Finding the longest isometric cycle in a graph

Proposed by : CHALOPIN Jérémie
Presented by : EL MASDOURI Reda

Aix-Marseille University,
Faculty of sciences
Luminy

Séminaire Tutoré
Finding the longest isometric cycle in a graph

The plan

1. Motivation
2. Preliminaries
3. Main results
4. LIC Algorithm
5. Complexity analysis
Finding the longest isometric cycle in a graph

Motivation

Some historic results

- In combinatory, the most fundamental problem is: under which conditions a graph posses some kind of cycles?
- The famous "seven bridges of Königsberg" problem.
- Finding a Hamiltonian cycle in a graph is $NP$-complete problem.
- Garey and Johnson proved that finding the longest cycle and the longest induced cycle in a graph are $NP$-complete problems.
- We can found the longest isometric cycle in a graph in a polynomial time.
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A simple graph is a graph that does not have more than one edge between any two vertices and no edge starts and ends at the same vertex.

A graph is said to be connected if every pair of vertices in the graph is connected.

A walk is a sequence of vertices and edges of a graph.

If the first and last vertices of a walk are the same we say that is cyclic.

A path is a walk in which all vertices and also all edges are distinct.

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Finding the longest isometric cycle in a graph

Preliminaries

Definitions

- The length of a walk is the number of edges in it.
- The distance between two vertices \( u \) and \( v \) in a graph \( G \) is the number of edges in a shortest path connecting them, denoted by \( d_G(u, v) \), and \( d_G(u, u) = 0 \), and \( d_G(u, v) = \infty \) if \( u \) and \( v \) are in different components of \( G \).
- A cycle in a graph \( G \) is isometric if the distance between two vertices in the cycle is equal to their distance in \( G \).
- A subgraph \( H \) of a graph \( G \) is said an isometric subgraph if for every \( u \) and \( v \) in \( V(H) \), we have \( d_H(u, v) = d_G(u, v) \).

Notice that an isometric subgraph is an induced subgraph.
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Notice that an isometric subgraph is an induced subgraph.
Let $G$ be a undirected graph, the $p$-th power $G^p$ of $G$ is another graph that has the same set of vertices, but in which two vertices are adjacent when their distance in $G$ is at most $p$.

$$G^p = (V(G), \{(u, v) : d_G(u, v) \leq p\})$$

Example

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$$G^p = (V(G), \{(u, v) : d_G(u, v) \leq p\})$$
Let $G$ a simple, connected, unweighted and undirected graph, and let $k \geq 3$. From $G$ we construct the graph $G_k$ such that:

- $V(G_k) = \{(u, v) / d_G(u, v) = \lfloor \frac{k}{2} \rfloor \land u, v \in V(G)\}$
- $E(G_k) = \{((u, v), (a, b)) / (u, a) \in E(G) \land (v, b) \in E(G)\}$

**Lemma**

$$d_{G_k}((u, v), (a, b)) \geq \max \{d_G(u, a), d_G(v, b)\}$$

**Proof:**
If $P = \{(u, v), (a_1, b_1), \ldots, (a, b)\}$ is the shortest path from $(u, v)$ to $(a, b)$ in $G_k$, then $P_1 = \{u, a_1, \ldots, a\}$ and $P_2 = \{v, b_1, \ldots, b\}$ are paths in $G$. Thus $P, P_1$ and $P_2$ all have the same length. So the result.
Auxiliary graph

Let $G$ a simple, connected, unweighted and undirected graph, and let $k \geq 3$. From $G$ we construct the graph $G_k$ such that:

- $V(G_k) = \left\{ (u, v) \mid d_G(u, v) = \left\lfloor \frac{k}{2} \right\rfloor \land u, v \in V(G) \right\}$
- $E(G_k) = \left\{ ((u, v), (a, b)) \mid (u, a) \in E(G) \land (v, b) \in E(G) \right\}$

**Lemma**

$$d_{G_k} ((u, v), (a, b)) \geq \max \left\{ d_G(u, a), d_G(v, b) \right\}$$

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Let $G$ a simple, connected, unweighted and undirected graph, and let $k \geq 3$. From $G$ we construct the graph $G_k$ such that:

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If $P = \{ (u, v), (a_1, b_1), ..., (a, b) \}$ is the shortest path from $(u, v)$ to $(a, b)$ in $G_k$, then $P_1 = \{ u, a_1, ..., a \}$ and $P_2 = \{ v, b_1, ..., b \}$ are paths in $G$. Thus $P, P_1$ and $P_2$ all have the same length. So the result.
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Preliminaries

Particular case

**Lemma**

A cycle of length $k$ is an isometric cycle if and only if the distance in $G$ between every pair of diametrically opposite vertices $u$ and $v$ in the cycle is $\left\lfloor \frac{k}{2} \right\rfloor$.

Proof:

$\Rightarrow$) $\checkmark$

$\Leftarrow$) If the cycle is not isometric, then there exist a pair of vertices $u$ and $v$ such that their distance in the graph is smaller than the distance in the cycle. Let $x$ be a vertex in the cycle diametrically opposite to $u$ such that $v$ lies on the shortest path from $u$ to $x$ in the cycle, Then

$$\left\lfloor \frac{k}{2} \right\rfloor = d_G(u, x) \leq d_G(u, v) + d_G(x, v) < d_{cycle}(u, v) + d_{cycle}(v, x) = d_{cycle}(u, x) = \left\lfloor \frac{k}{2} \right\rfloor$$
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$$\left\lfloor \frac{k}{2} \right\rfloor = d_G(u, x) \leq d_G(u, v) + d_G(x, v) < d_{\text{cycle}}(u, v) + d_{\text{cycle}}(v, x) = d_{\text{cycle}}(u, x) = \left\lfloor \frac{k}{2} \right\rfloor$$
The vertices of $G_k$ represent pairs of vertices that might end up as diametrically opposite vertices in an isometric cycle of $G$.

$(u, v)$ and $(a, b)$ are adjacent if the pairs that they represent could be adjacent pairs of diametrically opposite vertices in an isometric cycle of $G$. 
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Finding the longest isometric cycle in a graph

Main results

Lemma

If $k$ is even and there is an isometric cycle of length $k$ in $G$ going through $u$ and $v$ then $d_{G_k}((u,v), (v,u)) = \frac{k}{2}$

Proof

Let $C = \left\{ c_1, ..., c_{\frac{k}{2}}, c_{\frac{k}{2}+1}, ..., c_k \right\}$ be an isometric cycle in $G$ of length $k$, with $c_1 = u$
As $(u,v) \in G_k$ then $d_G(u,v) = \frac{k}{2}$, and we know that $C$ is isometric thus $c_{\frac{k}{2}+1} = v$.
The vertices $(c_1, c_{\frac{k}{2}+1}), (c_2, c_{\frac{k}{2}+2}), ..., (c_k, c_k), (c_{\frac{k}{2}+1}, c_1)$ are in $G_k$ and there is an edge between each consecutive pairs of them, thus $d_{G_k}((u,v), (v,u)) \leq \frac{k}{2}$.
By the first lemma one has $d_{G_k}((u,v), (v,u)) \geq \frac{k}{2}$.
Finding the longest isometric cycle in a graph

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The vertices $(c_1, c_{\frac{k}{2}+1}), (c_2, c_{\frac{k}{2}+2}), \ldots, (c_{\frac{k}{2}}, c_k), (c_{\frac{k}{2}+1}, c_1)$ are in $G_k$ and there is an edge between each consecutive pairs of them, thus $d_{G_k}((u,v), (v,u)) \leq \frac{k}{2}$.

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Finding the longest isometric cycle in a graph
Lemma

If \( k \) is even and \( d_{G_k}((u,v),(v,u)) = \frac{k}{2} \), then there is an isometric cycle of length \( k \) in \( G \) going through \( u \) and \( v \).

Proof

Let \( d_{G_k}((u,v),(v,u)) = \frac{k}{2} \) and \( P = \{(u,v), (a_2, b_2), ..., (a_{\frac{k}{2}-1}, b_{\frac{k}{2}-1}), (v,u)\} \) be a shortest path between \((u,v)\) and \((v,u)\).

