AN ANTI-LOCALLY-NAMELESS APPROACH TO FORMALIZING QUANTIFIERS

Raffaele Di Donna
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Aix-Marseille University
Motivations

- Formalization of syntactic results by proof assistants.
Motivations

- Formalization of syntactic results by proof assistants.
- Avoiding $\alpha$-equivalence.
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TRICKY CASES OF CUT ELIMINATION
For some $y$ not free in $\Gamma$ nor in $\forall x A$ we have:

\[
\frac{\Gamma \vdash A[y/x]}{\Gamma \vdash \forall x A} \quad (\forall I) \quad \frac{\Delta, A[t/x] \vdash B}{\Delta, \forall x A \vdash B} \quad (\forall L)
\]

\[
\frac{\Delta, \forall x A \vdash B \quad (\text{cut})}{\Gamma, \Delta \vdash B}
\]
CASE OF A GENERALIZATION VERSUS AN INSTANTIATION

For some $y$ not free in $\Gamma$ nor in $\forall x A$ we have:

\[
\begin{align*}
\vdash \pi \\
\Gamma \vdash A[y/x] & \quad (\forall I) \\
\Gamma \vdash \forall x A \\
\vdash \rho \\
\Delta, A[t/x] \vdash B & \quad (\forall L) \\
\Delta, \forall x A \vdash B & \quad (\text{cut}) \\
\Gamma, \Delta \vdash B
\end{align*}
\]

Cut elimination step:

\[
\begin{align*}
\vdash \pi[t/y] \\
\Gamma \vdash A[t/x] & \quad (\text{cut}) \\
\vdash \rho \\
\Delta, A[t/x] \vdash B & \quad (\text{cut}) \\
\Gamma, \Delta \vdash B
\end{align*}
\]
**FIRST EXAMPLE**

\[
\begin{align*}
& Px, Py \vdash Py \quad (ax) \\
& Px, \forall x Px \vdash Py \quad (\forall L) \\
& Px, \forall x Px \vdash \forall x Px \quad (\forall I) \\
& Px, \forall x Px \vdash Py \lor \forall x Px \quad (\lor IR) \\
& Px, \forall x Px \vdash \forall y (Py \lor \forall x Px) \quad (\forall I) \\
& Px, \forall x Px \vdash \forall y (Py \lor \forall x Px) \quad (\forall I) \\
& \forall y (Py \lor \forall x Px) \vdash Px \lor \forall x Px \quad (\forall L) \\
& Px, \forall x Px \vdash Px \lor \forall x Px \\ & \phantom{Px, \forall x Px \vdash \forall y (Py \lor \forall x Px)} \quad (cut)
\end{align*}
\]
FIRST EXAMPLE

\[
\begin{align*}
&\frac{\text{Px, Px} \vdash \text{Px}}{\text{Px, } \forall x \text{Px} \vdash \text{Px}} \quad (\forall L) \\
&\frac{\text{Px, } \forall x \text{Px} \vdash \text{Px}}{\text{Px, } \forall x \text{Px} \vdash \forall x \text{Px}} \quad (\forall I) \\
&\frac{\text{Px, } \forall x \text{Px} \vdash \text{Px} \lor \forall x \text{Px}}{\text{Px, } \forall x \text{Px} \vdash \forall x \text{Px}} \quad (\lor IR) \\
&\frac{\text{Px} \lor \forall x \text{Px} \vdash \text{Px} \lor \forall x \text{Px}}{\text{Px, } \forall x \text{Px} \vdash \text{Px} \lor \forall x \text{Px}} \quad (\text{ax}) \\
&\frac{\text{Px, } \forall x \text{Px} \vdash \text{Px} \lor \forall x \text{Px}}{\text{Px} \lor \forall x \text{Px} \vdash \text{Px} \lor \forall x \text{Px}} \quad (\text{cut}) \\
&\text{Px, } \forall x \text{Px} \vdash \text{Px} \lor \forall x \text{Px}
\end{align*}
\]
For some $y$ not free in $\Delta, A$ nor in $\forall x B$ we have:

\[
\frac{
\frac{\pi}{\Gamma \vdash A} 
\quad \frac{\Delta, A \vdash B[y/x]}{\Delta, A \vdash \forall x B}
}{\Gamma, \Delta \vdash \forall x B}
\]  

$\forall I$  

$\text{(cut)}$
For some $y$ not free in $\Delta$, $A$ nor in $\forall x B$ we have:

$$
\Gamma \vdash A \\
\Delta, A \vdash B[y/x] \\
\Gamma, \Delta \vdash \forall x B
$$

Cut elimination step:

$$
\Gamma \vdash A \\
\Delta, A \vdash B[y/x] \\
\Gamma, \Delta \vdash B[y/x] \\
\Gamma, \Delta \vdash \forall x B
$$
SECOND EXAMPLE

\[
\begin{align*}
\frac{Py, Px \vdash Px}{Py, \forall x Px \vdash Px} & \quad (\forall L) \\
\frac{Py, \forall x Px \vdash \forall x Px}{Py, \forall x Px \vdash \forall x Px} & \quad (\forall I)
\end{align*}
\]

\[
\begin{align*}
\frac{Py, \forall x Px \vdash \forall x Px}{Py, \forall x Px \vdash \forall x Px} & \quad (cut)
\end{align*}
\]
SECOND EXAMPLE

\[
\begin{align*}
\frac{Py, Px \vdash Px}{Py, \forall xPx \vdash Px} & \quad (\forall I) \\
\frac{Py, \forall xPx \vdash \forall xPx}{Py, \forall xPx \vdash Py} & \quad (\forall I) \\
\frac{Py \vdash Py}{\forall xPx \vdash Py} & \quad (\forall I) \\
\frac{Py, \forall xPx \vdash Py}{Py, \forall xPx \vdash \forall xPx} & \quad (\forall I)
\end{align*}
\]
Introduce a notion of *nice* derivation: each occurrence of the \((\forall I)\) rule has its own variable.
One possible solution

- Introduce a notion of *nice* derivation: each occurrence of the $(\forall I)$ rule has its own variable.
- Every derivation is *equivalent* to a *nice* derivation.
One possible solution

- Introduce a notion of *nice* derivation: each occurrence of the \((\forall I)\) rule has its own variable.
- Every derivation is *equivalent* to a *nice* derivation.
- We can apply any cut elimination step to a *nice* derivation.
One possible solution

- Introduce a notion of *nice* derivation: each occurrence of the \((\forall I)\) rule has its own variable.
- Every derivation is *equivalent* to a *nice* derivation.
- We can apply any cut elimination step to a *nice* derivation.
- But we have to resort to variable renaming several times...
A DIFFERENT APPROACH
TWO KINDS OF VARIABLES

Gentzen’s approach, with $e$ not occurring in $\Gamma$ nor $A$:

$$\frac{\Gamma \vdash A[e/x]}{\Gamma \vdash \forall x A} \quad (\forall I)$$
TWO KINDS OF VARIABLES

Gentzen’s approach, with $e$ not occurring in $\Gamma$ nor $A$:

$$\frac{\Gamma \vdash A[e/x]}{\Gamma \vdash \forall x A} \quad (\forall I)$$

- $e$-variables (“eigen”) in $\mathcal{E}$, denoted $e, e', e_1, \ldots$
  - Constants at the level of terms and formulas.
  - Variables at the level of proofs.
Gentzen’s approach, with $e$ not occurring in $\Gamma$ nor $A$:

\[
\Gamma \vdash A[e/x] \\
\Gamma \vdash \forall x A
\]  

- **e-variables** (“eigen”) in $\mathcal{E}$, denoted $e, e', e_1, \ldots$
  - Constants at the level of terms and formulas.
  - Variables at the level of proofs.
- **f-variables** (“formula”) in $\mathcal{V}$, denoted $x, y, z, \ldots$
\textbf{Terms and Substitution}

\textit{E}-terms are given by:

\[ t ::= x \mid e \mid gt \ldots t \]
$\mathcal{E}$-terms are given by:

$$t ::= x \mid e \mid gt \ldots t$$

An $\mathcal{E}$-term is **f-closed** if it contains no f-variable.
\[t ::= x \mid e \mid gt \ldots t\]

An \(E\)-term is \(f\)-closed if it contains no \(f\)-variable.

