in the residual field k. Moreover, $\bar{\rho}_\Sigma = \bar{\rho}_{1,\Sigma} = \overline{U_n \rho_{2,\Sigma}' U_n^{-1}}$ is reducible, non abelian, thus we have as in Section 2.3.2:

Lemma 2.4.9. *The residual representations have the form*

$$\bar{\rho}_{1,\Sigma} = \begin{pmatrix} \lambda & 0 \\ \lambda u_1 & \lambda^{-1} \end{pmatrix}, \bar{\rho}_{2,\Sigma} = \begin{pmatrix} \lambda & \lambda^{-1} u_2 \\ 0 & \lambda^{-1} \end{pmatrix}$$

with $\lambda \in H^1(\Sigma, k^*), u_1 \in H^1(\Sigma, \lambda^{-2}), u_2 \in H^1(\Sigma, k_{\lambda^2})$ *non trivial.*

Proof. The expression of $\bar{\rho}_{2,\Sigma}$ follows from the conjugacy formula $\rho_1 = U_n \rho_2' U_n^{-1}$ when restricted on $\pi_1(\Sigma)$, the u_i's are non trivial because the residuals representations are not abelian.

□

The former sequence becomes residually $\bar{\mathcal{H}}$:

$$0 \to k \xrightarrow{\bar{d}_1} H^1(M_1, \mathrm{Ad} \circ \bar{\rho}_1) \oplus H^1(M_2, \mathrm{Ad} \circ \bar{\rho}_2) \xrightarrow{\bar{D}_2} H^1(\Sigma, \mathrm{Ad} \circ \bar{\rho}_{1,\Sigma}) \xrightarrow{\bar{\delta}} k \to 0$$

with $\bar{d}_1(1) = (v_1, v_2)$, and $\bar{D}_2(\zeta_1, \zeta_2) = z_{2,\Sigma} F$, where $z_{2,\Sigma}$ denotes the lower-left entry of ζ_2, restricted to $\pi_1(\Sigma)$.
Again (see Section 2.3.2), the triangularity of the adjoint action of $\bar{\rho}_{i,\Sigma}$ provides the following splittings:

$$0 \to K_1 \to \mathrm{Ad} \circ \bar{\rho}_{1,\Sigma} \to k_{\lambda^2} \to 0$$

$$0 \to K_2 \to \mathrm{Ad} \circ \bar{\rho}_{2,\Sigma} \to k_{\lambda^{-2}} \to 0$$

$$0 \to k_{\lambda^{-2}} \to K_1 \to k \to 0$$

and thus the exact sequences of k-vector spaces:

$$0 \to H^1(\Sigma, K_2) \to H^1(\Sigma, \mathrm{Ad} \circ \bar{\rho}_{2,\Sigma}) \xrightarrow{p} H^1(\Sigma, \lambda^{-2}) \to 0$$

$$0 \to H^0(\Sigma, k) \to H^1(\Sigma, \lambda^{-2}) \to H^1(\Sigma, K_1) \to H^1(\Sigma, k) \to 0$$

and

$$0 \to H^1(\Sigma, K_1) \to H^1(\Sigma, \mathrm{Ad} \circ \bar{\rho}_{1,\Sigma}) \to ...$$

We denote by j the composition $H^1(\Sigma, \lambda^{-2}) \to H^1(\Sigma, K_1) \to H^1(\Sigma, \mathrm{Ad} \circ \bar{\rho}_{1,\Sigma})$.

Lemma 2.4.10. *The space* $\ker j$ *is one dimensional, more precisely, it is generated by the image of* $H^0(\Sigma, k)$ *in* $H^1(\Sigma, \mathrm{Ad} \circ \bar{\rho}_{1,\Sigma})$*, that is by* $\partial_{1,\Sigma} H = -2u_1 F$*.*

Proof. We compute $\partial_{1,\Sigma} H = \bar{\rho}_{1,\Sigma} H \bar{\rho}_{1,\Sigma}^{-1} - H$, and obtain the claimed result. □

The inclusion $\Sigma \subset M_2$ induces $i : H^1(M_2, \mathrm{Ad} \circ \bar{\rho}_2) \to H^1(\Sigma, \mathrm{Ad} \circ \bar{\rho}_{2,\Sigma})$.

Lemma 2.4.11. *The map* $j \circ p \circ i : H^1(M_2, \mathrm{Ad} \circ \bar{\rho}_2) \to H^1(\Sigma, \mathrm{Ad} \circ \bar{\rho}_{1,\Sigma})$ *has a kernel of dimension at least* $-\chi(\Sigma)/2 + 1$*.*

Proof. Notice that we know $\dim H^1(\Sigma, \mathrm{Ad} \circ \bar{\rho}_\Sigma) = -3\chi(\Sigma)$, $\dim H^1(\Sigma, \lambda^2) = -\chi(\Sigma)$ and $\dim H^1(M_i, \mathrm{Ad} \circ \bar{\rho}_i) = -3/2\chi(\Sigma)$. As p is onto, $\ker p$ has dimension equal to $\dim H^1(\Sigma, \mathrm{Ad} \circ \bar{\rho}_{2,\Sigma}) - \dim H^1(\Sigma, \lambda^2) = -3\chi(\Sigma) - (-\chi(\Sigma)) = -2\chi(\Sigma)$. If i is injective and if $\ker p$ and $\mathrm{im}\, i$ intersect transversally, then $\dim \ker p \cap \mathrm{im}\, i = -\chi(\Sigma)/2$. We define the integer s by the formula $\dim \ker p \cap \mathrm{im}\, i = -\chi(\Sigma)/2 + s$. Finally, by the preceding lemma $\ker j$ has dimension 1. Moreover, $p \circ i(v_2) = u_1 \neq 0 \in \ker j$ hence the dimension of $\ker(j \circ p \circ i) = -\chi(\Sigma)/2 + 1 + s$. If $s = -\chi(\Sigma)$, then $p \circ i = 0$, but $u_1 \neq 0$, hence the result. □

Now we can give a proof of Theorem 2.0.2:

Proof. First we compute r_0, the dimension of the first homology group of $\bar{\mathcal{H}}$, i.e. $H^1(\bar{\mathcal{H}}) = \ker \bar{D}_2 / \mathrm{im}\, \bar{d}_1$. From the preceding lemma, $\dim \ker \bar{D}_2 \geq -\chi(\Sigma)/2 + 1 + (-3/2\chi(\Sigma)) = -2\chi(\Sigma) + 1$. Hence $r_0 \geq -2\chi(\Sigma)$. Now the higher order maps $\frac{d^i}{d^i t}|_{t=0}(D_2)$, for $i = 1, \ldots, n-1$, remains zero when restricted to $\ker(\bar{D}_2)$, hence each $r_i, i = 1, \ldots, n-1$ is greater than $-2\chi(\Sigma)$. Let $r = \sum_{i \geq n} r_i$, we have from Lemma 2.4.8 that $\det(D_2) \geq -2n\chi(\Sigma) + r$, and $v(\mathrm{tor}(\mathcal{H}_t)) \leq 2n\chi(\Sigma)$. Finally, observe that $t^{n\,\mathrm{rk}(d_2)} \mathrm{tor}(\mathcal{H}_t) = \mathrm{tor}(\mathcal{H})$. Since $\mathrm{rk}(d_2) = -3\chi(\Sigma) - 1$, the theorem is proved. □

2.4.3 The case when ρ_Σ is residually abelian

In this section we treat the case when the residual reducible representation $\bar{\rho}_\Sigma$ is abelian. This case is motivated by the example of the figure-eight knot complement, see Remark 3.3.1. In fact, if Σ is two holed torus, as soon as the boundary curve will be mapped to $\pm \mathrm{Id}$ by $\bar{\rho}_\Sigma$, then the representation $\bar{\rho}_\Sigma$ will be residually abelian. Computational evidences suggest that it is always the case. Remark that in this case, the eigenvalues of $\bar{\rho}_\Sigma(\partial\Sigma)$ are necessarily ± 1 because they are roots of unity with order that divides the number of boundary components of the surface Σ.
The main difference with the previous case is the following:

Lemma 2.4.12. *The $\mathcal{O}_v$-module $H^1(\Sigma, \mathrm{Ad} \circ \rho_\Sigma)_v$ is isomorphic to $\mathcal{O}_v^{-3\chi(\Sigma)} \oplus \mathcal{O}_v/(t^m)$ for some positive integer m.*

Proof. Again $H^0(\Sigma, \mathrm{Ad} \circ \rho_\Sigma)$ is trivial because ρ_Σ is not abelian, but in this case we have $H^0(\Sigma, \mathrm{Ad} \circ \bar{\rho}_\Sigma) \simeq k$. The Universal Coefficients theorem gives

$$0 \to H^0(\Sigma, \mathrm{Ad} \circ \rho_\Sigma)_v \otimes_{\mathcal{O}_v} \mathcal{O}_v/(t) \to H^0(\Sigma, \mathrm{Ad} \circ \bar{\rho}_\Sigma) \to$$

$$\mathrm{Tor}(H^1(\Sigma, \mathrm{Ad} \circ \rho_\Sigma)_v, \mathcal{O}_v/(t)) \to 0$$

It follows that the torsion part of the module $H^1(\Sigma, \mathrm{Ad} \circ \rho_\Sigma)v$ is isomorphic to $\mathcal{O}_v/(t^m)$ for some m, and the lemma follows. □

Again, we pick basis c_i, c_Σ of the complexes of $k(Y)$-vector spaces $C^*(M_i, \mathrm{Ad} \circ \rho_i)$, $C^*(\Sigma, \mathrm{Ad} \circ \rho_\Sigma)$ and basis h_i, h_Σ of their first homology group (the only one that is not trivial). We want to use the formula:

$$\mathrm{tor}(M, \mathrm{Ad} \circ \rho) = \frac{\mathrm{tor}(M_1)\,\mathrm{tor}(M_2)}{\mathrm{tor}(\Sigma)}\,\mathrm{tor}(\mathcal{H})$$

Proposition 2.4.13. *The terms* $\mathrm{tor}(M_1)$ *and* $\mathrm{tor}(M_2)$ *lie in* $\mathcal{O}_v^*$*, but* $\mathrm{tor}(\Sigma)$ *has valuation* m*.*

Proof. The first part of the statement works in the same way than the proof of Proposition 2.4.4.
Recall that Σ has the same homotopy type that a wedge of $-\chi(\Sigma)+1$ circles, hence the complex $C^*(\Sigma, \mathrm{Ad} \circ \rho_\Sigma)$ looks like

$$\mathrm{sl}_2(k(Y)) \to \mathrm{sl}_2(k(Y))^{-\chi(\Sigma)+1}$$

and to compute its torsion it is enough to compute the determinant of the co-restricted isomorphism $k(Y)^3 \to k(Y)^3$. But this map induces a morphism of $\mathcal{O}_v$-modules $\mathcal{O}_v^3 \to \mathcal{O}_v^3$ whose co-kernel is isomorphic to $\mathcal{O}_v/(t^m)$, hence the result. □

Again, we need to focus on the term $\mathrm{tor}(\mathcal{H})$, where the exact sequence $\mathcal{H}$ is:

$$0 \to H^1(M, \mathrm{Ad} \circ \rho) \to H^1(M_1, \mathrm{Ad} \circ \rho_1) \oplus H^1(M_2, \mathrm{Ad} \circ \rho_2)$$

$$\to H^1(\Sigma, \mathrm{Ad} \circ \rho_\Sigma) \to H^2(M, \mathrm{Ad} \circ \rho) \to 0$$

We consider rather the sequence $\mathcal{H}_t$:

$$0 \to k(Y) \xrightarrow{d_1} H^1(M_1, \mathrm{Ad} \circ \rho_1) \oplus H^1(M_2, \mathrm{Ad} \circ \rho_2') \xrightarrow{t^n.d_2} H^1(\Sigma, \mathrm{Ad} \circ \rho_\Sigma) \xrightarrow{\delta} k(Y) \to 0$$

for n the number of parallel components in Σ, and the very same arguments that in the previous section lead to the lemma:

Lemma 2.4.14 (Lemma 2.4.7). *Let Δ_2 be the restriction of td_2 as above Lemma 2.4.7. Then*

$$\mathrm{tor}(\mathcal{H}_t) = \frac{1}{\det \Delta_2} c, \text{ with } c \in \mathcal{O}_v^*$$

Since $\bar{\rho}_\Sigma = \bar{\rho}_{1,\Sigma}$ is abelian, so is $\bar{\rho}_{2,\Sigma}$, hence we can write $\bar{\rho}_\Sigma(\gamma) = \begin{pmatrix} \lambda(\gamma) & 0 \\ 0 & \lambda^{-1}(\gamma) \end{pmatrix}$ for all $\gamma \in \pi_1(\Sigma)$, with $\lambda \in H^1(\Sigma, k^*)$. Consequently we have the decomposition

$$H^1(\Sigma, \mathrm{Ad} \circ \bar{\rho}_\Sigma) = H^1(\Sigma, \lambda^2) \oplus H^1(\Sigma, k) \oplus H^1(\Sigma, \lambda^{-2})$$

Lemma 2.4.15. *The dimension of the kernel of the induced map*

$$H^1(M_2, \mathrm{Ad} \circ \bar{\rho}_2) \to H^1(\Sigma, \mathrm{Ad} \circ \bar{\rho}_\Sigma)$$
$$\zeta_2 = \begin{pmatrix} x_2 & y_2 \\ z_2 & -x_2 \end{pmatrix} \mapsto z_{2,\Sigma} F$$

is $d \geq -\chi(\Sigma)/2$.

