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ON THE PERIOD OF SEQUENCES $(A^n(p))$ IN INTUITIONISTIC PROPOSITIONAL CALCULUS

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§0. Abstract. In classical propositional calculus for each proposition $A(p)$ the following holds: $\vdash A(p) \leftrightarrow A^3(p)$. In this paper we consider what remains of this in the intuitionistic case. It turns out that for each proposition $A(p)$ the following holds: there is an $n \in \mathbf{N}$ such that

$$\vdash A^n(p) \leftrightarrow A^{n+2}(p).$$

As a byproduct of the proof we give some theorems which may be useful elsewhere in propositional calculus.

§1. Finite order. Let Λ be a language for intuitionistic propositional calculus with atoms $a, b, c, \ldots$, constants $\top, \bot$, connectives $\wedge, \vee, \rightarrow$ and auxiliary symbols) and (. The formulas $\neg A$ and $A \leftrightarrow B$ are introduced as abbreviations for $A \rightarrow \bot$ and $(A \rightarrow B) \wedge (B \rightarrow A)$. Let Ω be the Heyting algebra for this language Λ with as objects equivalence classes

$$[A] = \{B \mid \vdash A \leftrightarrow B\}$$

and with the ordering induced by $\vdash$.

Let $A(p)$ be a formula, which may contain extra parameters $q, r, s, \ldots$. We can interpret $A(p)$ as a map from Ω to Ω sending $[B]$ to $[A(B)]$. We begin by considering the order of $A(p)$ as a map.

Define $A^0(p) = p$ and $A^{n+1}(p) = A(A^n(p))$.

1.1. PROPOSITION. *In classical propositional calculus we have for all $A(p)$*

$$\vdash_C A(p) \leftrightarrow A^3(p).$$

PROOF. Use the definability of Boolean functions. $\square$

So in the classical case $A(p)$ has order at most 3 and the length of the loop is at most 2.

Let $\Gamma \cup \{A(p), B, C\}$ be a set of formulas. Then the Substitution Lemma gives that if $\Gamma \vdash B \leftrightarrow C$ then $\Gamma \vdash A(B) \leftrightarrow A(C)$. By using Proposition 1.1 and iterated substitution we get: for all $A(p)$ and for all $m \geq 1$ we have

$$\vdash_C A^m(p) \leftrightarrow A^{m+2}(p).$$

Proposition 1.1 does not hold in the intuitionistic case. Consider $A(p) = \neg p \vee \neg\neg p$. Then we only have $\vdash A^2(p) \leftrightarrow A^3(p)$. This weaker result

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suggests what to look for in the intuitionistic situation. We shall prove that for each formula $A(p)$ there is an $n \in \mathbf{N}$ such that $\vdash A^n(p) \leftrightarrow A^{n+2}(p)$. Then for all $m \geq n$ we get

$$\vdash A^m(p) \leftrightarrow A^{m+2}(p).$$

1.2. LEMMA. *For all $A(p)$ and for all $s, m, n \in \mathbf{N}$ such that $s \leq m$ we have*
i) $A(\top), A^s(p) \vdash A^m(p)$.
ii) $A(\top), A^s(p) \vdash (A^{n+1}(p) \to A^n(p)) \to A^n(p)$.
PROOF. i) is proved by induction on $(m - s)$. Use $\vdash A^s(p) \leftrightarrow (A^s(p) \leftrightarrow \top)$.
ii) By i) we get $A(\top) \vdash A^n(p) \to A^{n+1}(p)$, so

$$\Gamma = \{A(\top), A^{n+1}(p) \to A^n(p)\} \vdash A^n(p) \leftrightarrow A^{n+1}(p).$$

By iterated substitution in $A(p)$ we get $\Gamma \vdash A^i(p) \leftrightarrow A^{i+1}(p)$, $n \leq i < s$.
Therefore Γ, $A^s(p) \vdash A^n(p)$. $\square$

1.3. DEFINITION. Let $A(p)$ be a formula and let Γ be a set of formulas. Then $A(p)$ has *bound n over Γ* if there is a sequence $\top = B_0(p), B_1(p), \ldots, B_n(p)$ of formulas satisfying the following condition: for each proposition variable $C(p) = a$ or $C(p) = p$ in $A(p)$ and for each implication subformula $C(p) = D(p) \to E(p)$ of $A(p)$ there is an $i \leq n$ such that $\Gamma \vdash C(\top) \leftrightarrow B_i(\top)$.