Obviously, \( W = \{u, a_2, a_3, ..., v, b_2, ..., u\} \) is a cyclic walk of length \( k \), and it's a subgraph in \( G \).

We suppose that there is a pair of vertices \( a \) and \( b \) in \( W \) with \( d_G(a, b) < d_W(a, b) \).

Let \( x \) be a vertex in \( W \) such that \((a, x)\) or \((x, a)\) is in \( P \).

As \( x \) and \( a \) are diametrically opposite in \( W \), there is a walk of length \( \frac{k}{2} \) from \( a \) to \( x \) going through \( b \), thus \( d_W(a, b) + d_W(b, x) \leq \frac{k}{2} \).

Thus, \( \frac{k}{2} = d_G(a, x) \leq d_G(a, b) + d_G(b, x) < d_W(a, b) + d_W(b, x) \leq \frac{k}{2} \).

Impossible.
Finding the longest isometric cycle in a graph

Main results

Lemma

If \( k \) is even and \( d_{G_k}((u,v),(v,u)) = \frac{k}{2} \), then there is an isometric cycle of length \( k \) in \( G \) going through \( u \) and \( v \).

Proof

Let \( d_{G_k}((u,v),(v,u)) = \frac{k}{2} \) and \( P = \{(u,v),(a_2,b_2),\ldots,(a_{\frac{k}{2}-1},b_{\frac{k}{2}-1}),(v,u)\} \) be a shortest path between \((u,v)\) and \((v,u)\).

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Thus, \( \frac{k}{2} = d_G(a,x) \leq d_G(a,b) + d_G(b,x) < d_W(a,b) + d_W(b,x) \leq \frac{k}{2} \), impossible.
Finding the longest isometric cycle in a graph

Main results

Corollary

If $k$ is even, there is an isometric cycle of length $k$ in $G$ if and only if there is a pair of vertices $u$ and $v$ with $d_{G_k}((u, v), (v, u)) = \frac{k}{2}$
The analogous results when $k$ is odd are more technical.

**Definition**

For a vertex $(u, v) \in G_k$ we define the set

$$M'_k(u, v) = \{(u, x)/(u, x) \in V(G_k) \land (v, x) \in E(G)\}$$

**Lemma**

If $k$ is odd and there is an isometric cycle of length $k$ in $G$, going through $u$ and $v$, then $d_{G_k}((u, v), (v, x)) = \left\lfloor \frac{k}{2} \right\rfloor$ where $(v, x)$ is a vertex in $M'_k(v, u)$.

**Proof**

Assume $G$ has an isometric cycle $C$ of length $k$,

$$C = \{c_1, c_2, \ldots, c_{\left\lfloor \frac{k}{2} \right\rfloor}, c_{\left\lfloor \frac{k}{2} \right\rfloor}+1, \ldots, c_k\}$$

with $c_1 = u, c_{\left\lfloor \frac{k}{2} \right\rfloor}+1 = v$ and $c_k = x$.

As the last proof the vertices $(c_1, c_{\left\lfloor \frac{k}{2} \right\rfloor}+1), \ldots, (c_{\left\lfloor \frac{k}{2} \right\rfloor}, c_k-1)$ and $(c_{\left\lfloor \frac{k}{2} \right\rfloor}+1, c_k)$ are in $G_k$, also there is an edge between each consecutive vertices, moreover $(c_{\left\lfloor \frac{k}{2} \right\rfloor}+1, c_k) \in M'_k(v, u)$ because $c_{\left\lfloor \frac{k}{2} \right\rfloor}+1 = v$ and $(c_k, c_1) \in E(G)$. Then $d_{G_k}((u, v), (v, x)) \leq \left\lfloor \frac{k}{2} \right\rfloor$.

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If \( k \) is odd and there is an isometric cycle of length \( k \) in \( G \), going through \( u \) and \( v \), then \( d_{G_k}((u, v), (v, x)) = \left\lfloor \frac{k}{2} \right\rfloor \) where \((v, x)\) is a vertex in \( M'_k(v, u) \).