The substitution \(t[u/x]\) of an \(f\)-variable \(x\) by an \(E\)-term \(u\) in an \(E\)-term \(t\) is defined by induction on \(t\):

\[
\begin{align*}
x[u/x] &= u \\
y[u/x] &= y & \text{(if } x \neq y) \\
e[u/x] &= e \\
(gt_1 \ldots t_k)[u/x] &= g(t_1[u/x]) \ldots (t_k[u/x])
\end{align*}
\]
\( \exists \)-formulas are given by:

\[
A ::= Pt \ldots t | A \star A | \forall x A | \exists x A
\]
**Formulas and substitution**

$\varepsilon$-formulas are given by:

$$A ::= Pt \ldots t \mid A \star A \mid \forall x A \mid \exists x A$$

The substitution $A[u/x]$ of an f-variable $x$ by an $\varepsilon$-term $u$ in an $\varepsilon$-formula $A$ is defined by induction on $A$:

$$(Pt_1 \ldots t_k)[u/x] = P(t_1[u/x]) \ldots (t_k[u/x])$$

$$(B \star C)[u/x] = (B[u/x]) \star (C[u/x])$$

$$(\forall x B)[u/x] = \forall x B$$

$$(\forall y B)[u/x] = \forall y (B[u/x]) \quad (\text{if } x \neq y)$$

$$(\exists x B)[u/x] = \exists x B$$

$$(\exists y B)[u/x] = \exists y (B[u/x]) \quad (\text{if } x \neq y)$$
If $t$ is an $f$-closed term:

\[
\frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x A} \quad (\exists I)
\]

\[
\frac{\Gamma, A[t/x] \vdash B}{\Gamma, \forall x A \vdash B} \quad (\forall L)
\]

Notice that $Pt \not\vdash \exists x P x$ is provable if and only if $t$ is $f$-closed.

Not a problem: provability of $f$-closed formulas (i.e. with no free occurrence of $f$-variable) in our system corresponds to provability of formulas in usual intuitionistic sequent calculus.
If $t$ is an $f$-closed term:

$$\frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x A} \quad (\exists I) \quad \frac{\Gamma, A[t/x] \vdash B}{\Gamma, \forall x A \vdash B} \quad (\forall L)$$

Notice that $Pt \to \exists x Px$ is provable if and only if $t$ is $f$-closed.
If \( t \) is an \textbf{f-closed} term:

\[
\frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x A} \quad (\exists I)
\]

\[
\frac{\Gamma, A[t/x] \vdash B}{\Gamma, \forall x A \vdash B} \quad (\forall L)
\]

Notice that \( Pt \rightarrow \exists x Px \) is provable if and only if \( t \) is f-closed.

Not a problem: provability of f-closed formulas (i.e. with no free occurrence of f-variable) in our system corresponds to provability of formulas in usual intuitionistic sequent calculus.
If $e$ does not occur in $\Gamma, A, C$:

\[
\frac{\Gamma \vdash A[e/x]}{\Gamma \vdash \forall x A} \quad (\forall I)
\]

\[
\frac{\Gamma, A[e/x] \vdash B}{\Gamma, \exists x A \vdash B} \quad (\exists L)
\]
If $e$ does not occur in $\Gamma, A, C$:

\[
\begin{align*}
\Gamma \vdash A[e/x] & \quad (\forall I) \\
\Gamma \vdash \forall xA & \\
\end{align*}
\]

\[
\begin{align*}
\Gamma, A[e/x] \vdash B & \quad (\exists I) \\
\Gamma, \exists xA \vdash B & \\
\end{align*}
\]

The e-variable $e$ is called the eigenvariable of these rules.
If \( e \) does not occur in \( \Gamma, A, C \):

\[
\frac{\Gamma \vdash A[e/x]}{\Gamma \vdash \forall x A} \quad (\forall I)
\]

\[
\frac{\Gamma, A[e/x] \vdash B}{\Gamma, \exists x A \vdash B} \quad (\exists L)
\]

The e-variable \( e \) is called the eigenvariable of these rules.

We wish for uniqueness of eigenvariables, but this is not possible due to proof transformations with duplications involved.
THE ANTI-LOCALLY-NAMELESS APPROACH
We fix $\mathcal{E} = \mathbb{N}$ and we lift eigenvariables when crossing a generalization rule upwards.

\[
\frac{\Gamma \vdash A^{[0/x]} \uparrow}{\Gamma, \forall x A \vdash B^{[0/x]} \uparrow} \quad (\forall I) \quad \frac{\Gamma \vdash A^{[0/x]} \uparrow, B^{[0/x]} \vdash \Gamma^{[0/x]} \uparrow}{\Gamma, \exists x A \vdash B \uparrow} \quad (\exists L)
\]
A CONCRETE EXAMPLE

\[
\begin{align*}
\text{(ax)} & \quad P00 \vdash P00 \\
\text{(\forall L)} & \quad \forall yP0y \vdash P00 \\
\text{(\forall L)} & \quad \forall x\forall yPxy \vdash P00 \\
\text{(\forall L)} & \quad \forall x\forall yPxy \vdash P01 \\
\text{(\forall L)} & \quad \forall x\forall yPxy \vdash \forall yPy0 \\
\text{(\forall L)} & \quad \forall x\forall yPxy \vdash \forall x(Pxx \land \forall yPyx) \\
\text{ax} & \quad P01 \vdash P01 \\
\text{(\forall L)} & \quad \forall yP0y \vdash P01 \\
\text{(\forall L)} & \quad \forall x\forall yPxy \vdash P01 \\
\text{(\forall L)} & \quad \forall x\forall yPxy \vdash \forall yPy0 \\
\text{(\land I)} & \quad \forall x\forall yPxy \vdash \forall x(Pxx \land \forall yPyx)
\end{align*}
\]
Given a function \( r: \mathcal{E} \rightarrow \mathcal{E} \)-terms, we define the parallel substitution \( t[r] \) by induction on \( t \):

\[
\begin{align*}
x[r] & = x \\
e[r] & = r(e) \\
(gt_1 \ldots t_k)[r] & = g(t_1[r]) \ldots (t_k[r])
\end{align*}
\]

We also have \( A[r] \) defined by induction on \( A \):

\[
\begin{align*}
(Pt_1 \ldots t_k)[r] & = P(t_1[r]) \ldots (t_k[r]) \\
(B \rightarrow C)[r] & = (B[r]) \rightarrow (C[r]) \\
(\forall xB)[r] & = \forall x(B[r]) \\
(\exists xB)[r] & = \exists x(B[r])
\end{align*}
\]
We say that $r$ is f-closed if $r(e)$ is f-closed for all $e \in \mathcal{E}$. 
We say that $r$ is f-closed if $r(e)$ is f-closed for all $e \in \mathcal{E}$.

**Lemma (Composition).**

\[
t[r][s] = t[e \mapsto r(e)[s]]
\]
\[
A[r][s] = A[e \mapsto r(e)[s]]
\]

**Lemma (Commutation).** If $r$ is f-closed, then:

\[
t[u/x][r] = t[r][u[r]/x]
\]
\[
A[u/x][r] = A[r][u[r]/x]
\]
Let $S : \mathbb{N} \to \mathbb{N}$-terms be defined by $S(n) = n + 1$. We define:

$$t' = t[S]$$
$$A' = A[S]$$
Let $S: \mathbb{N} \to \mathbb{N}$-terms be defined by $S(n) = n + 1$. We define:

$$t^\uparrow = t[S]$$
$$A^\uparrow = A[S]$$

And given $r: \mathbb{N} \to \mathbb{N}$-terms, we define $\uparrow r: \mathbb{N} \to \mathbb{N}$-terms by:

$$\uparrow r(n) = \begin{cases} 
0 & \text{if } n = 0 \\
 r(n - 1)^\uparrow & \text{otherwise}
\end{cases}$$
Let $S: \mathbb{N} \rightarrow \mathbb{N}$-terms be defined by $S(n) = n + 1$. We define:

$$t \uparrow = t[S]$$
$$A \uparrow = A[S]$$

And given $r: \mathbb{N} \rightarrow \mathbb{N}$-terms, we define $\uparrow r: \mathbb{N} \rightarrow \mathbb{N}$-terms by:

$$\uparrow r(n) = \begin{cases} 0 & \text{if } n = 0 \\ r(n-1) \uparrow & \text{otherwise} \end{cases}$$

By construction $\uparrow$ is f-closed and $\uparrow r$ is f-closed if $r$ is f-closed.
Lemma *(Lifting).*

\[ t^{\uparrow [\uparrow r]} = t[r]^{\uparrow} \]
\[ A^{\uparrow [\uparrow r]} = A[r]^{\uparrow} \]
Lemma (Lifting).