Observe that such an n always exists.

1.4. THEOREM. *Let $A(p)$ and $B(p)$ be formulas, let Γ be a set of formulas, and let $\Gamma_s = \Gamma \cup \{A(\top), A^s(p)\}$ for some $s \in \mathbf{N}$. Let $A(p) \wedge B(p)$ have bound n over Γ_s. Then at least one of the following cases holds for a new variable q.*
i) $\Gamma_s, A^{2n}(p) \to q \vdash (B(q) \leftrightarrow B(\top)) \wedge (B(\top) \to q)$.
ii) $\Gamma_s, A^{2n}(p) \to q \vdash B(q) \leftrightarrow q$.
iii) $\Gamma_s, A^{2n}(p) \to q \vdash B(q)$.

PROOF. By induction on the bound n. We may assume that $B(p)$ is a subformula of $A(p)$ by replacing $A(p)$ by the equivalent formula $A(p) \wedge (B(p) \vee \top)$. In that case $A(p)$ has bound n over Γ_s.

The case $n = 0$. Since the bound of $A(p)$ over Γ is equal to $n = 0$ we have $\Gamma_s \vdash a \leftrightarrow \top$ for all proposition variables $a \neq p$ and $\Gamma_s \vdash B(\top) \leftrightarrow \top$ for all implication subformulas $B(p)$. From $\Gamma_s \vdash a$ for all proposition variables $a \neq p$ it follows that each subformula $B(q)$ of $A(q)$ is equivalent to a formula of the Rieger-Nishimura lattice. The property $\Gamma_s \vdash B(\top)$ for implication subformulas $B(p)$ implies that if there is a subformula $B(p)$ such that $\Gamma_s \vdash B(q) \leftrightarrow (q \to \bot)$, then $\Gamma_s \vdash (\top \to \bot) \leftrightarrow \top$ and Γ_s is inconsistent. So if Γ_s is consistent, then for each subformula $B(p)$ we have $\Gamma_s \vdash B(q) \leftrightarrow \bot$ or $\Gamma_s \vdash B(q) \leftrightarrow q$ or $\Gamma_s \vdash B(q)$.

Induction step on n. We prove the induction step by induction on the length of the subformula $B(p)$. Let $\Delta_{s,m} = \Gamma_s \cup \{A^m(p) \to q\}$.

The case for length $= 1$. If $B(p) = p$, $B(p) = \top$ or if $B(p) = \bot$, then we easily verify ii), iii) or i) with $m = 0$ instead of $m = 2n$. Assume $B(p) = a$ for some variable $a \neq p$. If $\Gamma_s \vdash a$, then iii) holds. Assume $\Gamma_s \nvdash a$. Take $\Gamma'_s = \Gamma_s \cup \{a\}$. Then over the theory Γ'_s the formula $A(p)$ has a lower bound and we apply induction on n. For the subformula $A(p)$ of $A(p)$ itself one of the following statements holds for all $s \geq 0$:

$$\Delta'_{s,2n-2} = \Delta_{s,2n-2} \cup \{a\} \vdash A(q),$$
$$\Delta'_{s,2n-2} \vdash A(q) \leftrightarrow q.$$

Substitute $q = A^{2n-1}(p)$ in the relations above. With Lemma 1.2 this gives us $\Gamma_s \cup \{a\} \vdash A^{2n}(p)$. So $\Gamma_s \vdash a \to A^{2n}(p)$ and this implies that i) holds for $B(p) = a$.

Induction step on the length. Write $B(p) = C(p) \square D(p)$ where $C(p)$ and $D(p)$ satisfy one of the conditions i), ii) and iii) and where $\square$ is one of the connectives $\wedge$, $\vee$ or $\to$. Then we can make the following tables.