**Proof**

Assume \( G \) has an isometric cycle \( C \) of length \( k \),
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C = \{c_1, c_2, ..., c_{\lfloor \frac{k}{2} \rfloor}, c_{\lfloor \frac{k}{2} \rfloor} + 1, ..., c_k\}
\]
with \( c_1 = u, c_{\lfloor \frac{k}{2} \rfloor} + 1 = v \) and \( c_k = x \).

As the last proof the vertices \( (c_1, c_{\lfloor \frac{k}{2} \rfloor} + 1), ..., (c_{\lfloor \frac{k}{2} \rfloor}, c_{k-1}) \) and \( (c_{\lfloor \frac{k}{2} \rfloor} + 1, c_k) \)
are in \( G_k \), also there is an edge between aech consecutive vertices, moreover
\( (c_{\lfloor \frac{k}{2} \rfloor} + 1, c_k) \in M'_k(v, u) \) because \( c_{\lfloor \frac{k}{2} \rfloor} + 1 = v \) and \((c_k, c_1) \in E(G)\).

Then \( d_{G_k}((u, v), (v, x)) \leq \left\lfloor \frac{k}{2} \right\rfloor \)
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If $k$ is odd and there is an isometric cycle of length $k$ in $G$, going through $u$ and $v$, then $d_{G_k}((u, v), (v, x)) = \left\lfloor \frac{k}{2} \right\rfloor$ where $(v, x)$ is a vertex in $M'_k(v, u)$.

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Assume $G$ has an isometric cycle $C$ of length $k$,

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As the last proof the vertices $(c_1, c_{\left\lfloor \frac{k}{2} \right\rfloor}+1), \ldots, (c_{\left\lfloor \frac{k}{2} \right\rfloor}, c_k)$ and $(c_{\left\lfloor \frac{k}{2} \right\rfloor}+1, c_k)$ are in $G_k$, also there is an edge between each consecutive vertices, moreover

$$c_{\left\lfloor \frac{k}{2} \right\rfloor}+1, c_k \in M'_k(v, u)$$

because $c_{\left\lfloor \frac{k}{2} \right\rfloor}+1 = v$ and $(c_k, c_1) \in E(G)$.

Then $d_{G_k}((u, v), (v, x)) \leq \left\lfloor \frac{k}{2} \right\rfloor$

The result holds so.
The analogous results when $k$ is odd are more technical.

**Definition**

For a vertex $(u, v) \in G_k$ we define the set

\[ M'_k(u, v) = \{(u, x) / (u, x) \in V(G_k) \land (v, x) \in E(G)\} \]

**Lemma**

If $k$ is odd and there is an isometric cycle of length $k$ in $G$, going through $u$ and $v$, then $d_{G_k}((u, v), (v, x)) = \left\lfloor \frac{k}{2} \right\rfloor$ where $(v, x)$ is a vertex in $M'_k(v, u)$.

**Proof**

Assume $G$ has an isometric cycle $C$ of length $k$,

\[ C = \left\{ c_1, c_2, ..., c_{\lfloor \frac{k}{2} \rfloor}, c_{\lfloor \frac{k}{2} \rfloor+1}, ..., c_k \right\} \text{ with } c_1 = u, c_{\lfloor \frac{k}{2} \rfloor+1} = v \text{ and } c_k = x. \]

As the last proof the vertices $\left( c_1, c_{\lfloor \frac{k}{2} \rfloor+1} \right), ..., \left( c_{\lfloor \frac{k}{2} \rfloor}, c_{k-1} \right) \text{ and } \left( c_{\lfloor \frac{k}{2} \rfloor+1}, c_k \right)$ are in $G_k$, also there is an edge between each consecutive vertices, moreover $\left( c_{\lfloor \frac{k}{2} \rfloor+1}, c_k \right) \in M'_k(v, u)$ because $c_{\lfloor \frac{k}{2} \rfloor+1} = v \text{ and } (c_k, c_1) \in E(G)$.