\[ t^{\uparrow[\uparrow r]} = t[r] \uparrow \]
\[ A^{\uparrow[\uparrow r]} = A[r] \uparrow \]

Proof. By the composition lemma, we have:

\[ t^{\uparrow[\uparrow r]} = t[n \mapsto (n + 1)[\uparrow r]] = t[n \mapsto r(n) \uparrow] = t[r] \uparrow \]

Analogous for formulas.
Given a derivation $\pi$ of $\Gamma \vdash A$, we define a derivation $\pi[r]$ of $\Gamma[r] \vdash A[r]$ for every f-closed $r$ by induction on the size of $\pi$. 
Given a derivation $\pi$ of $\Gamma \vdash A$, we define a derivation $\pi[r]$ of $\Gamma[r] \vdash A[r]$ for every f-closed $r$ by induction on the size of $\pi$.

$$\pi_0 \quad \vdots$$

$$\pi = \frac{\Gamma \vdash B^\uparrow[0/x]}{\Gamma \vdash \forall x B} (\forall I)$$
Substitution in proofs, case of a generalization

Given a derivation $\pi$ of $\Gamma \vdash A$, we define a derivation $\pi[r]$ of $\Gamma[r] \vdash A[r]$ for every f-closed $r$ by induction on the size of $\pi$.

\[
\begin{align*}
\pi_0 & \\
\therefore & \\
\pi & = \begin{array}{c}
\Gamma \vdash \forall x B \\
\Gamma \vdash B \uparrow[0/x] \\
\Gamma \vdash B \uparrow[0/x][r]
\end{array} \\
& \text{(\forall I)}
\end{align*}
\]

\[
\pi[r] = \begin{array}{c}
\Gamma \uparrow[r] \vdash B \uparrow[0/x][r]
\end{array}
\]
Given a derivation $\pi$ of $\Gamma \vdash A$, we define a derivation $\pi[r]$ of $\Gamma[r] \vdash A[r]$ for every f-closed $r$ by induction on the size of $\pi$.

\[
\pi_0 \\
\pi = \frac{\Gamma \vdash B \uparrow[0/x]}{\Gamma \vdash \forall x B} \quad (\forall I)
\]

\[
\pi_0 [\uparrow r] \\
\pi[r] = \frac{\Gamma \uparrow[\uparrow r] \vdash B \uparrow[0/x][\uparrow r]}{}
\]
Given a derivation $\pi$ of $\Gamma \vdash A$, we define a derivation $\pi[r]$ of $\Gamma[r] \vdash A[r]$ for every f-closed $r$ by induction on the size of $\pi$.

$$\pi = \frac{\vdots \pi_0 \quad \Gamma \vdash B\uparrow[0/x]}{\Gamma \vdash \forall xB} \quad (\forall l)$$

$$\vdots \pi_0 [\uparrow r]$$

$$\pi[r] = \Gamma[\uparrow r] \vdash B[\uparrow r][0/x]$$
SUBSTITUTION IN PROOFS, CASE OF A GENERALIZATION

Given a derivation $\pi$ of $\Gamma \vdash A$, we define a derivation $\pi[r]$ of $\Gamma[r] \vdash A[r]$ for every f-closed $r$ by induction on the size of $\pi$.

\[
\begin{align*}
\pi & = \frac{\pi_0 \quad \Gamma \vdash B[0/x]}{\Gamma \vdash \forall x B} \quad (\forall I) \\
\pi[r] & = \frac{\pi_0[\uparrow r] \quad \Gamma \uparrow[\uparrow r] \vdash B[\uparrow r][0/x]}{\Gamma \uparrow[\uparrow r] \vdash B[\uparrow r][0/x]} 
\end{align*}
\]
Given a derivation $\pi$ of $\Gamma \vdash A$, we define a derivation $\pi[r]$ of $\Gamma[r] \vdash A[r]$ for every f-closed $r$ by induction on the size of $\pi$.

\[
\pi[r] = \begin{cases} 
\vdots \pi_0[r] \\
\Gamma[r] \vdash B[r][0/x] \\
\Gamma \vdash \forall x B \\
\end{cases} \quad (\forall l)
\]

\[
\pi_0[r] = \begin{cases} 
\vdots \pi_0[r] \\
\Gamma \vdash B[0/x] \\
\end{cases}
\]
Given a derivation $\pi$ of $\Gamma \vdash A$, we define a derivation $\pi[r]$ of $\Gamma[r] \vdash A[r]$ for every f-closed $r$ by induction on the size of $\pi$.

$$
\pi = \frac{\vdash \pi_0}{\Gamma \vdash \forall x B} \quad (\forall I)
$$

$$
\pi[r] = \frac{\vdash \pi_0[r]}{\Gamma[r] \vdash \forall x (B[r])} \quad (\forall I)
$$
Given a derivation $\pi$ of $\Gamma \vdash A$, we define a derivation $\pi[r]$ of $\Gamma[r] \vdash A[r]$ for every f-closed $r$ by induction on the size of $\pi$.

$$\pi = \frac{\vdash \pi_0}{\Gamma \vdash \forall x B} \quad (∀I)$$

$$\pi[r] = \frac{\vdash \pi_0[r]}{\Gamma[r] \vdash \forall x (B[r])} \quad (∀I)$$
Given a derivation $\pi$ of $\Gamma \vdash A$, we define a derivation $\pi[r]$ of $\Gamma[r] \vdash A[r]$ for every $f$-closed $r$ by induction on the size of $\pi$.

\[
\begin{align*}
\pi &= \dfrac{\vdots \pi_0}{\Gamma \vdash \forall x B} \quad (\forall I) \\
\pi[r] &= \dfrac{\vdots \pi_0[r]}{\Gamma[r] \vdash (\forall x B)[r]} \quad (\forall I)
\end{align*}
\]
Suppose $t$ is an f-closed term, in which case $t[r]$ is f-closed.

$$
\pi = \frac{\pi_0}{\Gamma \vdash B[t/x]} \quad (\exists!) \\
\Gamma \vdash \exists x B
$$
Suppose $t$ is an $f$-closed term, in which case $t[r]$ is $f$-closed.

\[
\pi[r] = \frac{\Gamma \vdash B[t/x]}{\Gamma \vdash \exists x B} \quad (\exists !)
\]

\[
\pi_0[r] = \frac{\pi_0}{\Gamma[r] \vdash B[t/x][r]}
\]
Suppose $t$ is an $f$-closed term, in which case $t[r]$ is $f$-closed.

$$
\begin{align*}
\pi &= \frac{\pi_0}{\Gamma \vdash B[t/x]} \\
\Gamma &\vdash \exists x B
\end{align*}
$$

$$
\begin{align*}
\pi[r] &= \frac{\pi_0[r]}{\Gamma[r] \vdash B[t/x][r]}
\end{align*}
$$
Suppose $t$ is an $f$-closed term, in which case $t[r]$ is $f$-closed.

$$
\pi = \frac{\pi_0}{\Gamma \vdash B[t/x]} \quad (\exists !)
$$

$$
\pi[r] = \frac{\pi_0[r]}{\Gamma[r] \vdash B[r][t[r]/x]}
$$
Suppose $t$ is an $f$-closed term, in which case $t[r]$ is $f$-closed.

$$\pi = \frac{\pi_0}{\Gamma \vdash B[t/x]} \quad (\exists I)$$

$$\pi[r] = \frac{\pi_0[r]}{\Gamma[r] \vdash B[r][t[r]/x]} \quad (\exists I)$$
Suppose $t$ is an $f$-closed term, in which case $t[r]$ is $f$-closed.

$$\pi = \frac{\Gamma \vdash B[t/x]}{\Gamma \vdash \exists x B} \quad (\exists I)$$

$$\pi[r] = \frac{\Gamma[r] \vdash B[r][t[r]/x]}{\Gamma[r] \vdash \exists x (B[r])} \quad (\exists I)$$
Suppose $t$ is an f-closed term, in which case $t[r]$ is f-closed.

$$
\pi = \frac{\Gamma \vdash B[t/x]}{\Gamma \vdash \exists x B} \quad (\exists I)
$$

$$
\pi[r] = \frac{\Gamma[r] \vdash B[r][t[r]/x]}{\Gamma[r] \vdash (\exists x B)[r]} \quad (\exists I)
$$
BACK TO CUT ELIMINATION
Given an \( \mathbb{N} \)-term \( v \), we define \( \downarrow v : \mathbb{N} \to \mathbb{N} \)-terms by:

\[
\downarrow v(n) = \begin{cases} 
v & \text{if } n = 0 \\
 n - 1 & \text{otherwise}
\end{cases}
\]
Given an $\mathbb{N}$-term $\nu$, we define $\nu\downarrow: \mathbb{N} \rightarrow \mathbb{N}$-terms by:

$$\nu\downarrow(n) = \begin{cases} 
\nu & \text{if } n = 0 \\
n - 1 & \text{otherwise}
\end{cases}$$

By construction $\nu\downarrow$ is f-closed if $\nu$ is f-closed.
Lemma (Deletion).

\[ t \uparrow[v\downarrow] = t \]
\[ A \uparrow[v\downarrow] = A \]
Lemma (Deletion).

\[ t^{n \downarrow} = t \]
\[ A^{n \downarrow} = A \]

Proof. By the composition lemma, we have:

\[ t^{n \downarrow} = t[n \mapsto (n + 1)^{n \downarrow}] = t[n \mapsto n] = t \]

Analogous for formulas.
Lemma (Substitution). If $\pi$ is a derivation of $\Gamma \vdash A^{[0/x]}$ and $t$ is an $f$-closed term, then $\pi[t\triangleright]$ is a derivation of $\Gamma \vdash A[t/x]$. 