Proof. We observe that this map factors through the maps $H^1(M_2, \mathrm{Ad} \circ \bar{\rho}_2) \hookrightarrow H^1(\Sigma, \mathrm{Ad} \circ \bar{\rho}_\Sigma) \xrightarrow{p} H^1(\Sigma, \lambda^{-2}) \hookrightarrow H^1(\Sigma, \mathrm{Ad} \circ \bar{\rho}_\Sigma)$. Since $\ker p = H^1(\Sigma, \lambda^2) \oplus H^1(\Sigma, k)$ we find that the dimension d is at least

$$\begin{aligned} d &\geq \dim H^1(M_2, \mathrm{Ad} \circ \bar{\rho}_2) + \dim H^1(\Sigma, \lambda) + \dim H^1(\Sigma, k) - \dim H^1(\Sigma, \mathrm{Ad} \circ \bar{\rho}_\Sigma) \\ &= -3/2\chi(\Sigma) - \chi(\Sigma) - \chi(\Sigma) + 1 - (-3\chi(\Sigma) + 1) \\ &= -\chi(\Sigma)/2 \end{aligned}$$

□

Lemma 2.4.16. *The valuation of the torsion $v(\mathcal{H}_t)$ is smaller than $n(2\chi(\Sigma) + 1)$.*

Proof. By Lemma 2.4.8, the map

$$\Delta_2 : (H^1(M_1, \mathrm{Ad} \circ \rho_1) \oplus H^1(M_2, \mathrm{Ad} \circ \rho_2'))/(\mathrm{im}\, H^1(M, \mathrm{Ad} \circ \rho)) \to \mathrm{im}(\Delta_2)$$

has determinant greater that the sum of the dimensions of the kernels of the residual maps

$$\bar{D}_2 = \Delta_2|_{t=0} : (H^1(M_1, \mathrm{Ad} \circ \bar{\rho}_1) \oplus H^1(M_2, \mathrm{Ad} \circ \bar{\rho}_2))/k \to H^1(\Sigma, \mathrm{Ad} \circ \bar{\rho}_\Sigma)$$
$$[\zeta_1, \zeta_2] \mapsto z_{2,\Sigma} F$$

and

$$\frac{d^i}{dt^i}|_{t=0}\Delta_2 : \ker(\frac{d^{i-1}}{dt^{i-1}}|_{t=0}\Delta_2) \to H^1(\Sigma, \mathrm{Ad} \circ \bar{\rho}_\Sigma)$$

for $i = 1, \dots, n-1$.
But $\dim \ker \bar{D}_2$ is at least $\dim H^1(M_1, \mathrm{Ad} \circ \bar{\rho}_1) + d - 1 = -2\chi(\Sigma) - 1$, and the maps $\frac{d^i}{dt^i}|_{t=0}\Delta_2$ are identically zero on $\ker \bar{D}_2$, hence the sum of the kernels dimensions is at least $-n(2\chi(\Sigma) + 1)$. The lemma follows now from Lemma 2.4.7 above. □

Lemma 2.4.17. *The valuation of the sequence $\mathcal{H}$ is $v(\mathrm{tor}(\mathcal{H})) \le -n\chi(\Sigma)$*

Proof. It follows from the fact that $t^{\mathrm{rk}(d_2)} \mathrm{tor}(\mathcal{H}_t) = \mathrm{tor}(\mathcal{H})$, and $\mathrm{rk}(d_2) = -3\chi(\Sigma) - 1$ □

Proof of Theorem 2.0.2, second part. Using Milnor's formlula, we know that

$$v(\mathrm{tor}(M, \mathrm{Ad} \circ \rho)) = v(\mathrm{tor}(\mathcal{H})) - v(\mathrm{tor}(\Sigma)) \le -n\chi(\Sigma) - m$$

□

2.4.4 Back to Examples

1. The trefoil knot.

   The incompressible surface Σ is an annulus, hence $\rho_\Sigma : \mathbb{Z} \to \mathrm{SL}_2(k(t))$ is abelian, and the theorem cannot apply. Nevertheless the torsion has a pole of order one at ideal points corresponding to Σ, hence the equality of theorem remains true.

2. The figure-eight knot.

   There are two incompressible surfaces Σ_1 and Σ_2 that are two-holed tori, and the torsion vanishes at order 1 at each ideal point. Again the equality $1 = -\chi(\Sigma_i) - 1$ holds for $m = n = 1$

3. The knot 5.2.

   There are two incompressible surfaces Σ_1 and Σ_2 (see Figure 1 in the introduction), and we have $\chi(\Sigma_1) = -4, \chi(\Sigma_2) = -2$. At the ideal points, the torsion vanishes at order 1 (corresponds to Σ_1) and 3 (corresponds to Σ_2), thus the equality of Theorem 2.0.2 holds.

4. The knot 6.1.

   Again, there are two incompressible surface, the first of Euler characteristic -2 (a two-holed torus), and the second of Euler characteristic -6 (a two-holed genus 3 surface). At the corresponding ideal points, the vanishing order of $\mathrm{tor}(M, \mathrm{Ad} \circ \rho)$ is 1, respectively 5, that corresponds again with the equality case of the theorem.

Chapter 3

The acyclic torsion

Here M is a compact connected orientable 3-manifold, whose boundary is a torus. In this chapter we pick X a one-dimensional component of irreducible type of the character variety $X(M)$, and denote by $k(X)$ its function field. Here we consider the action of the fundamental group $\pi_1(M)$ on the 2-dimensional vector space $k(X)^2$ through the tautological representation $\rho : \pi_1(M) \to \mathrm{SL}_2(k(X))$, and the corresponding cohomological complex $C^*(M, \rho)$. When this complex is acyclic (for instance when X is a component of the character variety of a hyperbolic manifold that carries the holonomy character, say a *geometric component*), the Reidemeister torsion $\mathrm{tor}(M, \rho)$ of this complex defines a rational function on the complex curve X. We prove that the zeros of this function are precisely the characters $\chi_{\bar{\rho}} \in X$ where the residual cohomology group $H^1(M, \bar{\rho})$ are not trivial, and that this function is regular (it has no poles on X). Moreover, we consider $\bar{X}$ a smooth projective model, and show that an ideal point of $\bar{X}$ is a pole of the torsion function if it detects an incompressible surface that splits M into two handlebodies and which satisfies the following condition: the boundary slope γ is known (see [CCG$^+$94]) to be mapped by the limit representation on a matrix with eigenvalue a root of unity. Moreover the order of this root of unity divides the minimal number of boundary components of any component of the incompressible surface Σ associated to this ideal point by the construction of Section 1.3. We assume that the root of unity is 1, and we will say that the boundary curve has *trivial eigenvalues*. Such an assumption is motivated by numerical computations, and by the fact that it occurs in many known examples. In fact it has been a difficult task to find roots of unity different of ± 1 with this construction, see [Dun99] for the first known example. The results of this chapter are the core of the paper [Ben17], we summarize them in the following theorems:

Theorem 3.0.1. *Let X be a one dimensional component of the character variety $X(M)$ and $\rho : \pi_1(M) \to \mathrm{SL}_2(k(X))$ the tautological representation. Assume that the complex $C^*(M, \rho)$ is acyclic. Then* $\mathrm{tor}(M, \rho)$ *is a regular function on X. Moreover its order at a point $\chi \in X$ is given by the length of the torsion module $H^2(M, \rho)_v$, where v is the valuation associated to χ on the function field $k(X)$. In particular it vanishes if and only if $H^1(M, \bar{\rho})$ is non trivial, where $\bar{\rho}$ is the residual representation $\bar{\rho} : \pi_1(M) \xrightarrow{\rho} \mathrm{SL}_2(\mathcal{O}_v) \xrightarrow{mod\ t} \mathrm{SL}_2(k)$ whose character is χ.*

Theorem 3.0.2. *Let $x \in \hat{X}$ be an ideal point in the smooth projective model of X, and assume that an associated incompressible surface Σ is a union of parallel homeomorphic copies Σ_i such that $M \setminus \Sigma_i$ is a (union of) handlebodie(s). If the curve $\gamma = \partial\Sigma \in \pi_1(M)$ has trivial eigenvalues, then the torsion function* $\mathrm{tor}(M, \rho)$ *has a pole at x.*

The proof of this theorem recovers two cases, treated separately in Section 3.3.1, where the components of the incompressible surface split M, and Section 3.3.2, where they do not. The hypothesis on the boundary eigenvalues is discussed in the beginning of Section 3.3.1, on the other hand it is automatically satisfied in Section 3.3.2.
If M is hyperbolic, the question whether the torsion defines a non-constant function is motivated by the fact that the torsion is known to be locally constant on the character variety of torus knots. The first computation of non constant torsion function has been performed by Teruaki Kitano in [Kit94] for the figure-eight knot. In [DFJ12], Nathan Dunfield, Stefan Friedl and Nicholas Jackson study the twisted Alexander polynomial of hyperbolic knots. It is a Laurent polynomial $\Delta_\rho(t)$ associated to the character χ of a representation of the fundamental group, and it follows from its definition that $\Delta_\chi(1) = \mathrm{tor}(M, \rho)(\chi)$. It is conjectured from numerical evidences that this polynomial considered at the holonomy representation carries a lot of topological information on the manifold M, as fiberness, chirality, genus and volume. If the torsion was constant, it would suggest a disproof of this conjecture. As an immediate corollary we obtain that the twisted Alexander polynomial is not constant.
Recall that a connected surface $\Sigma \in M$ is free if $M \setminus S$ is a union of handlebodies.

