A bit of nondeterminism makes pushdown automata expressive and concise

joint work with Shibashis Guha, Ismaël Jecker, Martin Zimmermann

Karoliina Lehtinen
6.1.2022

CNRS, Aix-Marseille Univ, LIS
Determinism vs Nondeterminism
Determinism vs nondeterminism for pushdown automata

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- **Expressivity:** $\text{D-CFL} \subset \text{GFG-CFL} \subset \text{CFL}$
- **Solving Games:** $\text{ExpTime}$ undecidable
- **Universality:** $\text{ExpTime}$ undecidable
- **Succinctness:** (at least) Exponential, non-recursive, already on finite words!
## Determinism vs nondeterminism for pushdown automata

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- **Expressivity**:
  - **D-PDA (Deterministic PDA)**:
    - Expresses **D-CFL**
  - **PDA (Pushdown Automaton)**:
    - Expresses **CFL**

- **Solving Games**:
  - **ExpTime**
  - **Undecidable**

- **Universality**:
  - **ExpTime**
  - **Undecidable**

- **Succinctness**:
  - **(at least) Exponential**
  - **Non-recursive**

 Already on finite words!
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\(\text{D-CFL} \subset\ GFG-CFL \subset\ \text{CFL}\)
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- D-CFL: Deterministic Context-Free Languages
- GFG-CFL: Generalized Greibach Form Context-Free Languages
- CFL: Context-Free Languages
- ExpTime: Exponential Time
- undecidable
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A run is accepting if the last letter is $a$. 
An automaton $A$ is history deterministic if there is a resolver:

$$ r: \Delta^* \times \Sigma \to \Delta $$

that induces an accepting run for all words in $L(A)$.

Equivalently, Eve wins the following game on $A$:

- Adam chooses letters $a_i \in \Sigma$
- Eve responds with transitions $\tau_i$ over $a_i$
- Eve wins if $a_0 a_1 ... \in L(A)$ or $\tau_0 \tau_2 ...$ is accepting.
An automaton $A$ is *history deterministic* if there is a resolver:

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Equivalently, Eve wins the following game on $A$:

- Adam chooses letters $a_i \in \Sigma$
- Eve responds with transitions $\tau_i$ over $a_i$
- Eve wins if $a_0a_1... \notin L(A)$ or $\tau_0\tau_2...$ is accepting.
History-determinism
History-deterministic Pushdown Automaton

\[ \{a^i\$a^i$b^k\mid k \leq \max(i, j)\} \notin \text{DCFL} \]
\{ a^i a^i b^k \mid k \leq \max(i, j) \} \notin \text{DCFL}
What are History-Deterministic automata good for?

Environment

System
What are History-Deterministic automata good for?
What are History-Deterministic automata good for?
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The System wins if \((a_0)^n(b_0)^n(a_1)^n(b_1)^n \ldots\) satisfies the specification.
What are History-Deterministic automata good for?

The System wins if the scheduling is fair.
What are History Deterministic automata good for?

Solving games with winning condition $L(\mathcal{A})$:

- Undecidable for $\mathcal{A}$ a PDA;
What are History Deterministic automata good for?

Solving games with winning condition $L(A)$:

- Undecidable for $A$ a PDA;
- **ExpTime** for $A$ a DPDA;
What are History Deterministic automata good for?

Solving games with winning condition $L(A)$:

- Undecidable for $A$ a PDA;
- $\text{ExpTime}$ for $A$ a DPDA;
- $\text{ExpTime}$ for $A$ a HD-PDA.
Solving games with HD winning conditions

Play is $w = (a_0 b_0) (a_1 b_1) ...$ and run $\rho$ on $w$ in $A$. Eve wins if $\rho$ is accepting.

If $A$ is HD-PDA, then Eve wins whenever she wins the game on $L(A)$. This game has a DPA winning condition, i.e. solvable in ExpTime.
Solving games with HD winning conditions

Play is \( w = (a_0 b_0)(a_1 b_1) \ldots \) and run \( \rho \) on \( w \) in \( A \).