Then $d_{G_k}((u, v), (v, x)) \leq \left\lfloor \frac{k}{2} \right\rfloor$

The result holds so.
Lemma

If \( k \) is odd and the distance in \( G_k \) between \((u, v)\) and \((v, x)\) \(\in M_k^t(v, u)\) is \(\lfloor \frac{k}{2} \rfloor\), then there is an isometric cycle of length \(k\) in \(G\) going through \(u\) and \(v\).

Proof

Let \(d_{G_k}((u, v), (v, x)) = \lfloor \frac{k}{2} \rfloor\) and let's \(P = \{(u, v), (a_2, b_2), \ldots, (a_{\lfloor \frac{k}{2} \rfloor}, b_{\lfloor \frac{k}{2} \rfloor}), (v, x)\}\) a shortest path between \((u, v)\) and \((v, x)\).

One remark \(W = \{u, a_2, \ldots, a_{\lfloor \frac{k}{2} \rfloor}, v, b_2, \ldots, b_{\lfloor \frac{k}{2} \rfloor}, x, u\}\) is a subgraph of \(G\) and cyclic walk.

Suppose that there is a pair of vertices \(a\) and \(b\) in \(W\) with \(d_G(a, b) < d_W(a, b)\).

We can found a vertex \(z \in V(W)\) such that either \((a, z)\) or \((z, a)\) is in \(P\) and \(d_W(a, b) + d_W(b, z) \leq \lfloor \frac{k}{2} \rfloor\).

Then, \(d_G(a, z) \leq d_G(a, b) + d_G(b, z) < d_W(a, b) + d_W(b, z) \leq \lfloor \frac{k}{2} \rfloor\), contradiction.
Lemma

If $k$ is odd and the distance in $G_k$ between $(u, v)$ and $(v, x) \in M_k^i(v, u)$ is $\left\lfloor \frac{k}{2} \right\rfloor$, then there is an isometric cycle of length $k$ in $G$ going through $u$ and $v$.

Proof

Let $d_{G_k}((u, v), (v, x)) = \left\lfloor \frac{k}{2} \right\rfloor$ and let's $P = \{(u, v), (a_2, b_2), ..., (a_{\left\lfloor \frac{k}{2} \right\rfloor}, b_{\left\lfloor \frac{k}{2} \right\rfloor}), (v, x)\}$ a shortest path between $(u, v)$ and $(v, x)$.

One remark $W = \{u, a_2, ..., a_{\left\lfloor \frac{k}{2} \right\rfloor}, v, b_2, ..., b_{\left\lfloor \frac{k}{2} \right\rfloor}, x, u\}$ is a subgraph of $G$ and cyclic walk.

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Then, $d_G(a, z) \leq d_G(a, b) + d_G(b, z) < d_W(a, b) + d_W(b, z) \leq \left\lfloor \frac{k}{2} \right\rfloor$, contradiction.
Finding the longest isometric cycle in a graph

Main results

Corollary

If $k$ is odd, there is an isometric cycle of length $k$ in $G$ if and only if there are vertices $u, v$ and $x$ so that $(v, x) \in M'_k(v, x)$ and $d_{G_k}((u, v), (v, x)) = \lfloor \frac{k}{2} \rfloor$

Main Theorem

$G$ has an isometric cycle of length $k$ if and only if there are vertices $u, v, x \in V(G)$ so that $(v, x) \in M_k(v, u)$ and $d_{G_k}((u, v), (v, x)) = \lfloor \frac{k}{2} \rfloor$

Where: $M_k(u, v) = \begin{cases} (u, v) & \text{if } k \text{ is even} \\ M'_k(u, v) & \text{if } k \text{ is odd} \end{cases}$
Finding the longest isometric cycle in a graph

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\end{cases}
\]
Finding the longest isometric cycle in a graph

LIC Algorithm

Algorithm

For a given $k$, showing if a graph $G$ has an isometric cycle using $G_k$, we check if there is a pair of vertices $(u, v)$ and $(v, x)$ in $V(G_k)$ such that $(v, x) \in M_k(v, u)$ and $d_{G_k}((u, v), (v, x)) = \left\lfloor \frac{k}{2} \right\rfloor$.