Corollary 3.0.3. *Let M be a hyperbolic manifold and X be a geometric component of its* $\mathrm{SL}_2(\mathbb{C})$ *character variety. Assume that an ideal point of X detects an incompressible surface which is connected or union of parallel free copies, and such that* *the eigenvalue of its boundary curve is 1. Then the torsion function is not constant on the component X.*

The relation with twisted Alexander polynomial suggests an other interpretation of the vanishing locus of the torsion: given an irreducible representation $\rho : \pi_1(M) \to \mathrm{SL}_2(k)$, one can construct a semi-simple representation $\tilde{\rho} : \pi_1(M) \to \mathrm{SL}_3(k)$ by defining $\tilde{\rho} = \left(\begin{smallmatrix} \rho & 0 \\ 0 & 1 \end{smallmatrix}\right)$. A classical dimensional argument (see [HP15, Section 5] for instance), shows that if $\tilde{\rho}$ is deformable into irreducible representations in the character variety $X(M, \mathrm{SL}_3(k))$ then it necessarily exists a reducible, non semi-simple representation $\tilde{\rho}'$ with the same character, namely $\tilde{\rho}' = \left(\begin{smallmatrix} \rho & z \\ 0 & 1 \end{smallmatrix}\right)$ where $z : \pi_1(M) \to k^2$ represents a non trivial class in $H^1(M, \rho)$. More generally, given $\lambda \in k^*$ and a surjective abelianization map $\varphi : \pi_1(M) \to \mathbb{Z}$, it is proven in [HP15] that if the representation $\tilde{\rho}_\lambda = \left(\begin{smallmatrix} \lambda^\varphi \rho & 0 \\ 0 & \lambda^{-2\varphi} \end{smallmatrix}\right)$ is deformable into irreducible representations, then the Twisted Alexander Polynomial $\Delta_\rho(\lambda^3)$ vanishes. A converse statement is proved in the case when λ^3 is a simple root of the Twisted Alexander Polynomial. Our Theorem 3.0.1 is a first step in a proof of a general converse statement when $\lambda = 1$, saying that if $\Delta_\rho(1) = 0$ then there exists a non trivial $z \in H^1(M, \rho)$, hence a non semi-simple $\tilde{\rho}'$ as above, generalizing an basic fact from the SL_2-case.
This chapter divides into three sections, in Section 1 we define the torsion as a rational function on a component on the character variety, in Section 2 we prove Theorem 3.0.1, and in Section 3 we prove Theorem 3.0.2.

3.1 The torsion function

Let X be a one-dimensional, reduced component of irreducible type of the character variety $X(M)$, recall that a component of irreducible type means that it contains the character of an irreducible representation. In Section 1.1.7 we defined $\rho : \pi_1(M) \to \mathrm{SL}_2(k(X))$ the tautological representation, up to conjugation. Hence the torsion of the twisted complex $C^*(M, \rho)$ of $k(X)$-vector spaces is well-defined, and is an element of the homological determinant vector space $\mathrm{tor}(M, \rho) \in \det(H^*(M, \rho))$. The first statement of the following proposition follows directly from this definition. Recall that X is a geometric component if it carries the holonomy character of a hyperbolic structure on M. In Section 1.3.4, we have seen that for any $\chi \in X$ there is a valuation v on $k(X)$, and a choice of a convergent tautological representation $\rho : \pi_1(M) \to \mathrm{SL}_2(\mathcal{O}_v)$. The reduction mod t map $\mathcal{O}_v \to k$ defines a residual representation $\bar{\rho} : \pi_1(M) \to \mathrm{SL}_2(k)$ whose character is the point χ. We denote by $H^*(M, \rho)_v$ the cohomological $\mathcal{O}_v$-modules and by $H^*(M, \bar{\rho})$ the residual cohomological k-vector spaces.

Proposition 3.1.1. *If the complex $C^*(M, \rho)$ is acyclic, then the Reidemeister function $\mathrm{tor}(M, \rho) \in k(X)^*$ defines a rational function on the curve X. In particular, it is the case if X is a geometric component.*

Proof. We just have to show that if X is a geometric component, then $H^i(M, \rho) = 0$ for all i. Since M has the homotopy type of a two-dimensional CW complex, it has no homology in rank greater that 2.
The space of invariants $H^0(M, \rho) = \{z \in k(X)^2 | \rho(\gamma) z = z, \forall \gamma \in \pi_1(M)\}$ is non trivial if and only if $\mathrm{Tr}(\rho(\gamma)) = 2$ for all $\gamma \in \pi_1(M)$, but the tautological representation is irreducible, thus $H^0(M, \rho) = 0$.
We know that the Euler characteristic $\chi(M)$ is zero, hence it is now enough to prove that $H^1(M, \rho) = 0$. The Universal Coefficients Theorem provides isomorphisms $H^1(M, \rho)_v \otimes k(X) \simeq H^1(M, \rho)$ and $H^1(M, \bar{\rho}) \simeq H^1(M, \rho)_v \otimes k$, hence it is enough to show that for some $\chi \in X$, one has $H^1(M, \bar{\rho}) = 0$. It follows from Ragunathan's vanishing theorem (see for instance [MP12, Theorem 0.2]) that it is the case if χ is the character of a holonomy representation. □

Remark 3.1.2. As soon as there exists a character $\chi \in X$ such that $H^1(M, \bar{\rho})$ is trivial, the proposition applies and the torsion defines a well-defined function on the curve X. If follows from the semi-continuity of the dimension of $H^1(M, \bar{\rho})$ on X that in this case, $H^1(M, \bar{\rho})$ is trivial for all but a finite numbers $\chi \in X$. It has been the way to define almost everywhere the torsion function on X, the novelty here is that it is defined *a priori* even at characters χ with non trivial first cohomology groups. Indeed, we will show in the next section that the torsion vanishes exactly in those points.

3.2 The case of a finite character

3.2.1 Proof of Theorem 3.0.1

In this section we give a proof of the following theorem:

Theorem (Theorem 3.0.1)**.** *Let X be a one dimensional component of the character variety $X(M)$ and $\rho : \pi_1(M) \to \mathrm{SL}_2(k(X))$ the tautological representation. Assume that the complex $C^*(M, \rho)$ is acyclic. Then $\mathrm{tor}(M, \rho)$ is a regular function on X.*

Moreover its vanishing order at a point $\chi \in X$ is given by the length of the torsion module $H^2(M,\rho)_v$, where v is the valuation associated to x on the function field $k(X)$. In particular it vanishes if and only if $H^1(M,\bar{\rho})$ is non trivial, where $\bar{\rho}$ is the residual representation $\bar{\rho} : \pi_1(M) \xrightarrow{\rho} \mathrm{SL}_2(\mathcal{O}_v) \xrightarrow{mod\ t} \mathrm{SL}_2(k)$.

The main tool of the proof is the following theorem already used in Section 2.3. Recall that a complex of $\mathcal{O}_v$-modules C^* such that $C^* \otimes k(X)$ is an exact complex is said rationally exact, and that the length of a torsion $\mathcal{O}_v$-module $\oplus_k \mathcal{O}_v/(t^{n_k})$ equals $\sum n_k$.

Theorem 3.2.1. *[GKZ94, Theorem 30] Let $\chi \in X$ a character, and v a valuation on $k(X)$ associated to χ. If C^* is a rationally exact based complex of $\mathcal{O}_v$-modules with basis $\{c^i\}$, then*

$$v(\mathrm{tor}(C^* \otimes k(X), \{c^i\})) = \sum_k (-1)^k \,\mathrm{length}(H^k(C^*))$$

Proof of Theorem 3.0.1. Since the complex $C^*(M,\rho)$ is acyclic, the theorem above applies. Now notice that the $H^i(M,\rho)_v$ are torsion modules. As a submodule of a free module, $H^0(M,\rho)_v$ is trivial. Then Proposition 1.4.2 implies that no character $\chi \in X$ is central, in particular $H^0(M,\bar{\rho})$ is trivial. But the Universal Coefficients Theorem provide the isomorphisms $H^0(M,\bar{\rho}) \simeq H_0(M,\bar{\rho})^*$, and $H_0(M,\bar{\rho}) \simeq H_0(M,\rho)_v \otimes k$, thus we have proved that $H_0(M,\rho)_v$ is trivial. Again by the U.C. Theorem we have $\mathrm{Ext}(H^1(M,\rho)_v, \mathcal{O}_v) \simeq H_0(M,\rho)_v = \{0\}$, and we conclude that $H^1(M,\rho)_v \simeq \mathrm{Ext}(H^1(M,\rho)_v, \mathcal{O}_v) = \{0\}$ because its a torsion module. In conclusion we have proved the first part of the theorem

$$v(\mathrm{tor}(M,\rho)) = \mathrm{length}(H^2(M,\rho)_v)$$

Now $H^2(M,\rho)_v$ being trivial is equivalent to $H^2(M,\bar{\rho})$ being trivial which is the same that $H^1(M,\bar{\rho})$ being trivial, and the theorem is proved. □

3.2.2 Some computations and examples

We compute the torsion function on a series of examples of twist knots, and determine its zeros on the character variety. A presentation of the fundamental group of the Whitehead link can be computed to be

$$\pi_1(L_{5_1^2}) = \langle a, b, \lambda |\ b = \lambda a \lambda^{-1}, [\lambda^{-1}, a^{-1}][\lambda^{-1}, a][\lambda, a][\lambda, a^{-1}] = 1 \rangle$$

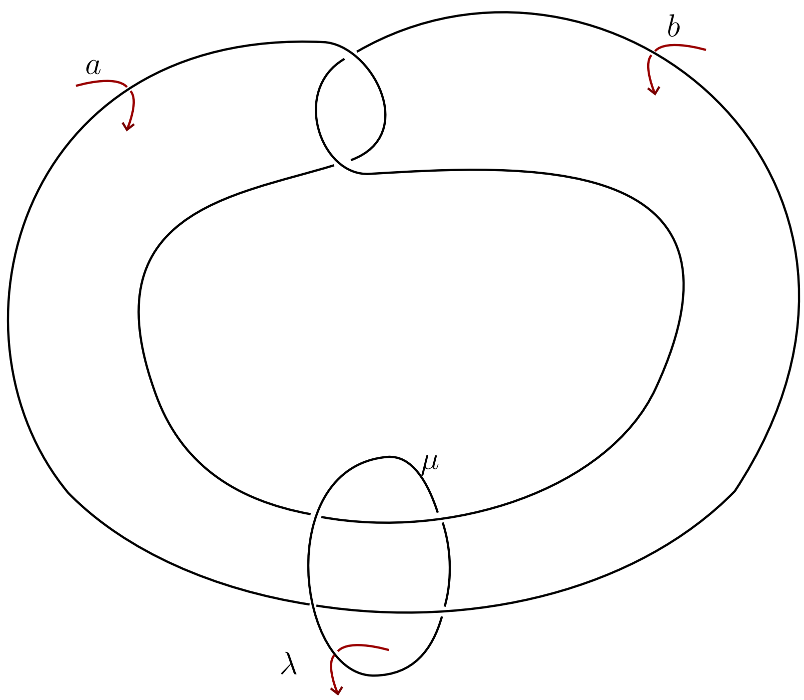


Figure 3.1 – A diagram of the Whitehead link. a, b and λ are depicted generators of the fundamental group, and μ is a counter-clockwise oriented longitude of the circle component.

The $J(2,2n)$-twists knots, $n \in \mathbb{Z}$, are obtained as $\frac{1}{n}$ Dehn filling along the circle component. The additional relation is thus $\mu^n = \lambda$, where $\mu = ba^{-1}b^{-1}a$. Notice that the second relation in the presentation above is $[\lambda, \mu] = 1$, hence is redondant whence $\mu^n = \lambda$. Figure 3.2 shows positive and negative twist knots, for $n = \pm 1$. Hence we obtain the following presentation of twist knot group $\pi_1(J(2,2n)) = \langle a, b | (ba^{-1}b^{-1}a)^n a = b(ba^{-1}b^{-1}a)^n \rangle$, or $\langle a, \lambda | \mu^n = \lambda \rangle$ where the curve μ is the curve $ba^{-1}b^{-1}a = [\lambda, a][\lambda, a^{-1}]$.

We define a tautological representation of the character variety $X(2,2n)$ of the twist knots $J(2,2n)$ by

$$\rho(a) = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix}, \rho(b) = \begin{pmatrix} s & 0 \\ y - s^2 - s^{-2} & s^{-1} \end{pmatrix}$$

We will use the variable $x = s + s^{-1}$. A direct computation shows (see [Kit96], for instance) that $\mathrm{tor}(M, \rho) = \frac{\det(\rho(\frac{\partial r}{\partial \lambda}))}{\det(\rho(a) - 1)} = \frac{\det(\rho(\frac{\partial r}{\partial \lambda}))}{(2-x)}$, where ρ is extended linearly to the ring $\mathbb{Z}[\pi_1]$.