Proof. We have that $\pi[t\triangleright]$ is a derivation of $\Gamma[t\triangleright] \vdash A^{[0/x]}[t\triangleright]$. 

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Lemma (Substitution). If $\pi$ is a derivation of $\Gamma \vdash A^{[0/x]}$ and $t$ is an $f$-closed term, then $\pi[t\Downarrow]$ is a derivation of $\Gamma \vdash A[t/x]$.

Proof. We have that $\pi[t\Downarrow]$ is a derivation of:

$$\Gamma^{[t\Downarrow]} \vdash A^{[0/x]}[t\Downarrow]$$
Lemma (Substitution). If $\pi$ is a derivation of $\Gamma \vdash A^{[0/x]}$ and $t$ is an f-closed term, then $\pi[t\downarrow]$ is a derivation of $\Gamma \vdash A[t/x]$.

Proof. We have that $\pi[t\downarrow]$ is a derivation of:

$$\Gamma^{[t\downarrow]} \vdash A^{[0/x]}[t\downarrow]$$
Lemma (Substitution). If $\pi$ is a derivation of $\Gamma \vdash A[0/x]$ and $t$ is an f-closed term, then $\pi[t\downarrow]$ is a derivation of $\Gamma \vdash A[t/x]$.

Proof. We have that $\pi[t\downarrow]$ is a derivation of:

\[ \Gamma[t\downarrow] \vdash A[t\downarrow][t/x] \]
Lemma (Substitution). If $\pi$ is a derivation of $\Gamma \vdash A^{[0/x]}$ and $t$ is an f-closed term, then $\pi[t\downarrow]$ is a derivation of $\Gamma \vdash A[t/x]$.

Proof. We have that $\pi[t\downarrow]$ is a derivation of:

$$\Gamma^{[t\downarrow]} \vdash A^{[t\downarrow][t/x]}$$
Lemma (Substitution). If $\pi$ is a derivation of $\Gamma \vdash A[0/x]$ and $t$ is an f-closed term, then $\pi[t\downarrow]$ is a derivation of $\Gamma \vdash A[t/x]$.

Proof. We have that $\pi[t\downarrow]$ is a derivation of:

$$\Gamma \vdash A[t/x]$$
Case of a generalization versus an instantiation revisited

}\[ \vdash \pi \]

\[ \Gamma \vdash A^\uparrow[0/x] \]

\[ \Gamma \vdash \forall x A \]  (\forall I)

\[ \vdash \rho \]

\[ \Delta, A^\uparrow[t/x] \vdash B \]

\[ \Delta, \forall x A \vdash B \]  (\forall L)

\[ \Gamma, \Delta \vdash B \]  (cut)
CASE OF A GENERALIZATION VERSUS AN INSTANTIATION REVISITED

\[
\begin{align*}
\therefore \pi & : \Gamma \uparrow \vdash A \uparrow[0/x] \\
\Gamma & \vdash \forall x A & (\forall I) \\
\therefore \rho & : \Delta, A[t/x] \vdash B \\
\Delta, \forall x A & \vdash B & (\forall L) \\
\therefore & \Gamma, \Delta \vdash B & \text{(cut)}
\end{align*}
\]

Cut elimination step:

\[
\begin{align*}
\therefore \pi[t\Downarrow] & : \Gamma \vdash A[t/x] \\
\therefore \rho & : \Delta, A[t/x] \vdash B \\
\therefore & \Gamma, \Delta \vdash B & \text{(cut)}
\end{align*}
\]
FIRST EXAMPLE REVISITED

\[
\begin{align*}
\frac{P_2, P_0 \vdash P_0}{\forall x P x \vdash P_0} & \quad (\forall L) \\
\frac{P_2, \forall x P x \vdash P_0}{P_1, \forall x P x \vdash \forall x P x} & \quad (\forall I) \\
\frac{P_1, \forall x P x \vdash P_0 \lor \forall x P x}{P_0, \forall x P x \vdash \forall y (P y \lor \forall x P x)} & \quad (\forall I) \\
\frac{P_0, \forall x P x \vdash \forall y (P y \lor \forall x P x)}{P_0 \lor \forall x P x \vdash P_0 \lor \forall x P x} & \quad (\forall L) \\
\frac{P_0 \lor \forall x P x \vdash P_0 \lor \forall x P x}{\forall y (P y \lor \forall x P x) \vdash P_0 \lor \forall x P x} & \quad (\forall L) \\
\frac{\forall y (P y \lor \forall x P x) \vdash P_0 \lor \forall x P x}{P_0, \forall x P x \vdash P_0 \lor \forall x P x} & \quad (\text{cut})
\end{align*}
\]
**First Example Revisited**

\[
\begin{align*}
\frac{P_1, P_0}{P_0} & \quad \text{(ax)} \\
\frac{P_1, \forall x P x}{P_0} & \quad \text{(\forall L)} \\
\frac{P_1, \forall x P x}{P_0, \forall x P x} & \quad \text{(\forall I)} \\
\frac{P_0, \forall x P x}{P_0, \forall x P x \vdash P_0 \lor \forall x P x} & \quad \text{(\lor IR)} \\
\frac{P_0 \lor \forall x P x}{P_0 \lor \forall x P x} & \quad \text{(ax)} \\
\frac{P_0 \lor \forall x P x}{P_0} & \quad \text{(cut)}
\end{align*}
\]
CASE OF A COMMUTATIVE CUT REVISITED

\[ \Gamma \vdash A \]

\[ \Delta \uparrow, A \uparrow \vdash B \uparrow[0/x] \quad (\forall I) \]

\[ \Delta, A \vdash \forall x B \quad (cut) \]

\[ \Gamma, \Delta \vdash \forall x B \]
**Case of a commutative cut revisited**

Cut elimination step:

\[
\begin{align*}
\vdash \pi & \quad \Delta \uparrow, A \uparrow \vdash B[0/x] \\
\Gamma \vdash A & \quad \Delta, A \vdash \forall x B \\
\hline
\Gamma, \Delta \vdash \forall x B
\end{align*}
\]

\((\forall l)\)

\[
\begin{align*}
\vdash \pi[\uparrow] & \quad \vdash \rho \\
\Gamma \uparrow \vdash A \uparrow & \quad \Delta \uparrow, A \uparrow \vdash B[0/x] \\
\hline
\Gamma \uparrow, \Delta \uparrow \vdash B[0/x]
\end{align*}
\]

\((cut)\)

\[
\begin{align*}
\hline
\Gamma \uparrow, \Delta \uparrow \vdash B[0/x] \\
\hline
\Gamma, \Delta \vdash \forall x B
\end{align*}
\]