Remark

By the first lemma we had $d_{G_k}((u, v), (v, x)) \geq \left\lfloor \frac{k}{2} \right\rfloor$. Thus we have to search for vertices satisfying just the inequality $d_{G_k}((u, v), (v, x)) \leq \left\lfloor \frac{k}{2} \right\rfloor$. 
For a given $k$, showing if a graph $G$ has an isometric cycle using $G_k$ we check if there is a pair of vertices $(u, v)$ and $(v, x)$ in $V(G_k)$ such that $(v, x) \in M_k(v, u)$ and $d_{G_k}((u, v), (v, x)) = \lfloor \frac{k}{2} \rfloor$.

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Finding the longest isometric cycle in a graph

LIC Algorithm

Longest isometric cycle (LIC) algorithm

Input: a graph $G = (V, E)$.
Output: The length of the longest isometric cycle in $G$, noted $ans$.

begin
  $ans := 0$
  Compute the distance matrix of $G$.
  if $G$ is a tree then
    return $ans$
  end-if
  for every $k$ from 3 to $n$ do
    $V_k := \emptyset$
    for every $u$ and $v$ in $V$ do
      if $d(u, v) = \lfloor k/2 \rfloor$ then
        $V_k := V_k \cup \{(u, v)\}$
      end-if
    end-for
    $E_k := \emptyset$
    for every $(u, v)$ and $(w, x)$ in $V_k$ do
      if $(u, w) \in E \land (v, x) \in E$ then
        $E_k := E_k \cup \{[(u, v), (w, x)]\}$
      end-if
    end-for
    $G_k := (V_k, E_k)$
    Compute $G_k^{\lfloor k/2 \rfloor} = (V_k, E_k^{\lfloor k/2 \rfloor})$
    for every triple of vertices $(u, v, x)$ in $V$ do
      if $(u, v) \in V(G_k) \land (v, x) \in M_k(v, u) \land [(u, v), (v, x)] \in E_k^{\lfloor k/2 \rfloor}$ then
        $ans := k$
      end-if
    end-for
  end-for
  return $ans$
end
Finding the longest isometric cycle in a graph

LIC Algorithm

Theorem
LIC algorithm computes the length of the longest isometric cycle of a graph $G$.

Proof
- If $G$ is a tree, it has no cycle and the algorithm return 0.
- If $G$ has a cycle, it must have an isometric cycle of length at least 3 and at most $n$.
- If $k'$ is the length of the longest isometric cycle in $G$.
  By the main theorem there exist vertices $(u, v) \in V(G_k)$ and $(v, x) \in M(v, u)$ such that $d_{G_k}((u, v), (v, x)) = \left\lfloor \frac{k}{2} \right\rfloor$, thus

$(u, v), (v, x) \in E_k^{\lfloor \frac{k}{2} \rfloor}$. So the algorithm return $ans = k'$ when $k = k'$.

Now if $k > k'$, the same theorem implies that there isn’t any vertices $(u, v), (v, x)$ who satisfy the above conditions. So the command $ans = k$ will not be executed.

Then the algorithm return $ans = k'$. 
Finding the longest isometric cycle in a graph

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LIC algorithm computes the length of the longest isometric cycle of a graph $G$.

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Finding the longest isometric cycle in a graph
Finding the longest isometric cycle in a graph

Complexity analysis

Folklor algorithm for computing graph powers

To compute $G^x$ we write $x$ to base 2 and let $d_{i+1}$ the $i$-th digit in this string counting from right to left.

Now we find $G^{2^k}$ for $2^k \leq x$ and compute the matrix product $\prod_{i=0}^{\lfloor \log(x) \rfloor} A_i$ where

$$A_i = \begin{cases} G^{2^i} & \text{if } d_i = 1 \\ id & \text{else} \end{cases}$$

If $n^\alpha$ is the time needed to multiply tow $n$ by $n$ matrices. Then the time complexity of folklore algorithm is $|V(G)|^\alpha \times \log(x)$. 
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If $n^\alpha$ is the time needed to multiply two $n$ by $n$ matrices. Then the time complexity of folklore algorithm is $|V(G)|^\alpha \times \log(x)$. 