— If $n > 0$, we obtain $\mathrm{tor}(M, \rho) = \frac{\det((1+\rho(\mu)+\ldots+\rho(\mu)^{n-1})(1-\rho(b)+\rho(ba^{-1})-\rho(ba^{-1}b^{-1}))-1)}{2-x}$

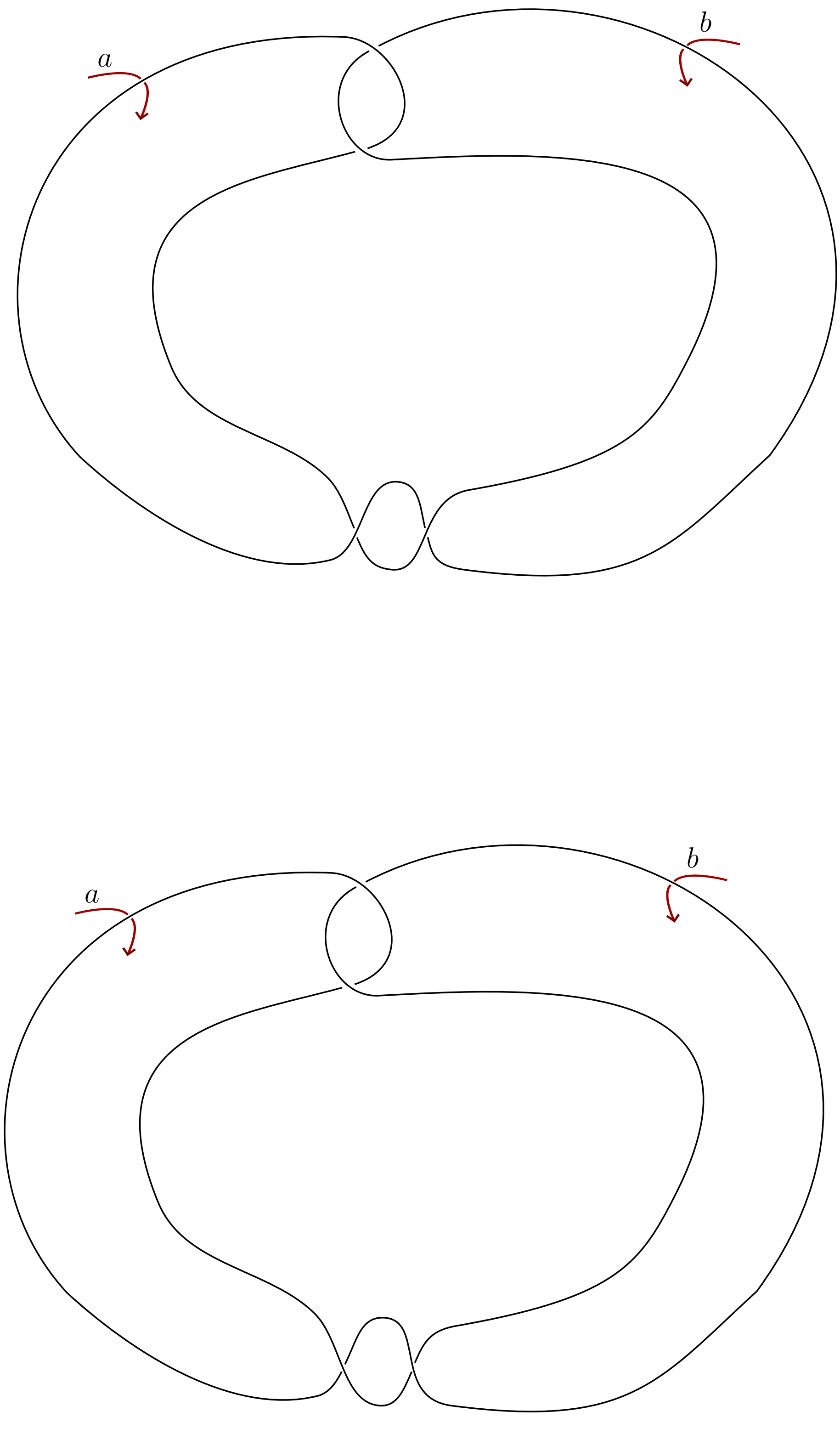


Figure 3.2 – The diagram above if $J(2,2)$, the trefoil knot. The one below is $J(2,-2)$, the figure-eight knot.

— If $n < 0$, we obtain $\mathrm{tor}(M,\rho) = \frac{\det((\rho(\mu)^{-1}+\ldots+\rho(\mu)^{n})(1-\rho(b)+\rho(ba^{-1})-\rho(ba^{-1}b^{-1})))}{2-x}$

The trefoil knot J(2,2)

The character variety equation is given by $X(2,2) = \{(x^2-y-2)(y-1) = 0\}$. The component of irreducible type X is thus $\{y-1=0\}$. We compute the torsion function in $\mathbb{C}[x,y]/(y-1)$, it is $\mathrm{tor}(M,\rho) = \frac{y-2x+3}{2-x} = 2$, the torsion is constant.

The figure-eight knot J(2,-2)

Let $X = \{2x^2+y^2-x^2y-y-1=0\}$ the component of irreducible type of $X(2,-2)$. We have $\mathrm{tor}(M,\rho) = (4x^2-x^2y+y^2-y-6x+3)/(2-x) = 2x-2$ in $\mathbb{C}[X]$, hence there is a zero at the point $\{x=1, y=1\}$, with multiplicity 2.

The knot 5_2: J(2,4)

Here $X = \{-x^2(y-1)(y-2)+y^3-y^2-2y+1=0\}$, and $\mathrm{tor}(M,\rho)$ has two double zeros when $x=y$ are roots of x^2-3x+1.

The knot 6_1: J(2,-4)

Here $X = \{x^4(y-2)^2 - x^2(y+1)(y-2)(2y-3) + (y^3-3y-1)(y-1) = 0\}$, and $\mathrm{tor}(M,\rho)$ has three double zeros when $x=y$ are roots of x^3-4x^2+3x+1.

Remark 3.2.2. We observe that each time we have found a zero for the torsion, it had multiplicity 2 and $\{\mathrm{tor}(M,\rho)=0\} \subset X \cap \{x=y\}$. We have checked that this inclusion is strict. We have no precise interpretation of those phenomenon, we think that it comes from the computation of the torsion on the character variety of the Whitehead link .

3.3 The torsion at ideal points

3.3.1 The split case

In this section we give a proof of the following theorem:

Theorem (Theorem 3.0.2, the split case). *Let $x \in \hat{X}$ be an ideal point in the smooth projective model of X, and assume that an associated incompressible surface Σ is a union of n parallel copies $\Sigma_1 \cup \ldots \cup \Sigma_n$ and that each copy splits M into two handlebodies. If the curve $\gamma = \partial\Sigma \in \pi_1(M)$ has trivial eigenvalues at x, then the* *torsion function* $\mathrm{tor}(M, \rho)$ *has a pole at* x*. In particular in this case the torsion function is non-constant.*

Recall from Section 1.3 that from an ideal point $x \in \hat{X}$ one can produce an incompressible surface $\Sigma \in M$. In this section we will make the following assumptions on Σ:

1. The surface Σ is a union of homeomorphic parallel copies $\Sigma_1 \cup \ldots \cup \Sigma_n$.
2. The complement of any Σ_i in M is the disjoint union of two handlebodies M_1 and M_2.
3. The eigenvalue of $\bar{\rho}(\partial\Sigma)$ is the trivial root of unity 1.

Remark 3.3.1. This assumptions are motivated by the fact that it is the way it appears in simple examples we can produce: for instance consider the figure-eight knot's classical diagram in Figure 3.3, and the non-orientable surface $\check{\Sigma}$ obtained by a "checkerboard" coloring. The boundary of its neighborhood is an orientable surface Σ, which turns out to be incompressible. It is detected by the point $\{x = \infty, y = 2\}$ of the component of irreducible type of the character variety of the figure-eight knot. It easy to see on the picture that its complement is the union of two genus 2 handlebodies, and a computation shows that the root of unity associated to the boundary curve $\partial\Sigma = uv^{-1}u^{-1}vuv^{-1}u^{-2}v^{-1}uvu^{-1}v^{-1}u^{-1}$ is 1.
We have performed numerous computations in the case of a two-holed torus with the help of the software SageMath. We have produced reducible representations of the free group on three generators, that come from irreducible representations of the closed genus two surface obtained by gluing together the boundary components. We have observed that the torsion will vanish whenever the image of the boundary curve has eigenvalues equal to 1 (and we proved the theorem in this case), on the other hand we produced several examples with eigenvalue -1 where the torsion did not vanish.

Recall from Section 1.3.6 that we have the splitting $M = M_1 \cup_\Sigma M_2$, it induces the following exact sequence of complexes of $k(X)$-vector spaces $0 \to C^*(M, \rho) \to C^*(M_1, \rho_1) \oplus C^*(M_2, \rho_2) \to C^*(\Sigma, \rho_\Sigma) \to 0$.

Notation. Since we have picked a base point $p \in \Sigma_1$ in Section 1.3.6, we will abuse of the notation $\pi_1(\Sigma)$ to designate $\pi_1(\Sigma_1)$. In the same way, we denote by $C^*(\Sigma, \rho_\Sigma)$ the twisted cohomological complex of Σ_1.

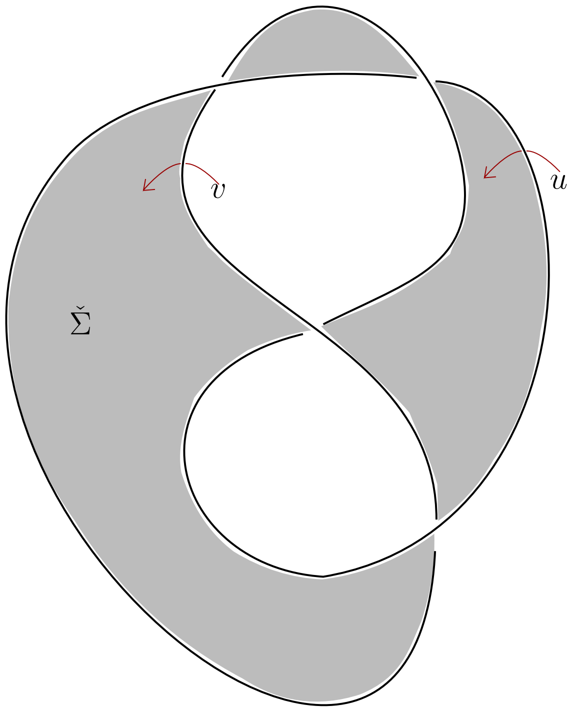


Figure 3.3 – The figure-eight knot, with a non orientable checkerboard surface $\check{\Sigma}$, and generating loops u, v of its fundamental group

Lemma 3.3.2. *One has the following isomorphism of $k(X)$-vector spaces:*

$$d : H^1(M_1, \rho_1) \oplus H^1(M_2, \rho_2) \xrightarrow{\sim} H^1(\Sigma, \rho_\Sigma)$$

Proof. Recall that $C^*(M, \rho)$ is acyclic and $H^j(M_i, \rho_i) = \{0\}$ for any $j \geq 2, i = 1, 2$ because M_i have the same homotopy type that a one-dimensional CW complex. Now $\bar{\rho}_i$ are irreducible by Lemma 1.3.15, hence ρ_i are irreducible and in particular there is $\gamma_i \in \pi_1(M_i)$ such that $\mathrm{Tr}(\rho_i(\gamma_i)) \neq 2$. Thus $H^0(M_i, \rho_i) = \{0\}$. The lemma follows now from the Mayer-Vietoris sequence. □

Lemma 3.3.3. *The $\mathcal{O}_v$-modules $H^1(M_1, \rho_1)_v, H^1(M_2, \rho_2')_v, H^1(\Sigma, \rho_\Sigma)_v$ are free of rank $-\chi(\Sigma_1)$, $-\chi(\Sigma_1)$ and $-2\chi(\Sigma_1)$ respectively.*

Proof. The Mayer-Vietoris sequence of Lemma 3.3.2 implies that $H^0(\Sigma, \rho_\Sigma) = \{0\}$ since $H^0(M_i, \rho_i) = H^0(M, \rho) = \{0\}$, and similarly $H^2(\Sigma, \rho) = \{0\}$. Now $H^1(\Sigma, \rho_\Sigma)_v$ is free of rank $-2\chi(\Sigma)$, and the lemma follows. □

We pick bases of the complexes of $k(X)$-vector spaces $C^*(M_1, \rho_1)$, $C^*(M_2, \rho_2)$ and $C^*(\Sigma, \rho_\Sigma)$ and of their homology groups $H^1(M, \rho_1)$, $H^1(M_2, \rho_2)$ and $H^1(\Sigma, \rho_\Sigma)$. We also pick a basis for the acyclic complex $C^*(M,\rho)$. We have the following formula due to Milnor [Mil66], that does not depends on the choices:

Proposition 3.3.4. *The torsion of the complex $C^*(M,\rho)$ can be expressed as*

$$\mathrm{tor}(M,\rho) = \frac{\mathrm{tor}(M_1,h_1)\,\mathrm{tor}(M_2,h_2)}{\mathrm{tor}(\Sigma,h_\Sigma)}\mathrm{tor}(\mathcal{H},h_1,h_2,h_\Sigma) \in k(X)^*$$

where $\mathcal{H}$ is the Mayer-Vietoris sequence of Lemma 3.3.2.