	$\wedge$	$D(p)$ i)	ii)	iii)
$C(p)$	i)	i)	i)	i)
	ii)	i)	ii)	ii)
	iii)	i)	ii)	iii)

	$\vee$	$D(p)$ i)	ii)	iii)
$C(p)$	i)	i)	ii)	iii)
	ii)	ii)	ii)	iii)
	iii)	iii)	iii)	iii)

	$\to$	$D(p)$ i)	ii)	iii)
$C(p)$	i)	*	iii)	iii)
	ii)	*	iii)	iii)
	iii)	i)	ii)	iii)

These tables express which condition will be satisfied by $B(p) = C(p) \square D(p)$. Most of them are easy to verify. There are two cases which are more involved. Both are marked by *.

Case (a): $B(p) = C(p) \to D(p)$, where $C(p)$ satisfies i) and $D(p)$ satisfies i). We have $\Delta_{s,2n} \vdash B(q) \leftrightarrow B(\top)$. If $\Gamma_s \vdash B(\top)$, then $B(p)$ satisfies iii). Assume $\Gamma_s \nvdash B(\top)$. Let $\Gamma'_s = \Gamma_s \cup \{B(\top)\}$. Then over the theory Γ'_s we find that $A(p)$ has a lower bound. Apply induction. For $A(p)$ as subformula of itself we have $\Delta'_{s,2n-2} = \Delta_{s,2n-2} \cup \{B(\top)\} \vdash A(q)$ or $\Delta'_{s,2n-2} \vdash A(q) \leftrightarrow q$. Substitute $q = A^{2n-1}(p)$. Then $\Gamma'_s \vdash A^{2n}(p)$. So $\Gamma_s \vdash B(\top) \to A^{2n}(p)$. Thus $B(p)$ satisfies i).

Case (b): $B(p) = C(p) \to D(p)$, where $C(p)$ satisfies ii) and $D(p)$ satisfies i). If $\Gamma_s \vdash B(\top)$, then $B(p)$ satisfies iii). So assume $\Gamma_s \nvdash B(\top)$. We shall prove that $B(p)$ satisfies i). We easily see that $\Delta_{s,2n} \vdash B(\top) \to q$ and $\Delta_{s,2n} \vdash B(\top) \to B(q)$. It remains to show $\Delta_{s,2n} \vdash B(q) \to B(\top)$. Let $\Gamma'_s = \Gamma_s \cup \{B(\top)\}$. Then $A(p)$ has a lower bound over Γ'_s; thus $\Delta'_{s,2n-2} = \Delta_{s,2n-2} \cup \{B(\top)\} \vdash A(q)$ or $\Delta'_{s,2n-2} \vdash A(q) \leftrightarrow q$. Substitute $q = A^{2n-2}(p)$. Then $\Gamma_s \vdash B(\top) \to A^{2n-1}(p)$. Let $\Delta''_{s,2n} = \Delta_{s,2n} \cup \{B(q)\}$. Then $\Delta''_{s,2n} \vdash q \to D(\top)$. Since $\Gamma_s \vdash (B(\top) \leftrightarrow D(\top))$ and $\Gamma_s \vdash (B(\top) \to A^{2n-1}(p))$, we have $\Delta''_{s,2n} \vdash q \to A^{2n-1}(p)$. Thus $\Delta''_{s,2n} \vdash A^{2n}(p) \to A^{2n-1}(p)$. Apply Lemma 1.2. Then we get $\Delta''_{s,2n} \vdash A^{2n}(p)$. Thus $\Delta''_{s,2n} \vdash q$ and $\Delta''_{s,2n} = \Delta_{s,2n} \cup \{B(q)\} \vdash D(\top)$. Thus $\Delta_{s,2n} \vdash B(q) \to B(\top)$.