\((\forall l)\)
SECOND EXAMPLE REVISITED

\[
\frac{P_1, P_0 \vdash P_0}{P_1, \forall x P x \vdash P_0} \quad (\forall L)
\]

\[
\frac{P_0 \vdash P_0}{\forall x P x \vdash P_0} \quad (\forall L)
\]

\[
\frac{P_0, \forall x P x \vdash \forall x P x}{P_0, \forall x P x \vdash \forall x P x} \quad (\text{cut})
\]
SECOND EXAMPLE REVISITED

\[
\frac{P_2, P_0 \vdash P_0}{P_2, \forall x P x \vdash P_0} \text{ (ax)}
\]

\[
\frac{P_2, \forall x P x \vdash P_0 \text{ (\forall L)}}{P_1, \forall x P x \vdash \forall x P x} \text{ (\forall I)}
\]

\[
\frac{P_0 \vdash P_0 \text{ (ax)}}{\forall x P x \vdash P_0 \text{ (\forall L)}}
\]

\[
\frac{P_1, \forall x P x \vdash P_0 \text{ (\forall I)}}{P_0, \forall x P x \vdash \forall x P x \text{ (\forall I)}}
\]
EXPRESSIVENESS ANALYSIS
Lemma. If the free variables of $\Gamma$ and $A$ are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{LJ} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{ALN} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$. 
Lemma. If the free variables of $\Gamma$ and $A$ are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{\text{LJ}} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{\text{ALN}} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ.
Lemma. If the free variables of $\Gamma$ and $A$ are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{LJ} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{ALN} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ. Suppose that $y$ is not free in $\Gamma$ nor in $\forall x B$.

\[
\begin{array}{c}
\vdots \\
\Gamma \vdash B[y/x] \\
\hline
\Gamma \vdash \forall x B \\
\end{array} \quad (\forall I)
\]
**Lemma.** If the free variables of $\Gamma$ and $A$ are among $x_1, \ldots, x_n$ and $\Gamma \vdash \text{LJ} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash \text{ALN} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

**Proof.** By induction on the size of the proof of $\Gamma \vdash A$ in LJ. Suppose that $y$ is not free in $\Gamma$ nor in $\forall x B$.

\[
\begin{align*}
\vdots \\
\Gamma \vdash B[y/x] \\
\Gamma \vdash \forall x B \quad (\forall I) \\
\vdots \\
\Gamma[t_1 \uparrow/x_1, \ldots, t_n \uparrow/x_n, 0/y] \vdash B[y/x][t_1 \uparrow/x_1, \ldots, t_n \uparrow/x_n, 0/y]
\end{align*}
\]
Lemma. If the free variables of $\Gamma$ and $A$ are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{\text{LJ}} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{\text{ALN}} A[t_1/x_1, \ldots, t_n/x_n]$ for any $f$-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ. Suppose that $y$ is not free in $\Gamma$ nor in $\forall x B$.

\[
\vdots \\
\Gamma \vdash B[y/x] \\
\Gamma \vdash \forall x B \quad (\forall I) \\
\vdots \\
\Gamma[t_1 \uparrow/x_1, \ldots, t_n \uparrow/x_n, 0/y] \vdash B[y/x][t_1 \uparrow/x_1, \ldots, t_n \uparrow/x_n, 0/y]
\]
Lemma. If the free variables of $\Gamma$ and $A$ are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{\text{LJ}} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{\text{ALN}} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ. Suppose that $y$ is not free in $\Gamma$ nor in $\forall xB$.

\[
\begin{align*}
\vdots \\
\Gamma \vdash B[y/x] \\
\Gamma \vdash \forall xB \quad (\forall I) \\
\vdots \\
\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash B[y/x][t_1/x_1, \ldots, t_n/x_n, 0/y]
\end{align*}
\]
Lemma. If the free variables of $\Gamma$ and $A$ are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{LJ} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{ALN} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ. Suppose that $y$ is not free in $\Gamma$ nor in $\forall x B$.

\[
\vdots
\]
\[
\Gamma \vdash B[y/x] \\
\frac{}{\Gamma \vdash \forall x B} \quad (\forall I)
\]
\[
\vdots
\]
\[
\Gamma[t_1 \uparrow/x_1, \ldots, t_n \uparrow/x_n] \vdash B[y/x][t_1 \uparrow/x_1, \ldots, t_n \uparrow/x_n, 0/y]
\]
Lemma. If the free variables of $\Gamma$ and $A$ are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{LJ} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{ALN} A[t_1/x_1, \ldots, t_n/x_n]$ for any $f$-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in $LJ$.
Suppose that $y$ is not free in $\Gamma$ nor in $\forall x B$.

\[
\vdots \\
\Gamma \vdash B[y/x] \\
\frac{}{\Gamma \vdash \forall x B} \quad \text{(}\forall I) \\
\vdots \\
\Gamma[t_1 \uparrow/x_1, \ldots, t_n \uparrow/x_n] \vdash B[y/x][t_1 \uparrow/x_1, \ldots, t_n \uparrow/x_n][0/y]
\]
Lemma. If the free variables of $\Gamma$ and $A$ are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{\text{LJ}} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{\text{ALN}} A[t_1/x_1, \ldots, t_n/x_n]$ for any $f$-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ. Suppose that $y$ is not free in $\Gamma$ nor in $\forall x B$.

\[
\vdots
\]
\[
\Gamma \vdash B[y/x]
\]
\[
\frac{\Gamma \vdash \forall x B}{(\forall I)}
\]
\[
\vdots
\]
\[
\Gamma[t_1\uparrow/x_1, \ldots, t_n\uparrow/x_n] \vdash B[y/x][t_1\uparrow/x_1, \ldots, t_n\uparrow/x_n][0/y]
\]
Lemma. If the free variables of $\Gamma$ and $A$ are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{\text{LJ}} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{\text{ALN}} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ.

Suppose that $y$ is not free in $\Gamma$ nor in $\forall x B$.

\[
\begin{align*}
\vdots \\
\Gamma \vdash B[y/x] \\
\Gamma \vdash \forall x B & \quad (\forall I) \\
\vdots \\
\Gamma[t_1 \uparrow/x_1, \ldots, t_n \uparrow/x_n] \vdash B[t_1 \uparrow/x_1, \ldots, t_n \uparrow/x_n][y/x][0/y]
\end{align*}
\]
Lemma. If the free variables of $\Gamma$ and $A$ are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{\text{LJ}} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{\text{ALN}} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ. Suppose that $y$ is not free in $\Gamma$ nor in $\forall xB$.

\[
\begin{array}{c}
\vdots \\
\Gamma \vdash B[y/x] \\
\Gamma \vdash \forall xB \quad (\forall I)
\end{array}
\]

\[
\begin{array}{c}
\vdots \\
\Gamma[t_1 \uparrow/x_1, \ldots, t_n \uparrow/x_n] \vdash B[t_1 \uparrow/x_1, \ldots, t_n \uparrow/x_n][y/x][0/y]
\end{array}
\]
**Lemma.** If the free variables of \( \Gamma \) and \( A \) are among \( x_1, \ldots, x_n \) and \( \Gamma \vdash_{LJ} A \), then \( \Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{ALN} A[t_1/x_1, \ldots, t_n/x_n] \) for any \( f \)-closed terms \( t_1, \ldots, t_n \).

**Proof.** By induction on the size of the proof of \( \Gamma \vdash A \) in \( LJ \). Suppose that \( y \) is not free in \( \Gamma \) nor in \( \forall x B \).

\[
\begin{align*}
\cdot & \\
\vdots & \\
\Gamma \vdash B[y/x] & (\forall I) \\
\Gamma \vdash \forall x B & \\
\cdot & \\
\Gamma[t_1 \uparrow/x_1, \ldots, t_n \uparrow/x_n] & \vdash B[t_1 \uparrow/x_1, \ldots, t_n \uparrow/x_n][0/x]
\end{align*}
\]
Lemma. If the free variables of \( \Gamma \) and \( A \) are among \( x_1, \ldots, x_n \) and \( \Gamma \vdash_{LJ} A \), then \( \Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{ALN} A[t_1/x_1, \ldots, t_n/x_n] \) for any f-closed terms \( t_1, \ldots, t_n \).

Proof. By induction on the size of the proof of \( \Gamma \vdash A \) in LJ. Suppose that \( y \) is not free in \( \Gamma \) nor in \( \forall x B \).

\[
\begin{align*}
\vdots \\
\Gamma \vdash B[y/x] & \quad \text{(\( \forall I \))} \\
\Gamma \vdash \forall x B \\
\vdots \\
\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash B[t_1/x_1, \ldots, t_n/x_n][0/x]
\end{align*}
\]
Lemma. If the free variables of $\Gamma$ and $A$ are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{\text{LJ}} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{\text{ALN}} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ.