The following is identical to Proposition 2.4.4, hence we refer to it for a proof.

Proposition 3.3.5. *The terms $\mathrm{tor}(M_1), \mathrm{tor}(M_2)$ and $\mathrm{tor}(\Sigma)$ lie in $\mathcal{O}_v^*$*

Remark 3.3.6. As a consequence of this proposition, it is enough to compute the term $\mathrm{tor}(\mathcal{H}, h_1^1, h_2^1, h_\Sigma^1)$, which is just the inverse of the determinant of the following map (see Section 1.5.3):

$$\begin{aligned}\theta : H^1(M_1,\rho_1)\oplus H^1(M_2,\rho_2') &\xrightarrow{\sim} H^1(\Sigma,\rho_\Sigma)\\ (Z_1,Z_2) &\mapsto (Z_1 - U_n Z_2)_{|\Sigma}\end{aligned}$$

that is θ is the composition of d with ϕ. We compute now $\det(\theta)$.

For this purpose we observe that the relation $\forall\gamma \in \pi_1(\Sigma), \rho_1(\gamma) = U_n\rho_2'(\gamma)U_n^{-1}$ implies that the corresponding residual representations have the following form, when restricted to $\pi_1(\Sigma)$:

$$\bar\rho_1(\gamma) = \begin{pmatrix}\lambda(\gamma) & 0\\ \lambda(\gamma)u_1(\gamma) & \lambda^{-1}(\gamma)\end{pmatrix}, \bar\rho_2(\gamma) = \begin{pmatrix}\lambda(\gamma) & \lambda^{-1}(\gamma)u_2(\gamma)\\ 0 & \lambda^{-1}(\gamma)\end{pmatrix}$$

for some $u_1 \in H^1(\Sigma,\lambda^{-2}), u_2 \in H^1(\Sigma,\lambda^2)$. Consequently one has the splittings

$$0 \to H^1(\Sigma,\lambda^{-1}) \xrightarrow{i_1} H^1(\Sigma,\bar\rho_{1,\Sigma}) \xrightarrow{p_1} H^1(\Sigma,\lambda) \to 0$$

$$0 \to H^1(\Sigma,\lambda) \xrightarrow{i_2} H^1(\Sigma,\bar\rho_{2,\Sigma}) \xrightarrow{p_2} H^1(\Sigma,\lambda^{-1}) \to 0$$

More precisely, if say $z_1 \in H^1(\Sigma,\bar\rho_1)$ has the form $z_1(\gamma) = \left(\begin{smallmatrix}x_1(\gamma)\\ y_1(\gamma)\end{smallmatrix}\right)$ then the first splitting is given explicitly by the morphisms $i_1(y_1) = \left(\begin{smallmatrix}0\\ y_1\end{smallmatrix}\right)$ and $p_1(\left(\begin{smallmatrix}x_1\\ y_1\end{smallmatrix}\right)) = x_1$.

Notation. Here, and in the sequel, we will denote by $\bar\rho_1$ and $\bar\rho_2$ the residual representations obtained from ρ_1 and ρ_2'. We will denote by $\bar\rho_{i,\Sigma}$ the restriction of

$\bar\rho_i : \pi_1(M_i) \to \mathrm{SL}_2(k)$ to $\pi_1(\Sigma)$ through the map $\pi_1(\Sigma) \to \pi_1(M_i)$ induced by inclusion. Similarly, we will denote by $\bar\rho_{2,\partial M_2}$ the restriction of $\bar\rho_2$ to $\pi_1(\partial M_2)$, and $\bar\rho_{2,\gamma}$ the restriction of $\bar\rho_2$ to any curve γ, in particular $\gamma = \partial\Sigma$.

We need to prove that the torsion has a pole at the ideal point x. Denote by v the valuation associated to x, that means that the determinant of the map

$$\begin{aligned} \theta : H^1(M_1,\rho_1) \oplus H^1(M_2,\rho_2') &\xrightarrow{\sim} H^1(\Sigma,\rho_\Sigma) \\ (Z_1, Z_2) &\mapsto (Z_1 - U_n Z_2)_{|\Sigma} \end{aligned}$$

has positive valuation. Consider the k-linear map $\bar\theta : H^1(M_1,\bar\rho_1) \oplus H^1(M_2,\bar\rho_2) \to H^1(\Sigma,\bar\rho_\Sigma)$ which is θ modulo t, it maps (z_1,z_2) onto $z_{1|\Sigma} - \left(\begin{smallmatrix} 0 \\ y_{2|\Sigma} \end{smallmatrix}\right)$.

Lemma 3.3.7. *The torsion has a pole at x iff $\bar\theta$ is not an isomorphism.*

Proof. It is clear from the fact that $\det(\bar\theta) = (\det\theta)(0)$, that is $v(\det(\theta)) \geq 0$ iff $\det(\bar\theta) = 0$. □

Let us prove that $\det(\bar\theta) = 0$. Let ∂M_2 be the boundary of M_2, and $\gamma = \partial\Sigma \subset \partial M_2$ the union of boundary curves of Σ. Since Σ is incompressible and $\partial\Sigma \subset \partial M$, all the components of $\partial\Sigma$ are parallel in ∂M because it is a torus, in particular they define the same free homotopy class. Assumption 3 implies that $\bar\rho_2(\gamma) = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$, hence $H^1(\gamma,\bar\rho_{2,\gamma})$ is not trivial. The long exact sequence for the coefficients of $C^*(\gamma,\bar\rho_{2,\gamma})$ ends with

$$\ldots \to H^1(\gamma,\bar\rho_{2,\gamma}) \xrightarrow{p_\gamma} H^1(\gamma,k) \to 0$$

There are two possibilities, namely $\rho_2(\gamma) = \mathrm{Id}$ or not, but in any case, the map $p_\gamma : H^1(\gamma,\bar\rho_{2,\gamma}) \to H^1(\gamma,k)$ is not zero.

On the other hand, the inclusion $\Sigma \subset \partial M_2$ provides the sequence

$$H^1(\partial M_2,\Sigma;\bar\rho_{2,\partial M_2}) \to H^1(\partial M_2,\bar\rho_{2,\partial M_2}) \to H^1(\Sigma,\bar\rho_{2,\Sigma}) \to H^2(\partial M_2,\Sigma;\bar\rho_{2,\partial M_2}) \to 0 \tag{3.1}$$

Denote by A the union of small annulus neighborhood of the components of γ in ∂M_2. By excision, we have $H^2(\partial M,\Sigma;\bar\rho_{2,\partial M_2}) \simeq H^2(A,\partial A;\bar\rho_{2,\gamma})$. Now Poincaré-Lefschetz duality implies that it is the same that $H_0(A,\bar\rho_{2,\gamma})$, and by homotopy this is $H_0(\gamma,\bar\rho_{2,\gamma})$. Again by duality we obtain $H^1(\gamma,\bar\rho_{2,\gamma})$.

We summarize that in the following commutative diagram:

$$\begin{array}{ccccccc}
 & & H^1(M_2,\bar{\rho}_2) & & & & \\
 & & \downarrow{\scriptstyle i_{\partial M_2}} & \searrow{\scriptstyle F} & & & \\
 & & H^1(\partial M_2,\bar{\rho}_{2,\partial M_2}) & & & & \\
 & & \downarrow{\scriptstyle i_{\Sigma}} & & & & \\
H^1(\Sigma,\lambda) & \xrightarrow{i_2} & H^1(\Sigma,\bar{\rho}_{2,\Sigma}) & \xrightarrow{p_2} & H^1(\Sigma,\lambda^{-1}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_{\partial\Sigma}} & & \downarrow{\scriptstyle i_{\partial\Sigma}} & & \\
 & & H^1(\gamma,\bar{\rho}_{2,\gamma})) & \xrightarrow{p_\gamma} & H^1(\gamma,k) & \longrightarrow & 0 \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array}$$

Lemma 3.3.8. *The composition map* $F : H^1(M_2,\bar{\rho}_2) \xrightarrow{i_{\partial M_2}} H^1(\partial M_2,\bar{\rho}_{2,\partial M_2}) \xrightarrow{i_{\Sigma}} H^1(\Sigma,\bar{\rho}_{2,\Sigma}) \xrightarrow{p_2} H^1(\Sigma,\lambda^{-1})$ *is not an isomorphism.*

Proof. The first observation is that $\dim H^1(M_2,\bar{\rho}_2) = \frac{\dim H^1(\Sigma,\bar{\rho}_{2,\Sigma})}{2} = \dim H^1(\Sigma,\lambda^{-1})$. Thus it suffices to prove that the map F is not onto. But if it was, it would have a non-trivial image in $H^1(\gamma,k)$ through the map $i_{\partial\Sigma}$. On the other hand, the vertical sequence $H^1(\partial M_2,\bar{\rho}_{2,\partial M_2}) \xrightarrow{i_{\partial M_2}} H^1(\Sigma,\bar{\rho}_{2,\Sigma}) \xrightarrow{i_{\partial\Sigma}} H^1(\gamma,\bar{\rho}_{2,\gamma})$ is exact by equation (3.1), and the commutativity of the diagram shows that $i_{\partial\Sigma}\circ F = 0$, and the lemma is proved. □

Proof of Theorem 3.0.2. We just have to observe that $\bar{\theta} : H^1(M_1,\bar{\rho}_1)\oplus H^1(M_2,\bar{\rho}_2) \to H^1(\Sigma,\bar{\rho}_{1,\Sigma})$ is the direct sum of

1. the injective map $H^1(M_1,\bar{\rho}_1) \to H^1(\Sigma,\bar{\rho}_{1,\Sigma})$ induced by inclusion
2. the map $H^1(M_2,\bar{\rho}_2) \to H^1(\Sigma,\bar{\rho}_{2,\Sigma}) \xrightarrow{p_2} H^1(\Sigma,\lambda^{-1}) \xrightarrow{i_1} H^1(\Sigma,\bar{\rho}_{1,\Sigma})$ which is $i_1\circ F$.