Eve wins if \( \rho \) is accepting.

If \( A \) is HD-PDA, then Eve wins whenever she wins the game on \( L(A) \).

This game has a DPA winning condition, i.e. solvable in ExpTime.
Solving games with HD winning conditions

\[ \text{Play is } w = (a_0, b_0)(a_1, b_1) \ldots \text{ and run } \rho \text{ on } w \text{ in } A. \]

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This game has a DPA winning condition, i.e. solvable in $\text{ExpTime}$. 
Solving games with HD winning conditions

Play is $w = \begin{pmatrix} a_0 \cr b_0 \end{pmatrix} \begin{pmatrix} a_1 \cr b_1 \end{pmatrix} \ldots$ and run $\rho$ on $w$ in $A$. 

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Solving games with HD winning conditions

Play is \( w = \left( a_0 b_0 \right) \left( a_1 b_1 \right) \ldots \) and run \( \rho \) on \( w \) in \( \mathcal{A} \).

Eve wins if \( \rho \) is accepting.

If \( \mathcal{A} \) is HD-PDA, then Eve wins whenever she wins the game on \( L(\mathcal{A}) \).
Solving games with HD winning conditions

Play is \( w = (a_0 b_0) (a_1 b_1) \ldots \) and run \( \rho \) on \( w \) in \( A \).

Eve wins if \( \rho \) is accepting.

If \( A \) is HD-PDA, then Eve wins whenever she wins the game on \( L(A) \).

This game has a \textit{DPA} winning condition, i.e. solvable in \textit{ExpTime}. 
What are history deterministic automata good for?

Solving games with context-free winning conditions:

- Undecidable for pushdown automata
What are history deterministic automata good for?

Solving games with context-free winning conditions:

- Undecidable for pushdown automata
- \text{ExpTime}-complete for history deterministic automata
What are history deterministic automata good for?

Solving games with context-free winning conditions:

- Undecidable for pushdown automata
- \textsc{ExpTime}-complete for history deterministic automata
- By reduction to solving games with deterministic winning condition.
What are history determinism automata good for?

- Solving games without determinism
What are history determinism automata good for?

- Solving games without determinism
- Succinctness
Succinctness of HD-PDA (idea)

\[ c_3 = 000$001$010$011$100$101$110$111 \]
c_3 = 000\$001\$010\$011\$100\$101\$110\$111

L_n = \{ w \in \{0, 1, \$, \#\} \mid w \neq c_n\# \}
Succinctness of HD-PDA (idea)

\[ c_3 = 000$001$010$011$100$101$110$111 \]

\[ L_n = \{ w \in \{0, 1, $, \#\} \mid w \neq c_n\# \} \]

- DPDA is of size \( O(2^n) \).
Succinctness of HD-PDA (idea)

\[ c_3 = 000$001$010$011$100$101$110$111 \]

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- DPDA is of size \(O(2^n)\).
- HD-PDA of size \(O(n)\).
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The bad news
Recognising HD-automata

- Undecidable whether an automaton is HD-PDA
- Undecidable whether a *language* is recognised by a HD-PDA.
What about closure properties?

No closure under:

- union
- intersection
- complementation
- set difference
- homomorphism
History-deterministic Visibly Pushdown Automata

- Recognisable in ExpTime
- Closure properties
- Exponentially more succinct than DVPA.
History-deterministic Visibly Pushdown Automata

- Recognisable in $\text{ExpTime}$
History-deterministic Visibly Pushdown Automata

- Recognisable in \textit{ExpTime}
- Closure properties
History-deterministic Visibly Pushdown Automata

- Recognisable in \( \text{ExpTime} \)
- Closure properties
- Exponentially more succinct than DVPA.
Conclusion

History-deterministic pushdown automata

- More expressive than DPDA
- More succinct than DPDA
- Decidable games and universality
- Poor closure properties and decidability
- HD-VPA to mitigate

Open:

- HD-PDA vs PDA succinctness gap
- Resolver complexity

[Good-for-games $\omega$-pushdown automata. LICS 2020. L., Zimmermann]


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