Suppose that $y$ is not free in $\Gamma$ nor in $\forall x B$.

\[
\begin{align*}
\vdots \\
\vdots \\
\Gamma \vdash B[y/x] \\
\Gamma \vdash \forall x B & \quad (\forall I) \\
\vdots \\
\Gamma[t_1/x_1, \ldots, t_n/x_n] \uparrow \vdash B[t_1/x_1, \ldots, t_n/x_n] \uparrow[0/x]
\end{align*}
\]
Lemma. If the free variables of $\Gamma$ and $A$ are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{LJ} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{ALN} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ. Suppose that $y$ is not free in $\Gamma$ nor in $\forall x B$.

\[
\vdots \\
\Gamma \vdash B[y/x] \\
\frac{\Gamma \vdash B[y/x]}{\Gamma \vdash \forall x B} \quad (\forall I)
\]

\[
\vdots \\
\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash B[t_1/x_1, \ldots, t_n/x_n][0/x] \\
\frac{\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash B[t_1/x_1, \ldots, t_n/x_n][0/x]}{\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash \forall x(B[t_1/x_1, \ldots, t_n/x_n])} \quad (\forall I)
\]
Lemma. If the free variables of $\Gamma$ and $A$ are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{LJ} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{ALN} A[t_1/x_1, \ldots, t_n/x_n]$ for any f-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ. Suppose that $y$ is not free in $\Gamma$ nor in $\forall x B$.

\[
\vdots
\]

\[
\Gamma \vdash B[y/x] \quad (\forall I)
\]

\[
\vdots
\]

\[
\Gamma \vdash \forall x B
\]

\[
\vdots
\]

\[
\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash B[t_1/x_1, \ldots, t_n/x_n] \vdash[0/x] \quad (\forall I)
\]

\[
\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash \forall x(B[t_1/x_1, \ldots, t_n/x_n])
\]
Lemma. If the free variables of $\Gamma$ and $A$ are among $x_1, \ldots, x_n$ and $\Gamma \vdash_{\text{LJ}} A$, then $\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash_{\text{ALN}} A[t_1/x_1, \ldots, t_n/x_n]$ for any $f$-closed terms $t_1, \ldots, t_n$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in LJ. Suppose that $y$ is not free in $\Gamma$ nor in $\forall x B$.

\begin{align*}
\vdots \\
\Gamma \vdash B[y/x] & \quad (\forall I) \\
\Gamma \vdash \forall x B \\
\vdots \\
\Gamma[t_1/x_1, \ldots, t_n/x_n] \uparrow \vdash B[t_1/x_1, \ldots, t_n/x_n] \uparrow[0/x] & \quad (\forall I) \\
\Gamma[t_1/x_1, \ldots, t_n/x_n] \vdash (\forall x B)[t_1/x_1, \ldots, t_n/x_n]
\end{align*}
Let $t$ be a term with free variables among $x_1, \ldots, x_n, y_1, \ldots, y_m$.

\[
\therefore \\
\frac{\Gamma \vdash B[t/x]}{\Gamma \vdash \exists x B} \quad (\exists l)
\]
Let $t$ be a term with free variables among $x_1, \ldots, x_n, y_1, \ldots, y_m$.

\[
\vdots
\begin{align*}
\Gamma &\vdash B[t/x] \\
\therefore \Gamma &\vdash \exists x B
\end{align*}
\]

\[
\vdots
\begin{align*}
\Gamma[t_i/x_i, 0/y_k] &\vdash B[t/x][t_i/x_i, 0/y_k]
\end{align*}
\]
Let $t$ be a term with free variables among $x_1, \ldots, x_n, y_1, \ldots, y_m$. 

\[
\Gamma \vdash B[t/x] \\
\Gamma \vdash \exists x B \\
\vdots \\
\Gamma[t_i/x_i, 0/y_k] \vdash B[t/x][t_i/x_i, 0/y_k]
\]
Let $t$ be a term with free variables among $x_1, \ldots, x_n, y_1, \ldots, y_m$.

\[
\Gamma \vdash B[t/x] \\
\Gamma \vdash \exists x B
\]

\[
\Gamma[t_i/x_i, 0/y_k] \vdash B[t_i/x_i, 0/y_k][t[t_i/x_i, 0/y_k]/x]
\]
Let $t$ be a term with free variables among $x_1, \ldots, x_n, y_1, \ldots, y_m$.

\[
\begin{align*}
\vdots \\
\Gamma \vdash B[t/x] \\
\Gamma \vdash \exists x B \\
\vdots \\
\Gamma[t_i/x_i, 0/y_k] \vdash B[t_i/x_i, 0/y_k][t[t_i/x_i, 0/y_k]/x]
\end{align*}
\]
Let $t$ be a term with free variables among $x_1, \ldots, x_n, y_1, \ldots, y_m$.

\[
\vdots
\]

\[\Gamma \vdash B[t/x] \quad (\exists I)\]

\[
\vdots
\]

\[
\Gamma[t_i/x_i] \vdash B[t_i/x_i][t[t_i/x_i, 0/y_k]/x]
\]

The other cases are trivial.
Let $t$ be a term with free variables among $x_1, \ldots, x_n, y_1, \ldots, y_m$.

\[
\vdots \\
\Gamma \vdash B[t/x] \\
\therefore \Gamma \vdash \exists x B \\
\vdots \\
\Gamma[t_i/x_i] \vdash B[t_i/x_i][t[t_i/x_i, 0/y_k]/x] \\
\therefore \Gamma[t_i/x_i] \vdash \exists x(B[t_i/x_i])
\]
Let $t$ be a term with free variables among $x_1, \ldots, x_n, y_1, \ldots, y_m$.

\[
\vdots \\
\Gamma \vdash B[t/x] \quad (\exists l)
\]

\[
\Gamma \vdash \exists x B
\]

\[
\vdots \\
\Gamma[t_i/x_i] \vdash B[t_i/x_i][t[t_i/x_i, 0/y_k]/x] \\
\Gamma[t_i/x_i] \vdash \exists x (B[t_i/x_i]) \quad (\exists l)
\]

The other cases are trivial.
Let $t$ be a term with free variables among $x_1, \ldots, x_n, y_1, \ldots, y_m$. 

$$
\frac{
\vdash B[t/x]
}{
\vdash \exists x B
}(\exists I)
$$

$$
\frac{
\vdash [t_i/x_i] B[t_i/x_i][t[t_i/x_i, 0/y_k]/x]
}{
\vdash [t_i/x_i] (\forall x B)[t_i/x_i]
}(\exists I)
$$
Let $t$ be a term with free variables among $x_1, \ldots, x_n, y_1, \ldots, y_m$.

\[
\Gamma \vdash B[t/x] \\
\Gamma \vdash \exists x B
\]

\[
\Gamma[t_i/x_i] \vdash B[t_i/x_i][t[t_i/x_i, 0/y_k]/x] \\
\Gamma[t_i/x_i] \vdash (\forall x B)[t_i/x_i]
\]

The other cases are trivial. \qed
Lemma. If the e-variables of $\Gamma$ and $A$ are among $e_1, \ldots, e_n$ and $\Gamma \vdash_{\text{ALN}} A$, then $\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash_{\text{LJ}} A[x_1/e_1, \ldots, x_n/e_n]$ for any variables $x_1, \ldots, x_n$ not bound in $\Gamma$ nor in $A$. 
Lemma. If the e-variables of $\Gamma$ and $A$ are among $e_1, \ldots, e_n$ and $\Gamma \vdash_{\text{ALN}} A$, then $\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash_{\text{LJ}} A[x_1/e_1, \ldots, x_n/e_n]$ for any variables $x_1, \ldots, x_n$ not bound in $\Gamma$ nor in $A$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in ALN.
Lemma. If the e-variables of $\Gamma$ and $A$ are among $e_1, \ldots, e_n$ and $\Gamma \vdash_{\text{ALN}} A$, then $\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash_{\text{LJ}} A[x_1/e_1, \ldots, x_n/e_n]$ for any variables $x_1, \ldots, x_n$ not bound in $\Gamma$ nor in $A$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in ALN.

\[
\begin{align*}
\vdash & \vdash B[0/x] \\
\vdash & \vdash B[0/x] \quad (\forall I) \\
\vdash & \vdash B \quad (\forall I) \\
\vdash & \vdash \forall x B \\
\end{align*}
\]
Lemma. If the e-variables of $\Gamma$ and $A$ are among $e_1, \ldots, e_n$ and $\Gamma \vdash_{ALN} A$, then $\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash_{LJ} A[x_1/e_1, \ldots, x_n/e_n]$ for any variables $x_1, \ldots, x_n$ not bound in $\Gamma$ nor in $A$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in ALN.

$$
\begin{array}{c}
\vdash B[0/x] \\
\Gamma \vdash \forall x B
\end{array}
\quad (\forall I)
$$

Let $y$ be a fresh variable.