The first map has maximal rank $-\chi(\Sigma)$, but the second has rank smaller than $-\chi(\Sigma)$ by Lemma 3.3.8. Hence $\bar{\theta}$ is not onto, hence $\det(\bar{\theta}) = 0$. By Lemma 3.3.7 we conclude that the torsion vanishes at x, and the theorem is proved. □

3.3.2 The non-split case

In this section we prove the following theorem:

Theorem (Theorem 3.0.2, the non-split case)**.** *Let $x \in \hat{X}$ be an ideal point in the smooth projective model of a component X of the character variety, that produces an incompressible surface S in M which is a union of parallel copies $S_1 \cup \ldots \cup S_n$ of a Seifert surface. Then the torsion function* $\mathrm{tor}(M, \rho)$ *has a pole at x, in particular it is non-constant.*

In this section we assume that the incompressible surface associated to x is a union of parallel Seifert surfaces. Recall from Section 1.3.7 that we fix a base-point $p \in S_1$, and that we identify $\pi_1(S)$ with $\pi_1(S_1)$. We have the following splitting $M = H \cup_{S_1 \cup S_n} V(S)$, where $V(S)$ is a neighborhood of S homeomorphic to $S_1 \times [0, 1]$ with $\partial V(S) = S_1 \cup S_n$. We identify as well $\pi_1(V(S))$ with $\pi_1(S)$. Given $\alpha : \pi_1(S_1) \to \pi_1(S_n)$, one can write the fundamental group of M as an HNN extension $\pi_1(M) = \langle \pi_1(H), v \mid v\gamma v^{-1} = \alpha(\gamma), \forall \gamma \in \pi_1(S)\rangle$.
We denote by $\rho_1 : \pi_1(S) \to \mathrm{SL}_2(\mathcal{O}_v)$ the restriction of ρ to $\pi_1(S)$, and by $\rho_n : \pi_1(S_n) \to \mathrm{SL}_2(k(X))$ its restriction to $\pi_1(S_n) = v\pi_1(S)v^{-1}$, hence we define $\rho_n(\gamma) = V_n \rho_1(\gamma) V_n^{-1}$ for $\gamma \in \pi_1(S)$. The splitting above induces the following exact sequence of twisted complexes:

$$0 \to C^*(M, \rho) \to C^*(H, \rho_H) \oplus C^*(S, \rho_1) \to C^*(S, \rho_1) \oplus C^*(S, \rho_n) \to 0$$

The following proposition recaps the series of lemmas in Section 3.3.1, we refer to the corresponding lemmas for proofs, that translate in exactly the same way here. We use the isomorphism $H^1(S, \rho_n) \to H^1(S, \rho_1)$, $Z \mapsto U_n \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) Z$.

Proposition 3.3.9. *The vanishing order of the torsion at the ideal point $x \in \bar{X}$ is given by computing $-v(\det \theta)$, where the isomorphism θ is given by*

$$\begin{aligned} \theta : H^1(H, \rho_H) \oplus H^1(S, \rho_1) &\to H^1(S, \rho_1) \oplus H^1(S, \rho_1) \\ (Z_1, Z_2) &\mapsto ((Z_1 - Z_2)|_S, U_n \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) (Z_1 - Z_2)|_S) \end{aligned}$$

Since we want to show that $v(\det \theta) > 0$, we focus on the residual map ; it is of the form

$$\begin{aligned} \bar{\theta} : H^1(H, \bar{\rho}_H) \oplus H^1(S, \bar{\rho}_S) &\to H^1(S, \bar{\rho}_S) \oplus H^1(S, \bar{\rho}_S) \\ (z_1, z_2) &\mapsto (z_1|_S - z_2|_S, -\begin{pmatrix} 0 \\ x_1|_S - x_2|_S \end{pmatrix}) \end{aligned}$$

where $z_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$. We show that it has a non trivial kernel.

Lemma 3.3.10. *The kernel of the map*

$$\begin{aligned} \psi : H^1(H, \bar{\rho}_H) &\to H^1(S, \bar{\rho}_S) \\ z_1 &\mapsto \begin{pmatrix} 0 \\ -x_1|_S \end{pmatrix} \end{aligned}$$

has dimension $d > -\chi(S)$.

Proof. The relation $\rho_n = V_n \rho_1 V_n^{-1}$ implies that the residual representation $\bar{\rho}_1$ has the form, for $\gamma \in \pi_1(S)$, $\bar{\rho}_1(\gamma) = \left(\begin{smallmatrix} \lambda(\gamma) & 0 \\ * & \lambda^{-1}(\gamma) \end{smallmatrix}\right)$. Hence we have the exact sequence

$$0 \to H^1(S, \lambda^{-1}) \to H^1(S, \bar{\rho}_1) \to H^1(S, \lambda) \to 0$$

We denote by $\rho'_1 : \pi_1(S) \to \mathrm{SL}_2(\mathcal{O}_v)$ the representation $\rho'_1(\gamma) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \rho_1(\gamma) \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and the map ψ is the composition

$$\begin{matrix} H^1(H, \bar{\rho}_H) & \to & H^1(S, \bar{\rho}'_1) & \to & H^1(S, \lambda) \simeq H^1(S, \lambda^{-1}) & \to & H^1(S, \bar{\rho}_1) \\ \begin{pmatrix} x \\ y \end{pmatrix} & \to & \begin{pmatrix} x|_S \\ y|_S \end{pmatrix} & \to & x|_S & \to & \begin{pmatrix} 0 \\ x|_S \end{pmatrix} \end{matrix}$$

We show that the composition $H^1(H, \bar{\rho}_H) \to H^1(S, \lambda^{-1})$ is not surjective, and it is enough to prove the lemma by a count of dimensions.
To see that it is not surjective, notice that it factors through the composition $H^1(H, \bar{\rho}_H) \to H^1(\partial H, \bar{\rho}_{\partial H}) \to H^1(S, \bar{\rho}'_1)$. Now we have the exact sequence of the pair $(\partial H, S)$:

$$0 \to C^*(\partial H, S; \rho_{\partial H}) \to C^*(H, \rho_{\partial H}) \to C^*(S, \rho'_1) \to 0$$

hence the long exact sequence in cohomology provides

$$H^1(\partial H, \bar{\rho}_{\partial H}) \to H^1(S, \bar{\rho}'_1) \to H^2(\partial H, S, \bar{\rho}_{\partial H}) \to 0$$

but by excision, $H^2(\partial H, S, \bar{\rho}_{\partial H}) \simeq H^2(\partial S \times [0,1], \partial S, \bar{\rho}_{\partial S})$, the latter is $H_0(\partial S \times [0,1], \bar{\rho}_{\partial S})$ by Poincaré-Lefschetz duality, which is $H^1(\partial S, \bar{\rho}_{\partial S})$.
We deduce the diagram:

$$
\begin{array}{ccccccc}
 & & H^1(H,\bar{\rho}_H) & & & & \\
 & & \downarrow & \overset{F}{\searrow} & & & \\
 & & H^1(\partial H,\bar{\rho}_{\partial H}) & & & & \\
 & & \downarrow & & & & \\
H^1(S,\lambda) & \longrightarrow & H^1(S,\bar{\rho}_1) & \longrightarrow & H^1(S,\lambda^{-1}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & H^1(\partial S,\bar{\rho}_{\partial S}) & \longrightarrow & H^1(\partial S,k) & \longrightarrow & 0 \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array}
$$

We conclude because the commutativity of the diagram and the exactness of the vertical arrows imply that the map F cannot surjective. $\square$

Proof of Theorem 3.0.2. We prove that the map residual map $\bar{\theta}$ is not an isomorphism, hence its determinant vanishes and it proves the theorem. To see that we show that it is not surjective, in particular the map

$$
\begin{aligned}
H^1(H,\bar{\rho}_H) \oplus H^1(S,\bar{\rho}_S) &\to H^1(S,\bar{\rho}_S) \\
(z_1,z_2) &\mapsto -\begin{pmatrix} 0 \\ x_1|_S - x_2|_S \end{pmatrix}
\end{aligned}
$$

is not surjective: the first part of the map has rank strictly less than $-\chi(S)$ by Lemma 3.3.10, and the second part has rank $-\chi(S)$. It proves the theorem. $\square$

Appendix A

Proof of Saito's theorem

Let us recall the statement of the theorem. The proof we present here is adapted from an unpublished version of [Sai94].

Theorem A.0.1. *Let R be a k-algebra, and $\chi : B[\Gamma] \to R$ an R-character. Assume that χ is irreducible, that is $\chi(\Delta_{\alpha,\beta}) \in R^\times$ for some $\alpha, \beta \in \Gamma$, and let $A, B \in \mathrm{SL}_2(R)$ such that $\operatorname{Tr} A = \chi(Y_\alpha), \operatorname{Tr} B = \chi(Y_\beta), \operatorname{Tr} AB = \chi(Y_{\alpha\beta})$. Then there exists a unique representation $\rho : \Gamma \to \mathrm{SL}_2(R)$ whose character is χ and such that $\rho(\alpha) = A$, $\rho(\beta) = B$.*

In the sequel we will denote by $\phi : \Gamma \to R$ the map $\phi(\gamma) = \chi(Y_\gamma)$.We first introduce the following definition:

Definition A.0.2. An R-valued *form* is a map $h : \Gamma \to R$ such that $\forall \gamma, \delta \in \Gamma, h(\gamma\delta) + h(\gamma\delta^{-1}) = \phi(\gamma)h(\delta)$. A form will be called *homogeneous* if $h(e) = 0$.

Remark A.0.3. By the trace formula, the map ϕ is a (non-homogeneous) form. If h is a homogeneous form, then clearly $h(\gamma) + h(\gamma^{-1}) = 0$, hence

$$h(\gamma\delta) + h(\delta\gamma) = \phi(\gamma)h(\delta) + \phi(\delta)h(\gamma) \tag{A.1}$$

Lemma A.0.4. *Let h be a homogeneous form, denote by $V_h = \{\gamma \in \Gamma : h(\gamma) = 0\}$. Then the following holds:*

1. $\forall \gamma \in V_h, \forall n \in \mathbb{Z}, \gamma^n \in V_h$.
2. $\forall \delta \in V_h, \forall \gamma \in \Gamma, h(\gamma\delta) = h(\delta\gamma^{-1})$.
3. *If $\delta_1, \ldots, \delta_m \in V_h$ such that $\delta_i\delta_j \in V_h$ for any $i < j$, then for any permutation $\sigma \in \mathcal{S}_m$ and any $\gamma \in \Gamma$, $h(\delta_1 \ldots \delta_m\gamma) = h(\delta_{\sigma(1)} \ldots \delta_{\sigma(m)}\gamma)$.*

Proof. 1. By definition $h(\gamma^n) = h(\gamma\gamma^{n-1}) = \phi(\gamma)h(\gamma^{n-1}) - h(\gamma^{n-2}) = 0$ by induction.

2. We have $h(\gamma\delta) = \phi(\gamma)h(\delta) - h(\gamma\delta^{-1})$ but $h(\delta) = 0$ and $h(\delta\gamma^{-1}) + h(\gamma\delta^{-1}) = 0$, hence $h(\gamma\delta) = h(\delta\gamma^{-1})$.

3. We first remark that Equation (A.1) implies that if $\delta_i, \delta_j, \delta_i\delta_j \in V_h$, so is $\delta_j\delta_i$, hence all the $\delta_i\delta_j$ lie in V_h. It is enough to show the result for $\sigma = \begin{pmatrix} i & i+1 \end{pmatrix}$ a transposition. Applying successively item 2 one finds $h(\delta_1 \dots \delta_n\gamma) = h(\delta_{i+2} \dots \delta_m\gamma\delta_1^{-1} \dots \delta_{i+1}^{-1})$. Then we apply again item 2 with $(\delta_{i+1}\delta_i)^{-1}$ and we obtain the equality $h(\delta_1 \dots \delta_m) = h(\delta_{i+1}\delta_i\delta_{i+2} \dots \delta_m\gamma\delta_1^{-1} \dots \delta_{i-1}^{-1})$, finally we conclude by applying item 2 again for getting back with $\delta_{i-1}, \dots \delta_1$ on the left hand-side.