This completes the proof of the induction steps. $\square$

EXAMPLE (PIET RODENBURG). Let $A(p) = ((p \to a) \to a) \vee (a \to p)$ and let $B(p) = p \to a$. Then we have $\vdash A(\top) \wedge A^2(p)$ and for all s we have

$$A(\top), A^s(p), A^2(p) \to q \vdash (B(q) \leftrightarrow B(\top)) \wedge (B(\top) \to q).$$

If we replace $A^2(p) \to q$ by $A(p) \to q$, then we can substitute $q = A(p)$ and $s = 2$, and we conclude that

$$\vdash B(A(p)) \to a.$$

Substitute $p = \bot$. Then we get $\vdash \neg\neg a \to a$. Contradiction. So the statement does not hold if we replace $A^2(p) \to q$ by $A(p) \to q$.

1.5. COROLLARY. *For each formula $A(p)$ there is an $m \in \mathbf{N}$ such that for all $s \in \mathbf{N}$ we have*

$$A(\top), A^s(p) \vdash A^m(p).$$

PROOF. Take $\Gamma = \varnothing, B(p) = A(p)$ and $q = A^{2n}(p)$ in Theorem 1.4. Then one of the following holds:

$$A(\top), A^s(p) \vdash A^{2n+1}(p) \wedge A^{2n}(p),$$
$$A(\top), A^s(p) \vdash A^{2n+1}(p) \leftrightarrow A^{2n}(p),$$
$$A(\top), A^s(p) \vdash A^{2n+1}(p).$$

So by Lemma 1.2 we have

$$A(\top), A^s(p) \vdash A^{2n+1}(p). \quad \square$$

1.6. LEMMA. *Given $A(p)$ and m such that for all s we have $A(\top), A^s(p) \vdash A^m(p)$. Then*

$$A(\top) \vdash A^m(p) \leftrightarrow A^{m+1}(p).$$

PROOF. By Lemma 1.2 we have $A(\top) \vdash A^m(p) \rightarrow A^{m+1}(p)$. Now take $s = m + 1$:

$$A(\top) \vdash A^m(p) \leftrightarrow A^{m+1}(p). \quad \square$$

1.7. LEMMA. *For all $A(p)$, m and n we have*
i) $A^{2m+1}(\top) \vdash A^n(\top)$,
ii) $A^{2m+2}(\top) \vdash A^{2n}(\top)$.
PROOF.

$$\text{i)}\quad \begin{array}{cl} A^{2m+1}(\top) \quad A^{2m}(\top)^{(1)} & (*) \\ \hline A(\top) & (1) \\ \hline A^{2m}(\top) \rightarrow A(\top) & (\#) \\ \hline A^{2m}(\top) \leftrightarrow A(\top) & \\ \hline A^{2m+1}(\top) \leftrightarrow A^2(\top) & \\ \hline A^2(\top) & \\ \hline A^{2m}(\top) & \\ \hline A(\top) & \\ \hline A^n(\top) & \end{array} \qquad \text{ii)}\quad \begin{array}{cl} A^{2m+2}(\top) \quad A^{2m+1}(\top)^{(1)} & \\ \hline A(\top) & (1) \\ \hline A^{2m+1}(\top) \rightarrow A(\top) & \\ \hline A^{2m+1}(\top) \leftrightarrow A(\top) & \\ \hline A^{2m+2}(\top) \leftrightarrow A^2(\top) & \\ \hline A^2(\top) & (\$) \\ \hline A^{2n}(\top) & \end{array}$$

(*) Use $\vdash A^{2m}(\top) \leftrightarrow (A^{2m}(\top) \leftrightarrow \top)$ and substitution.
(#) Apply Lemma 1.2i) with $p = \top$.
($) Apply Lemma 1.2i) to $A^2(\top)$ or use iterated substitution. $\square$

Observe that 1.7ii) is not an easy corollary of 1.7i). With Lemma 1.7 we can prove theorems like $\vdash A(\top) \leftrightarrow A^3(\top)$.