Finding the longest isometric cycle in a graph
The distance matrix of $G$ and finding out whether it's acyclic are computed in $O(n^3)$ time.

For a given $k$, the set $V_k$ is computed in $O(n^2)$.

We observe that $E_k$ is computed in $O(|V_k|^2)$.

By the folklore algorithm, computing $G^\lfloor \frac{k}{2} \rfloor_k$ from $G_k$ takes $O(|V(G)|^\alpha \times \log(\lfloor \frac{k}{2} \rfloor))$ times.

The last loop iterates over all triples $(u, v, x) \in V$, so it's calculated in $O(n^3)$. 
Finding the longest isometric cycle in a graph

Complexity analysis

Complexity analysis of LIC algorithm steps

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Finding the longest isometric cycle in a graph

Complexity analysis

Remark

\[ \sum_{k=3}^{n} |V_k| \leq 2n^2 \]

Proof

For given \( k_1 \) and \( k_2 \) with \( \lfloor \frac{k_1}{2} \rfloor \neq \lfloor \frac{k_2}{2} \rfloor \), then \( V_{k_1} \) and \( V_{k_2} \) are pairwise disjoint subsets of \( V^2 \).
If \( \lfloor \frac{k_1}{2} \rfloor = \lfloor \frac{k_2}{2} \rfloor \), then \( V_{k_1} = V_{k_2} \) by \( G_k \) definition.
We sum over all even and odd \( k \)s we obtain

\[
\sum_{k=3}^{n} |V_k| = \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} |V_{2k+1}| + \sum_{k=2}^{\lfloor \frac{n}{2} \rfloor} |V_{2k}|
\leq |V|^2 + |V|^2 = 2n^2
\]
**Theorem**

If $O(n^\alpha)$ is the time needed to multiply two $n$ by $n$ matrices and $\alpha \geq 2$, then LIC algorithm terminates in $O(n^\alpha \log(n))$ steps.

**Proof**

Let $T$ be the total number of steps performed by the algorithm. From the discussion above,

$$
T = O(n^3) + \sum_{k=3}^{n} \left[ O(n^2) + O(|V_k|^2) + O(|V_k|^\alpha \log\left\lfloor \frac{k}{2} \right\rfloor) + O(n^3) \right]
$$

Rearranging our terms and summing the terms independent on $k$, we obtain

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T = O(n^4) + \sum_{k=3}^{n} \left[ O(|V_k|^\alpha \log\left\lfloor \frac{k}{2} \right\rfloor) \right]
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As $\log\left\lfloor \frac{k}{2} \right\rfloor = O(\log(n))$, thus $T = O(n^4) + O(\log(n)) \sum_{k=3}^{n} O(|V_k|^\alpha)$

Also we have $n^\alpha$ is a convex function so we can put the summation inside the $O$, moreover $\sum_{k=3}^{n} |V_k|^\alpha \leq \left( \sum_{k=3}^{n} |V_k| \right)^\alpha$. 

Finding the longest isometric cycle in a graph

Complexity analysis

**Theorem**

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Finding the longest isometric cycle in a graph

Complexity analysis

Proof

This yields $T = O(n^4) + O(\log(n))O \left[ (\sum_{k=3}^{n} |V_k|)^\alpha \right]$. The remark implies the following simplification: $T = O(n^4) + O(\log(n)(2n^2)^\alpha)$. Finally, we have $2\alpha \geq 4$, thus $T = O(n^{2\alpha} \log(n))$.

Theorem [1]

Two $n$ by $n$ matrices can be multiplied in $O(n^{2.376})$ time.

Corollary

LIC algorithm runs in $O(n^{4.752} \log(n))$ time.
Proof

This yields \( T = O(n^4) + O(\log(n))O \left[ \left( \sum_{k=3}^{n} |V_k| \right)^\alpha \right] \).

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Theorem [1]

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References

Thank you for your attention