□

Lemma A.0.5. *Let h be a homogeneous form, with $h(\alpha) = h(\beta) = h(\alpha\beta) = 0$. Then for any $\gamma \in \Gamma, \chi(\Delta_{\alpha,\beta})h(\gamma) = 0$.*

Proof. For any $\gamma \in \Gamma$, let us compute $\phi(\alpha)h(\gamma) = h(\alpha\gamma) + h(\alpha^{-1}\gamma)$ hence

$$\phi(\alpha)^2h(\gamma) = \phi(\alpha)(h(\alpha\gamma + h(\alpha^{-1}\gamma)) = h(\alpha^2\gamma) + h(\alpha^{-2}\gamma) + 2h(\gamma)$$

In the very same way,

$$\phi(\beta)^2h(\gamma) = h(\beta^2\gamma) + h(\beta^{-2}\gamma) + 2h(\gamma)$$

and

$$\phi(\alpha\beta)^2h(\gamma) = h((\alpha\beta)^2\gamma) + h((\alpha\beta)^{-2}\gamma) + 2h(\gamma)$$

Now

$$\begin{aligned}\phi(\alpha)\phi(\beta)\phi(\alpha\beta)h(\gamma) &= \phi(\alpha\beta)\phi(\beta)(h(\alpha\gamma) + h(\alpha^{-1}\gamma))\\ &= \phi(\beta\alpha)(h\beta\alpha\gamma) + h(\beta\alpha^{-1}\gamma) + h(\beta^{-1}\alpha\gamma) + h(\beta^{-1}\alpha^{-1}\gamma))\\ &= h((\beta\alpha)^2\gamma) + h(\gamma) + h(\beta\alpha\beta\alpha^{-1}\gamma) + h(\alpha^{-2}\gamma)\\ &+ h(\alpha^2\gamma) + h(\beta^{-1}\alpha^{-1}\beta^{-1}\alpha\gamma) + h(\gamma) + h((\alpha\beta)^{-2}\gamma)\end{aligned}$$

Finally we obtain

$$\begin{aligned}\chi(\Delta_{\alpha,\beta})h(\gamma) &= \phi(\alpha)^2h(\gamma) + \phi(\beta)^2h(\gamma) + \phi(\alpha\beta)^2h(\gamma) - \phi(\alpha)\phi(\beta)\phi(\alpha\beta)h(\gamma) - 4h(\gamma)\\ &= h(\beta^2\gamma) - h(\beta\alpha\beta\alpha^{-1}\gamma) + h(\beta^{-2}\gamma) - h(\beta^{-1}\alpha^{-1}\beta^{-1}\alpha\gamma) + h((\alpha\beta)^2\gamma)\end{aligned}$$

$$- h((\beta\alpha)^2\gamma)$$

and using Item 3 of Lemma A.0.4 we easily see that the latter is zero, and the lemma is proved. ◻

Before we start with the proof of Saito's theorem, we prove the following lemma that will be used in the sequel. We fix the notations for this lemma and for the proof of the theorem: let α, β be elements of Γ satisfying the hypotheses of the theorem. For any $\gamma \in \Gamma$, we define the vector $T_\gamma = \begin{pmatrix} \phi(\gamma) & \phi(\alpha\gamma) & \phi(\beta\gamma) & \phi(\alpha\beta\gamma) \end{pmatrix}$ and the matrix $M = \begin{pmatrix} \phi(e) & \phi(\alpha) & \phi(\beta) & \phi(\alpha\beta) \\ \phi(\alpha) & \phi(\alpha^2) & \phi(\alpha\beta) & \phi(\alpha^2\beta) \\ \phi(\beta) & \phi(\alpha\beta) & \phi(\beta^2) & \phi(\alpha\beta^2) \\ \phi(\alpha\beta) & \phi(\alpha^2\beta) & \phi(\alpha\beta^2) & \phi(\alpha^2\beta^2) \end{pmatrix}$. A computation gives $\det(M) = -\chi(\Delta_{\alpha,\beta})^2$ hence $M \in \mathrm{GL}_4(R)$ in the following. We will denote by T_γ^t the column-vector transpose of T_γ.

Lemma A.0.6. *Assume that $\chi(\Delta_{\alpha,\beta}) \in R^\times$, then for any $\gamma, \delta \in \Gamma, \phi(\gamma\delta) = T_\gamma M^{-1} T_\delta^t$.*

Proof. For $\delta \in \Gamma$, we define $H_\delta : \Gamma \to R$ by $H_\delta(\gamma) = \phi(\gamma\delta) - T_\gamma M^{-1} T_\delta^t$. Then H_δ is a form since $H_\delta(\gamma\eta) + H_\delta(\gamma^{-1}\eta) = \phi(\gamma\eta\delta) + \phi(\gamma^{-1}\eta\delta) - (T_{\gamma\eta} + T_{\gamma^{-1}\eta})M^{-1}T_\delta^t$, and the trace relation $\phi(\gamma\gamma') + \phi(\gamma^{-1}\gamma') = \phi(\gamma)\phi(\gamma')$ implies that $H_\delta(\gamma\eta) + H_\delta(\gamma^{-1}\eta) = \phi(\gamma)\phi(\eta\delta) + \phi(\gamma)T_\eta M^{-1}T_\delta^t = \phi(\gamma)H_\delta(\eta)$.

Moreover, from $\begin{pmatrix} T_e \\ T_\alpha \\ T_\beta \\ T_{\alpha\beta} \end{pmatrix} = M$ we deduce that $H_\delta(e) = H_\delta(\alpha) = H_\delta(\beta) = H_\delta(\alpha\beta) = 0$, that is H_δ is homogeneous and satisfies the hypothesis of Lemma A.0.5, hence for any $\gamma \in \Gamma, \chi(\Delta_{\alpha,\beta})H_\delta(\gamma) = 0$ and we conclude that $H_\delta(\gamma) = 0$ for all $\delta \in \Gamma$, thus the lemma is proved. ◻

Proof of Saito's Theorem. Let us define

$$\begin{aligned} \rho : \Gamma &\to \mathcal{M}_2(R) \\ \gamma &\mapsto \begin{pmatrix} \mathrm{Id} & A & B & AB \end{pmatrix} M^{-1} T_\gamma^t \end{aligned}$$

Here we consider the vector $M^{-1}T_\gamma^t$'s entries to be 2×2 scalar matrices. We need to check that the map we defined does satisfy all the requirements of the theorem.

1. $\begin{pmatrix}\rho(e) & \rho(\alpha) & \rho(\beta) & \rho(\alpha\beta)\end{pmatrix} = \begin{pmatrix}\mathrm{Id} & A & B & AB\end{pmatrix} M^{-1} \begin{pmatrix}T_e^t & T_\alpha^t & T_\beta^t & T_{\alpha\beta}^t\end{pmatrix}$ but the right hand matrix is exactly the matrix M, hence $\rho(e) = \mathrm{Id}, \rho(\alpha) = A, \rho(\beta) = B, \rho(\alpha\beta) = AB$.

2. For any $\gamma \in \Gamma$, $\begin{pmatrix}\mathrm{Tr}(\rho(\gamma)) \\ \mathrm{Tr}(A\rho(\gamma)) \\ \mathrm{Tr}(B\rho(\gamma)) \\ \mathrm{Tr}(AB\rho(\gamma))\end{pmatrix} = T_\gamma^t$ since by definition, we have $\mathrm{Tr}(\rho(\gamma)) = \begin{pmatrix}\mathrm{Tr}\,\mathrm{Id} & \mathrm{Tr}\,A & \mathrm{Tr}\,B & \mathrm{Tr}\,AB\end{pmatrix} M^{-1}T_\gamma^t$ and thus the left hand side term is precisely $\begin{pmatrix}\mathrm{Tr}\,I & \mathrm{Tr}\,A & \mathrm{Tr}\,B & \mathrm{Tr}\,AB \\ \mathrm{Tr}\,A & \mathrm{Tr}\,A^2 & \mathrm{Tr}\,AB & \mathrm{Tr}\,A^2B \\ \mathrm{Tr}\,B & \mathrm{Tr}\,AB & \mathrm{Tr}\,B^2 & \mathrm{Tr}\,AB^2 \\ \mathrm{Tr}\,AB & \mathrm{Tr}\,A^2B & \mathrm{Tr}\,AB^2 & \mathrm{Tr}(AB)^2\end{pmatrix} M^{-1}T_\gamma^t$ but item 1 implies that the matrix $\begin{pmatrix}\mathrm{Tr}\,I & \mathrm{Tr}\,A & \mathrm{Tr}\,B & \mathrm{Tr}\,AB \\ \mathrm{Tr}\,A & \mathrm{Tr}\,A^2 & \mathrm{Tr}\,AB & \mathrm{Tr}\,A^2B \\ \mathrm{Tr}\,B & \mathrm{Tr}\,AB & \mathrm{Tr}\,B^2 & \mathrm{Tr}\,AB^2 \\ \mathrm{Tr}\,AB & \mathrm{Tr}\,A^2B & \mathrm{Tr}\,AB^2 & \mathrm{Tr}(AB)^2\end{pmatrix}$ is nothing but M.

3. Let us prove that for any $\gamma\delta \in \Gamma, \rho(\gamma\delta) = \rho(\gamma)\rho(\delta)$. Let $C = \begin{pmatrix}\mathrm{Tr}\,\rho(\gamma)\rho(\delta) \\ \mathrm{Tr}\,A\rho(\gamma)\rho(\delta) \\ \mathrm{Tr}\,B\rho(\gamma)\rho(\delta) \\ \mathrm{Tr}\,AB\rho(\gamma)\rho(\delta)\end{pmatrix}$.

   Then $\begin{pmatrix}\rho(\gamma)\rho(\delta) \\ A\rho(\gamma)\rho(\delta) \\ B\rho(\gamma)\rho(\delta) \\ AB\rho(\gamma)\rho(\delta)\end{pmatrix} = \begin{pmatrix}\rho(\gamma) \\ A\rho(\gamma) \\ B\rho(\gamma) \\ AB\rho(\gamma)\end{pmatrix}\rho(\delta) = \begin{pmatrix}\rho(\gamma) \\ A\rho(\gamma) \\ B\rho(\gamma) \\ AB\rho(\gamma)\end{pmatrix}\begin{pmatrix}I & A & B & AB\end{pmatrix} M^{-1}T_\delta^t = \begin{pmatrix}\rho(\gamma) & \rho(\gamma)A & \rho(\gamma)B & \rho(\gamma)AB \\ A\rho(\gamma) & A\rho(\gamma)A & A\rho(\gamma)B & A\rho(\gamma)AB \\ B\rho(\gamma) & B\rho(\gamma)A & B\rho(\gamma)B & B\rho(\gamma)AB \\ AB\rho(\gamma) & AB\rho(\gamma)A & AB\rho(\gamma)B & AB\rho(\gamma)AB\end{pmatrix} M^{-1}T_\delta^t$.

   Using item 2 and the trace relation we can compute the traces of the previous matrix, for instance

$$\begin{aligned}\mathrm{Tr}(AB\rho(\gamma)B) &= \mathrm{Tr}(AB)\,\mathrm{Tr}(\rho(\gamma)B) - \mathrm{Tr}(B{-}1A^{-1}\rho(\gamma)B) \\ &= \phi(\alpha\beta)\phi(\gamma\beta) - \mathrm{Tr}(A^{-1}\rho(\gamma)) \\ &= \phi(\alpha\beta\gamma\beta) + \phi(\alpha\gamma^{-1}) - \mathrm{Tr}(A\rho(\gamma^{-1}) \\ &= \phi(\alpha\beta\gamma\beta)\end{aligned}$$

We obtain $C = \begin{pmatrix} \phi(\gamma) & \phi(\gamma\alpha) & \phi(\gamma\beta) & \phi(\gamma\alpha\beta) \\ \phi(\alpha\gamma) & \phi(\alpha\gamma\alpha) & \phi(\alpha\gamma\beta) & \phi(\alpha\gamma\alpha\beta) \\ \phi(\beta\gamma) & \phi(\beta\gamma\alpha) & \phi(\beta\gamma\beta) & \phi(\beta\gamma\alpha\beta) \\ \phi(\alpha\beta\gamma) & \phi(\alpha\beta\gamma\alpha) & \phi(\alpha\beta\gamma\beta) & \phi(\alpha\beta\gamma\alpha\beta) \end{pmatrix} M^{-1}T_\delta^t.$

But from Lemma A.0.6 we know that $\phi(\gamma\delta) = T_\gamma M^{-1}T_\delta^t$ and so $C = \begin{pmatrix} \phi(\gamma\delta) \\ \phi(\alpha\gamma\delta) \\ \phi(\beta\gamma\delta) \\ \phi(\alpha\beta\gamma\delta) \end{pmatrix}$

and from the first item, $C = \begin{pmatrix} \mathrm{Tr}(\rho(\gamma\delta)) \\ \mathrm{Tr}(A\rho(\gamma\delta)) \\ \mathrm{Tr}(B\rho(\gamma\delta)) \\ \mathrm{Tr}(AB\rho(\gamma\delta)) \end{pmatrix}$.