1.8. LEMMA. *Given $A(p)$ and m such that $A(\top) \vdash A^m(p) \leftrightarrow A^{m+1}(p)$, then we have*

$$\vdash A^{m+1}(p) \leftrightarrow A^{m+3}(p).$$

PROOF. The places in the derivations below where we use our assumption $A(\top) \vdash A^m(p) \leftrightarrow A^{m+1}(p)$ are marked by (*). Observe that (*) is equivalent to: for all

$n \geq m$ we have $A(\top) \vdash A^n(p) \leftrightarrow A^m(p)$. First we show $\vdash A^{m+1}(p) \to A^{m+3}(p)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A^{m+1}(p)^{(4)} \quad A^m(p)^{(1)}}{A(\top)}(\#)}{A^{m+2}(p)}(\#)}{A^m(p) \to A^{m+2}(p)}(1) \qquad \dfrac{\dfrac{\dfrac{A^{m+1}(p)^{(2)} \quad A^{m+2}(p)^{(3)}}{A(\top)}(\#)}{A^m(p)}(2)(*)}{A^{m+2}(p) \to A^m(p)}(3)}{A^m(p) \leftrightarrow A^{m+2}(p)}}{A^{m+1}(p) \leftrightarrow A^{m+3}(p)}(\#)}{A^{m+3}(p)}(\$)}{A^{m+1}(p) \to A^{m+3}(p)}(4)$$

(#) Use substitution.
($) Use assumption (4).

Next we show $\vdash A^{m+3}(p) \to A^{m+1}(p)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A^{m+3}(p)^{(1)} \quad A^{m+2}(p)^{(2)}}{A(\top)}(\#)}{A^m(p)}(1)(*)}{A^{m+2}(p) \to A^m(p)}(2) \qquad \dfrac{\dfrac{\dfrac{\dfrac{A^{m+3}(p)^{(4)} \quad A^m(p)^{(3)}}{A^3(\top)}(\#)}{A^2(\top)}(\text{Lemma 1.7})}{A^{m+2}(p)}(\#)}{A^m(p) \to A^{m+2}(p)}(3)}{A^m(p) \leftrightarrow A^{m+2}(p)}}{A^{m+1}(p) \leftrightarrow A^{m+3}(p)}(\#)}{A^{m+1}(p)}(\$)}{A^{m+3}(p) \to A^{m+1}(p)}(4)$$

(#) Use substitution.
($) Use assumption (4). □

1.9. Theorem (Finite Order Theorem). *For all $A(p)$ there is an $m \in \mathbf{N}$ such that*

$$\vdash A^m(p) \leftrightarrow A^{m+2}(p).$$

Proof. Combine 1.5, 1.6 and 1.8. □

Observe that we also get a bound on m in Theorem 1.9. We say that $A(p)$ has bound n if $A(p)$ has bound n over $\Gamma = \varnothing$. Then by 1.4, 1.5 and 1.6 we get after substituting $q = A^{2n}(p)$ and $B(p) = A(p)$ that $A(\top) \vdash A^{2n+1}(p) \leftrightarrow A^{2n+2}(p)$. By Lemma 1.8 this gives

$$\vdash A^{2n+2}(p) \leftrightarrow A^{2n+4}(p).$$

§2. **Examples.** In this section we shall give some examples which show that the value m in Theorem 1.9 cannot be bounded.

2.1. EXAMPLE. Consider the formula

$$A(p) = (a_1 \vee (a_1 \to p)) \wedge (a_2 \vee (a_2 \to p)) \wedge \cdots \wedge (a_n \vee (a_n \to p)).$$

Then we can show $\vdash A(\top)$ and $\vdash A^{n+1}(p)$, but also $\nvdash A^n(p)$. Thus we do not have $\vdash A^n(p) \leftrightarrow A^{n+2}(p)$. We only show $\nvdash A^n(p)$.