$$
\begin{array}{c}
\vdash B[0/x][x_1/e_1, \ldots, x_n/e_n, y/0] \\
\vdash B[0/x][x_1/e_1, \ldots, x_n/e_n, y/0]
\end{array}
$$
Lemma. If the e-variables of $\Gamma$ and $A$ are among $e_1, \ldots, e_n$ and $\Gamma \vdash_{\text{ALN}} A$, then $\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash_{\text{LJ}} A[x_1/e_1, \ldots, x_n/e_n]$ for any variables $x_1, \ldots, x_n$ not bound in $\Gamma$ nor in $A$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in ALN.

\[
\begin{align*}
\vdash & \vdash_{\text{LJ}} B[0/x] \\
\Gamma & \vdash \forall x B \\
\end{align*}
\]

Let $y$ be a fresh variable.

\[
\begin{align*}
\vdash & \vdash_{\text{LJ}} B[0/x][x_1/e_1, \ldots, x_n/e_n, y/0] \\
\Gamma & \vdash_{\text{ALN}} A \\
\end{align*}
\]
Lemma. If the e-variables of \( \Gamma \) and \( A \) are among \( e_1, \ldots, e_n \) and \( \Gamma \vdash_{ALN} A \), then \( \Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash_{LJ} A[x_1/e_1, \ldots, x_n/e_n] \) for any variables \( x_1, \ldots, x_n \) not bound in \( \Gamma \) nor in \( A \).

Proof. By induction on the size of the proof of \( \Gamma \vdash A \) in ALN.

\[
\frac{\vdots}{\Gamma \vdash \forall x B} \quad (\forall I)
\]

Let \( y \) be a fresh variable.

\[
\frac{\vdots}{\Gamma \vdash[x_1/e_1] \ldots, x_n/e_n] \vdash B[0/x][x_1/e_1, \ldots, x_n/e_n, y/0]}
\]
Lemma. If the e-variables of \( \Gamma \) and \( A \) are among \( e_1, \ldots, e_n \) and \( \Gamma \vdash_{ALN} A \), then \( \Gamma[\frac{x_1}{e_1}, \ldots, \frac{x_n}{e_n}] \vdash_{LJ} A[\frac{x_1}{e_1}, \ldots, \frac{x_n}{e_n}] \) for any variables \( x_1, \ldots, x_n \) not bound in \( \Gamma \) nor in \( A \).

Proof. By induction on the size of the proof of \( \Gamma \vdash A \) in ALN.

\[
\frac{\vdots}{\Gamma \vdash B^\uparrow[0/x]} \quad (\forall I)
\]

Let \( y \) be a fresh variable.

\[
\frac{\vdots}{\Gamma^\uparrow[\frac{x_1}{e_1\uparrow}, \ldots, \frac{x_n}{e_n\uparrow}] \vdash B^\uparrow[0/x][\frac{x_1}{e_1\uparrow}, \ldots, \frac{x_n}{e_n\uparrow}, y/0]} \]
Lemma. If the e-variables of $\Gamma$ and $A$ are among $e_1, \ldots, e_n$ and $\Gamma \vdash_{ALN} A$, then $\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash_{LJ} A[x_1/e_1, \ldots, x_n/e_n]$ for any variables $x_1, \ldots, x_n$ not bound in $\Gamma$ nor in $A$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in ALN.

$$
\begin{align*}
\vdots \\
\Gamma \uparrow \vdash B \uparrow[0/x] \\
\Gamma \vdash \forall x B
\end{align*}
$$

(∀I)

Let $y$ be a fresh variable.

$$
\begin{align*}
\vdots \\
\Gamma \uparrow[x_1/e_1 \uparrow, \ldots, x_n/e_n \uparrow] \vdash B \uparrow[0/x][x_1/e_1 \uparrow, \ldots, x_n/e_n \uparrow][y/0]
\end{align*}
$$
Lemma. If the e-variables of $\Gamma$ and $A$ are among $e_1, \ldots, e_n$ and $\Gamma \vdash_{\text{ALN}} A$, then $\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash_{\text{LJ}} A[x_1/e_1, \ldots, x_n/e_n]$ for any variables $x_1, \ldots, x_n$ not bound in $\Gamma$ nor in $A$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in ALN.

\[
\Gamma \vdash A \quad \frac{\Gamma \vdash B[0/x]}{\Gamma \vdash \forall xB} \quad (\forall I)
\]

Let $y$ be a fresh variable.

\[
\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash B[0/x][x_1/e_1, \ldots, x_n/e_n][y/0]
\]
Lemma. If the e-variables of $\Gamma$ and $A$ are among $e_1, \ldots, e_n$ and $\Gamma \vdash_{\text{ALN}} A$, then $\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash_{\text{LJ}} A[x_1/e_1, \ldots, x_n/e_n]$ for any variables $x_1, \ldots, x_n$ not bound in $\Gamma$ nor in $A$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in ALN.

\[
\frac{
\vdash \Gamma \uparrow - B \uparrow[0/x]
}{
\Gamma \vdash \forall x B
}
\] (∀I)

Let $y$ be a fresh variable.

\[
\vdash \Gamma \uparrow[x_1/e_1 \uparrow, \ldots, x_n/e_n \uparrow] \vdash B \uparrow[x_1/e_1 \uparrow, \ldots, x_n/e_n \uparrow][0/x][y/0] \]
Lemma. If the e-variables of Γ and A are among $e_1, \ldots, e_n$ and $\Gamma \vdash_{\text{ALN}} A$, then $\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash_{\text{LJ}} A[x_1/e_1, \ldots, x_n/e_n]$ for any variables $x_1, \ldots, x_n$ not bound in $\Gamma$ nor in $A$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in ALN.

\[
\begin{align*}
\vdots \\
\Gamma^\uparrow \vdash B^\uparrow[0/x] \\
\hline
\Gamma \vdash \forall x B
\end{align*}
\]

(∀I)

Let $y$ be a fresh variable.

\[
\begin{align*}
\vdots \\
\Gamma^\uparrow[x_1/e_1^\uparrow, \ldots, x_n/e_n^\uparrow] \vdash B^\uparrow[x_1/e_1^\uparrow, \ldots, x_n/e_n^\uparrow][0/x][y/0]
\end{align*}
\]
Lemma. If the e-variables of \( \Gamma \) and \( A \) are among \( e_1, \ldots, e_n \) and \( \Gamma \vdash_{ALN} A \), then \( \Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash_{LJ} A[x_1/e_1, \ldots, x_n/e_n] \) for any variables \( x_1, \ldots, x_n \) not bound in \( \Gamma \) nor in \( A \).

Proof. By induction on the size of the proof of \( \Gamma \vdash A \) in ALN.

\[
\begin{align*}
\vdash & \Gamma^\uparrow \vdash B^\uparrow[0/x] \\
\Gamma^\uparrow \vdash & B^\uparrow[0/x] \quad \text{(\forall I)} \\
\Gamma \vdash & \forall x B
\end{align*}
\]

Let \( y \) be a fresh variable.

\[
\begin{align*}
\vdash & \Gamma^\uparrow[x_1/e_1, \ldots, x_n/e_n] \vdash B^\uparrow[x_1/e_1, \ldots, x_n/e_n][y/x] \\
\end{align*}
\]
Lemma. If the e-variables of $\Gamma$ and $A$ are among $e_1, \ldots, e_n$ and $\Gamma \vdash_{\text{ALN}} A$, then $\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash_{\text{LJ}} A[x_1/e_1, \ldots, x_n/e_n]$ for any variables $x_1, \ldots, x_n$ not bound in $\Gamma$ nor in $A$.

**Proof.** By induction on the size of the proof of $\Gamma \vdash A$ in ALN.

\[
\begin{align*}
\vdots \\
\Gamma \uparrow \vdash B \uparrow[0/x] \\
\hline
\Gamma \vdash \forall x B
\end{align*}
\tag{\forall I}
\]

Let $y$ be a fresh variable.

\[
\begin{align*}
\vdots \\
\Gamma \uparrow[x_1/e_1 \uparrow, \ldots, x_n/e_n \uparrow] \vdash B \uparrow[x_1/e_1 \uparrow, \ldots, x_n/e_n \uparrow][y/x]
\end{align*}
\]
**Lemma.** If the e-variables of $\Gamma$ and $A$ are among $e_1, \ldots, e_n$ and $\Gamma \vdash_{ALN} A$, then $\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash_{LJ} A[x_1/e_1, \ldots, x_n/e_n]$ for any variables $x_1, \ldots, x_n$ not bound in $\Gamma$ nor in $A$.