Let then $X = \rho(\gamma\delta) - \rho(\gamma)\rho(\delta)$, then the latter equality is equivalent to $\mathrm{Tr}(X) = \mathrm{Tr}(AX) = \mathrm{Tr}(BX) = \mathrm{Tr}(ABX) = 0$. We observe that M is a Gram matrix for $\{I, A, B, AB\}$ with respect to the non-degenerate bilinear form $(M, N) \mapsto \mathrm{Tr}(MN)$. As $\det(M) \in R^\times$ we know that $\{I, A, B, AB\}$ is a basis of $\mathcal{M}_2(R)$, and so $X = 0$, which proves that ρ is a group homomorphism.

4. A direct computation shows that $T_{\gamma^2} = \phi(\gamma)T_\gamma + T_e = 0$, in particular $\rho(\gamma)^2 - \mathrm{Tr}(\rho(\gamma))\rho(\gamma) + \rho(e) = 1$ and $\det(\rho(\gamma)) = 1$ for any $\gamma \in G$.

5. Finally, we prove the unicity as follows: for any representation ρ' that satisfies the statement of the theorem, we have $\begin{pmatrix} \mathrm{Tr}(\rho(\gamma)) \\ \mathrm{Tr}(A\rho(\gamma)) \\ \mathrm{Tr}(B\rho(\gamma)) \\ \mathrm{Tr}(AB\rho(\gamma)) \end{pmatrix} = \begin{pmatrix} \mathrm{Tr}(\rho'(\gamma)) \\ \mathrm{Tr}(A\rho'(\gamma)) \\ \mathrm{Tr}(B\rho'(\gamma)) \\ \mathrm{Tr}(AB\rho'(\gamma)) \end{pmatrix}$ for any $\gamma \in \Gamma$, hence $\rho = \rho'$ and the theorem is proved.

□

Bibliography

[Ben16] Leo Benard : Singularities of the reidemeister torsion form on the character variety. *arXiv preprint : 1612.06237*, 2016.

[Ben17] Leo Benard : Torsion function on character varieties. *ArXiv e-prints : 1711.08781*, 2017.

[Ber94] Robert W. Berger : Report on the torsion of the differential module of an algebraic curve. *In Algebraic geometry and its applications. Collections of papers from Shreeram S. Abhyankar's 60th birthday conference held at Purdue University, West Lafayette, IN, USA, June 1-4, 1990*, pages 285–303. New York: Springer-Verlag, 1994.

[Bro82] Kenneth S. Brown : Cohomology of groups. Graduate Texts in Mathematics, 87. New York-Heidelberg-Berlin: Springer-Verlag, 1982.

[Bur67] Gerhard Burde : Darstellungen von Knotengruppen. *Math. Ann.*, 173:24–33, 1967.

[BV13] Nicolas Bergeron et Akshay Venkatesh : The asymptotic growth of torsion homology for arithmetic groups. *J. Inst. Math. Jussieu*, 12(2): 391–447, 2013.

[BZ92] Jean-Michel Bismut et Weiping Zhang : *An extension of a theorem by Cheeger and Müller. With an appendix by François Laudenbach.* Paris: Société Mathématique de France, 1992.

[BZH14] Gerhard Burde, Heiner Zieschang et Michael Heusener : *Knots. 3rd fully revised and extented edition.* Berlin: Walter de Gruyter, 3rd fully revised and extented edition édition, 2014.

[CCG+94] Daryl Cooper, Marc Culler, Henri Gillet, Darren D Long et Peter B Shalen : Plane curves associated to character varieties of 3-manifolds. *Inventiones mathematicae*, 118(1):47–84, 1994.

[Che77] Jeff Cheeger : Analytic torsion and Reidemeister torsion. *Proc. Natl. Acad. Sci. USA*, 74:2651–2654, 1977.

[CS83] Marc Culler et Peter B Shalen : Varieties of group representations and splittings of 3-manifolds. *Annals of Mathematics*, pages 109–146, 1983.

[CS84] Marc Culler et Peter B Shalen : Bounded, separating, incompressible surfaces in knot manifolds. *Inventiones mathematicae*, 75(3):537–545, 1984.

[deR67] Georges deRham : Introduction aux polynômes d'un nœud. *Enseign. Math. (2)*, 13:187–194, 1967.

[DFJ12] Nathan M. Dunfield, Stefan Friedl et Nicholas Jackson : Twisted Alexander polynomials of hyperbolic knots. *Exp. Math.*, 21(4):329–352, 2012.

[DG16] Jerome Dubois et Stavros Garoufalidis : Rationality of the $SL(2, \mathbb{C})$-Reidemeister torsion in dimension 3. *Topol. Proc.*, 47:115–134, 2016.

[DHY09] Jérôme Dubois, Vu Huynh et Yoshikazu Yamaguchi : Non-abelian Reidemeister torsion for twist knots. *J. Knot Theory Ramifications*, 18(3):303–341, 2009.

[Dub06] Jérôme Dubois : Non abelian twisted Reidemeister torsion for fibered knots. *Can. Math. Bull.*, 49(1):55–71, 2006.

[Dun99] Nathan M. Dunfield : Examples of non-trivial roots of unity at ideal points of hyperbolic 3-manifolds. *Topology*, 38(2):457–465, 1999.

[FK91] Charles D Frohman et Eric P Klassen : Deforming representations of knot groups in su(2). *Commentarii Mathematici Helvetici*, 66(1):340–361, 1991.

[Fra35] Wolfgang Franz : Über die Torsion einer Überdeckung. *J. Reine Angew. Math.*, 173:245–254, 1935.

[Ful08] William Fulton : Algebraic curves. *An Introduction to Algebraic Geom*, 2008.

[GKZ94] I.M. Gelfand, M.M. Kapranov et A.V. Zelevinsky : *Discriminants, resultants, and multidimensional determinants.* Boston, MA: Birkhäuser, 1994.

[Gol84] William M. Goldman : The symplectic nature of fundamental groups of surfaces. *Adv. Math.*, 54:200–225, 1984.

[HP15] Michael HEUSENER et Joan PORTI : Representations of knot groups into $SL_n(\mathbb{C})$ and twisted Alexander polynomials. *Pac. J. Math.*, 277(2):313–354, 2015.

[HPSP01] Michael HEUSENER, Joan PORTI et Eva SUÁREZ-PEIRÓ : Deformations of reducible representations of 3-manifold groups into $sl_2(c)$. *Journal fur die Reine und Angewandte Mathematik*, pages 191–228, 2001.

[HT85] Allen E. HATCHER et William P. THURSTON : Incompressible surfaces in 2-bridge knot complements. *Invent. Math.*, 79:225–246, 1985.

[Jac80] William JACO : Lectures on three-manifold topology. Conference Board of the Mathematical Sciences. Regional Conference Series in Mathematics, 43. Providence, R.I.: American Mathematical Society (AMS). XII, 1980.

[Kit94] Teruaki KITANO : Reidemeister torsion of the figure-eight knot exterior for $SL(2,\mathbb{C})$-representations. *Osaka J. Math.*, 31(3):523–532, 1994.

[Kit96] Teruaki KITANO : Twisted Alexander polynomial and Reidemeister torsion. *Pac. J. Math.*, 174(2):431–442, 1996.

[KP96] Hanspeter KRAFT et Claudio PROCESI : *Classical Invariant Theory, a Primer.* 1996.

[Le14] Thang LE : Homology torsion growth and Mahler measure. *Comment. Math. Helv.*, 89(3):719–757, 2014.

[Liu06] Qing LIU : *Algebraic geometry and arithmetic curves. Transl. by Reinie Erné.* Oxford: Oxford University Press, 2006.

[LM85] Alexander LUBOTZKY et Andy R. MAGID : *Varieties of representations of finitely generated groups.*, volume 336. American Mathematical Society (AMS), Providence, RI, 1985.

[Mar15] Julien MARCHÉ : Character varieties in sl_2 and kauffman skein algebras. *arXiv preprint : 1510.09107*, 2015.

[MFK93] D. MUMFORD, J. FOGARTY et F. KIRWAN : *Geometric invariant theory. 3rd enl. ed.* Berlin: Springer-Verlag, 3rd enl. ed. édition, 1993.

[Mil66] John W. MILNOR : Whitehead torsion. *Bull. Am. Math. Soc.*, 72:358–426, 1966.

[MP12] Pere MENAL-FERRER et Joan PORTI : Twisted cohomology for hyperbolic three manifolds. *Osaka J. Math.*, 49(3):741–769, 2012.

[Mul78] Werner MULLER : Analytic torsion and R-torsion of Riemannian manifolds. *Adv. Math.*, 28:233–305, 1978.

[Mul93] Werner MULLER : Analytic torsion and R-torsion for unimodular representations. *J. Am. Math. Soc.*, 6(3):721–753, 1993.

[Oza00] Makoto OZAWA : Synchronism of an incompressible non-free Seifert surface for a knot and an algebraically split closed incompressible surface in the knot complement. *Proc. Am. Math. Soc.*, 128(3):919–922, 2000.

[Oza01] Makoto OZAWA : Additivity of free genus of knots. *Topology*, 40(4):659–665, 2001.

[Por97] Joan PORTI : *Torsion de Reidemeister pour les variétés hyperboliques.*, volume 612. American Mathematical Society (AMS), Providence, RI, 1997.

[Por15] Joan PORTI : Reidemeister torsion, hyperbolic three-manifolds, and character varieties. *ArXiv e-prints*, novembre 2015.

[Pro87] Claudio PROCESI : A formal inverse to the Cayley-Hamilton theorem. *J. Algebra*, 107:63–74, 1987.

[Prz83] Józef H. PRZYTYCKI : Incompressibility of surfaces after Dehn surgery. *Mich. Math. J.*, 30:289–308, 1983.

[PS00] Józef H. PRZYTYCKI et Adam S. SIKORA : On skein algebras and $Sl_2(\mathbb{C})$-character varieties. *Topology*, 39(1):115–148, 2000.

[Rei35] Kurt REIDEMEISTER : Homotopieringe und Linsenräume. *Abh. Math. Semin. Univ. Hamb.*, 11:102–109, 1935.

[Rol76] Dale ROLFSEN : Knots and links. Mathematical Lecture Series. 7. Berkeley, Ca.: Publish or Perish, Inc., 1976.

[RS71] D.B. RAY et I.M. SINGER : R-torsion and the Laplacian on Riemannian manifolds. *Adv. Math.*, 7:145–210, 1971.

[Sai94] Kyoji SAITO : Character variety of representations of a finitely generated group in sl_2. 1994.

[SB77] Jean-Pierre SERRE et Hyman BASS : Arbres, amalgames, sl_2 ; astérisque 46. 1977.

[Sha02] Peter B. SHALEN : Representations of 3-manifold groups. *In Handbook of geometric topology*, pages 955–1044. Amsterdam: Elsevier, 2002.

[Til03] Stephen Tillmann : *Finding hyperbolic structures of and surfaces in 3-manifolds*. 2003.

[Wei64] Andre Weil : Remarks on the cohomology of groups. *Annals of Mathematics*, 80(1):149–157, 1964.