Consider the following Kripke model.

$$\begin{array}{ll}
\alpha_{n+1} & \bullet \\
 & \mid\; p, a_1, a_2, a_3, \ldots, a_n \\
\alpha_n & \bullet \\
 & \mid\; a_1, a_2, a_3, \ldots, a_n \\
\alpha_{n-1} & \bullet \\
 & \mid\; a_2, a_3, \ldots, a_n \\
 & \vdots \\
 & \mid\; a_{n-1}, a_n \\
\alpha_2 & \bullet \\
 & \mid\; a_n \\
\alpha_1 & \bullet \\
 & \mid \\
\alpha_0 & \bullet
\end{array}$$

Then $\alpha_i \Vdash A^m(p)$ if and only if $i + m \geq n + 1$, so $\alpha_0 \nVdash A^n(p)$. Observe that we have $\vdash A(\top)$ and $\vdash A^{n+1}(p) \leftrightarrow A^{n+2}(p)$.

2.2. EXAMPLE. For $B(p) = A(p) \wedge (a_n \vee (p \to a_n))$, where $A(p)$ is as in 2.1, and for the Kripke model of 2.1 we again have $\alpha_i \Vdash B^k(p)$ if and only if $i + k \geq n + 1$ $(k \leq n + 1)$. But we only have $\alpha_0 \Vdash B^{n+1}(p)$ and $\alpha_0 \Vdash B^n(p) \leftrightarrow B^{n+2}(p)$, and not $\alpha_0 \Vdash B^n(p)$.

For special classes of formulas we can find a uniform bound on n such that for all formulas of that class we have $\vdash A^n(p) \leftrightarrow A^{n+2}(p)$.

2.3. PROPOSITION. *Let $A(p)$ have no extra variables or constants but $\top$ and $\bot$. Then we have*

$$\vdash A^2(p) \leftrightarrow A^4(p).$$

PROOF. *First proof.* The formula $A(p)$ is equivalent to a formula of the Rieger-Nishimura lattice. For almost all of these formulas we have $\vdash A(\top)$ and $\vdash A^2(p)$. The remaining cases are easy to verify. Of special interest are $A(p) = \neg p \vee \neg\neg p$ $(\vdash A^2(p) \leftrightarrow A^3(p))$ and $A(p) = \neg p$ $(\vdash A(p) \leftrightarrow A^3(p)$, i.e. $\vdash \neg p \leftrightarrow \neg\neg\neg p)$.

Second proof. We immediately see that a formula $A(p)$ with no variables but p has bound 1. For this special class of formulas when we go through the proof of Theorem 1.4 we find that we can take $m = 0$ instead of $m = 2n$, since if $\Gamma_s \nvdash B(\top)$ then $\Gamma_s \vdash \neg B(\top)$. It follows that $A(\top), A^s(p) \vdash A(p)$ for all s. Then apply 1.5, 1.6 and 1.8: $\vdash A^2(p) \leftrightarrow A^4(p)$. $\square$

2.4. THEOREM. *Let $A(p)$ have at most one sort of extra variable a and $\top$, and no $\bot$. Then we have*

$$\vdash A^3(p) \leftrightarrow A^5(p).$$

PROOF. The formula $A(p)$ is built up by $a, p, \top$ and the connectives. Therefore we have $\vdash A(\top)$ or $\vdash A(\top) \leftrightarrow a$.

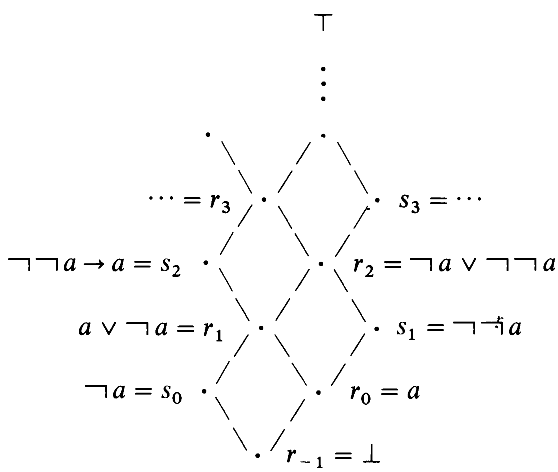
Assume $\vdash A(\top) \leftrightarrow a$. Then we have $A(\top) \vdash A(p) \leftrightarrow p$ or $A(\top) \vdash A(p)$. By Lemma 1.6 and Lemma 1.8 we get $\vdash A^2(p) \leftrightarrow A^4(p)$ and by substitution $\vdash A^3(p) \leftrightarrow A^5(p)$.