**Proof.** By induction on the size of the proof of $\Gamma \vdash A$ in ALN.

\[
\begin{align*}
\vdots \\
\Gamma \vdash B[0/x] & \quad (\forall I) \\
\Gamma \vdash \forall x B
\end{align*}
\]

Let $y$ be a fresh variable.

\[
\begin{align*}
\vdots \\
\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash B[x_1/e_1, \ldots, x_n/e_n][y/x]
\end{align*}
\]
Lemma. If the e-variables of \( \Gamma \) and \( A \) are among \( e_1, \ldots, e_n \) and \( \Gamma \vdash_{\text{ALN}} A \), then \( \Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash_{\text{LJ}} A[x_1/e_1, \ldots, x_n/e_n] \) for any variables \( x_1, \ldots, x_n \) not bound in \( \Gamma \) nor in \( A \).

Proof. By induction on the size of the proof of \( \Gamma \vdash A \) in ALN.

\[
\begin{align*}
\vdots \\
\Gamma \uparrow \vdash B \uparrow[0/x] \\
\Gamma \vdash \forall x B
\end{align*}
\]

(\(\forall I\))

Let \( y \) be a fresh variable.

\[
\begin{align*}
\vdots \\
\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash B[x_1/e_1, \ldots, x_n/e_n][y/x] \\
\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash \forall x(B[x_1/e_1, \ldots, x_n/e_n])
\end{align*}
\]

(\(\forall I\))
Lemma. If the e-variables of $\Gamma$ and $A$ are among $e_1, \ldots, e_n$ and $\Gamma \vdash_{\text{ALN}} A$, then $\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash_{\text{LJ}} A[x_1/e_1, \ldots, x_n/e_n]$ for any variables $x_1, \ldots, x_n$ not bound in $\Gamma$ nor in $A$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in ALN.

\[
\begin{align*}
\vdots \\
\Gamma \uparrow \vdash B \uparrow[0/x] \\
\Gamma \vdash \forall x B
\end{align*}
\]

(\forall I)

Let $y$ be a fresh variable.

\[
\begin{align*}
\vdots \\
\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash B[x_1/e_1, \ldots, x_n/e_n][y/x] \\
\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash \forall x (B[x_1/e_1, \ldots, x_n/e_n]) 
\end{align*}
\]

(\forall I)
Lemma. If the e-variables of $\Gamma$ and $A$ are among $e_1, \ldots, e_n$ and $\Gamma \vdash_{ALN} A$, then $\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash_{LJ} A[x_1/e_1, \ldots, x_n/e_n]$ for any variables $x_1, \ldots, x_n$ not bound in $\Gamma$ nor in $A$.

Proof. By induction on the size of the proof of $\Gamma \vdash A$ in ALN.

\[
\vdots
\]

\[
\Gamma \uparrow \vdash B \uparrow[0/x] \quad (\forall I)
\]

\[
\Gamma \vdash \forall x B
\]

Let $y$ be a fresh variable.

\[
\vdots
\]

\[
\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash B[x_1/e_1, \ldots, x_n/e_n][y/x] \quad (\forall I)
\]

\[
\Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash (\forall x B)[x_1/e_1, \ldots, x_n/e_n]
\]
Let $t$ be $f$-closed with $e$-variables among $e_1, \ldots, e_n, e'_1, \ldots, e'_m$.

\[
\begin{array}{c}
\vdots \\
\Gamma \vdash B[t/x] \\
\hline
\Gamma \vdash \exists x B \\
\end{array} \quad (\exists l)
\]
Let $t$ be $f$-closed with e-variables among $e_1, \ldots, e_n, e'_1, \ldots, e'_m$. 

\[
\frac{\Gamma \vdash B[t/x]}{\Gamma \vdash \exists x B} \quad (\exists l)
\]

\[
\frac{\vdots}{\Gamma \vdash \exists B}
\]

\[
\vdots
\]

\[
\Gamma[x_i/e_i, y/e'_k] \vdash B[t/x][x_i/e_i, y/e'_k]
\]
Let $t$ be f-closed with e-variables among $e_1, \ldots, e_n, e'_1, \ldots, e'_m$.

\[\begin{align*}
\Gamma & \vdash B[t/x] \\
\frac{}{\Gamma \vdash \exists x B} & (\exists I) \\
\Gamma & \vdash B[t/x][x_i/e_i, y/e'_k]
\end{align*}\]
Let $t$ be $f$-closed with $e$-variables among $e_1, \ldots, e_n, e'_1, \ldots, e'_m$.

Let $\Gamma \vdash B[t/x]$.

$\Gamma \vdash \exists x B$ (Existential Introduction)

Let $\Gamma[x_i/e_i, y/e'_k] \vdash B[x_i/e_i, y/e'_k][t[x_i/e_i, y/e'_k]/x]$.
Let $t$ be $f$-closed with $e$-variables among $e_1, \ldots, e_n, e'_1, \ldots, e'_m$. 

\[
\vdots \\
\Gamma \vdash B[t/x] \\
\frac{}{\Gamma \vdash \exists x B} \quad (\exists l) \\
\vdots \\
\Gamma[x_i/e_i, y/e'_k] \vdash B[x_i/e_i, y/e'_k][t[x_i/e_i, y/e'_k]/x] 
\]
Let $t$ be $f$-closed with $e$-variables among $e_1, \ldots, e_n, e'_1, \ldots, e'_m$.

\[\vdash B[t/x] \quad (\exists I)\]

\[\vdash \exists x B\]

\[\vdash B[x_i/e_i][t[x_i/e_i, y/e'_k]/x]\]
Let $t$ be $f$-closed with $e$-variables among $e_1, \ldots, e_n, e'_1, \ldots, e'_m$.

\[
\vdots \\
\vdots \\
\Gamma \vdash B[t/x] \\
\vdash \exists x B \\
\vdots \\
\Gamma[x_i/e_i] \vdash B[x_i/e_i][t[x_i/e_i, y/e'_k]/x] \\
\vdash \exists x (B[x_i/e_i])
\]
Let $t$ be $f$-closed with $e$-variables among $e_1, \ldots, e_n, e'_1, \ldots, e'_m$.

\[
\vdots
\]

\[
\Gamma \vdash B[t/x]
\]

\[
\frac{\vdash \exists xB}{\vdash \exists x(B[t/x])}
\quad (\exists \! I)
\]

\[
\vdots
\]

\[
\Gamma[x_i/e_i] \vdash B[x_i/e_i][t[x_i/e_i, y/e'_k]/x]
\]

\[
\frac{\Gamma[x_i/e_i] \vdash \exists x(B[x_i/e_i])}{\Gamma[x_i/e_i] \vdash \exists x(B[x_i/e_i])}
\quad (\exists \! I)
\]
Let $t$ be $f$-closed with $e$-variables among $e_1, \ldots, e_n, e'_1, \ldots, e'_m$.

$$
\vdots
\Gamma \vdash B[t/x] \\
\Gamma \vdash \exists x B \\
\vdots

\Gamma[x_i/e_i] \vdash B[x_i/e_i][t[x_i/e_i, y/e'_k]/x] \\
\Gamma[x_i/e_i] \vdash (\exists x B)[x_i/e_i]
$$

(\exists I)
Proposition (Embedding LJ). If the free variables of \( \Gamma \) and \( A \) are among \( x_1, \ldots, x_n \), then:

\[
\Gamma \vdash_{\text{LJ}} A \iff \Gamma[x_1/1, \ldots, n/x_n] \vdash_{\text{ALN}} A[x_1/1, \ldots, n/x_n]
\]
**Proposition (Embedding LJ).** If the free variables of $\Gamma$ and $A$ are among $x_1, \ldots, x_n$, then:

$$\Gamma \vdash_{LJ} A \iff \Gamma[1/x_1, \ldots, n/x_n] \vdash_{ALN} A[1/x_1, \ldots, n/x_n]$$

**Proposition (Embedding ALN).** If $\Gamma$ and $A$ are f-closed, their e-variables are among $e_1, \ldots, e_n$ and if $x_1, \ldots, x_n$ are variables not bound in $\Gamma$ nor in $A$, then:

$$\Gamma \vdash_{ALN} A \iff \Gamma[x_1/e_1, \ldots, x_n/e_n] \vdash_{LJ} A[x_1/e_1, \ldots, x_n/e_n]$$

Lorenzo Tortora de Falco e Vito Michele Abrusci. *Pagina dedicata al libro Logica*.