Assume $\vdash A(\top)$. The formula $A(p)$ has bound 1. By Corollary 1.5 this implies $A^s(p) \vdash A^3(p)$ for all s. Take $s = 5$ and use Lemma 1.2i). Then we get $\vdash A^3(p) \leftrightarrow A^5(p)$. □

2.5. Example. The following shows that Theorem 2.4 does not hold if we allow $\perp$ to occur in $A(p)$. Let $r_{-1}, r_0, r_1, r_2, \ldots$ and $s_0, s_1, s_2, \ldots$ be the following sequences of formulas:

$$r_{-1} = \perp, \quad r_0 = a, \quad s_0 = \neg a, \quad r_1 = a \vee \neg a,$$
$$r_m = s_{m-1} \vee s_{m-2} \quad (m \geq 2),$$
$$s_m = s_{m-1} \rightarrow r_{m-2} \quad (m \geq 1).$$

If we add $\top$, then these sequences form the Rieger-Nishimura lattice with the ordering induced by $\vdash$.



Now take as $A(p)$ the following formula, which only uses a, p, $\perp$ and the connectives:

$$A(p) = (r_0 \vee (r_0 \rightarrow p)) \wedge (r_2 \vee (r_2 \rightarrow p)) \wedge \cdots \wedge (r_{2n} \vee (r_{2n} \rightarrow p)).$$

Then for odd $k < 2n$ (including $k = -1$) we have $\vdash A(r_k) \leftrightarrow r_{k+2}$ and $\vdash A^{n+2}(p)$ (thus $\vdash A^{n+2}(p) \leftrightarrow A^{n+4}(p)$).

So if we include $\perp$ we no longer have a uniform bound on n as in Theorem 2.4.

2.6. Example. In the classical situation we have

$$\vdash_C A(p) \leftrightarrow A^3(p).$$

This provides us with uniform interpolants: if we have $A(p) \vdash_C B$, then $A(p) \vdash_C A(A(\top))$ and $A(A(\top)) \vdash_C B$. The interpolant $A(A(\top))$ in which p does not occur does not depend on the choice of B.

This procedure no longer works in the intuitionistic case. Let $A(p)$ be the following formula:

$$A(p) = (a_1 \vee (a_1 \to p)) \wedge \cdots \wedge (a_n \vee (a_n \to p)) \wedge (p \to b) \wedge ((p \to a_n) \vee (c \to p) \vee c).$$

Consider the following Kripke model.

$$\begin{array}{l|l}
\alpha_{n+3} & p, a_1, a_2, \ldots, a_n, b, c \\
\alpha_{n+2} & a_1, a_2, \ldots, a_n, b, c \\
\alpha_{n+1} & a_2, \ldots, a_n, b, c \\
 & \vdots \\
\alpha_4 & a_{n-1}, a_n, b, c \\
\alpha_3 & a_n, b, c \\
\alpha_2 & b, c \\
\alpha_1 & c \\
\alpha_0 &
\end{array}$$

Then we have $\alpha_0 \Vdash A^k(p) \leftrightarrow a_k$ for $1 \leq k \leq n$, $\alpha_0 \Vdash A^{n+1}(p)$, $\alpha_0 \Vdash A^{n+m+1}(p) \leftrightarrow b$ for odd $m > 0$ and $\alpha_0 \Vdash A^{n+m+1}(p) \leftrightarrow c$ for even $m > 0$. So this model shows that $A^{n+1}(p) \nvdash A(\top) \vee A^2(\top)$. Thus also $A(p) \nvdash A^2(\top)$. In the model we have $\alpha_0 \Vdash A(a_n)$, so there still is the possibility that $\bigvee\!\!\!\bigvee_{B(p)} A(B(\top))$ works as a uniform interpolant, where $B(p)$ ranges over the subformulas of $A(p)$.

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