The algebraic λ-calculus is a conservative extension of the ordinary λ-calculus

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A long story short

Theorem

the algebraic λ-calculus is a conservative extension of the ordinary λ-calculus:

$$\textit{if } M, N \in \Lambda \textit{ and } M \simeq_{\mathbf{A}} N \textit{ then } M \simeq_{\beta} N.$$

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If $\mathbf{A}$ *is positive, then*
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What is the algebraic λ-calculus about?

- λ-calculus with linear combinations of terms
  (β-reduction modulo vector space equations)

- A generic framework for studying various forms of non-determinism
  (plain/counting/probabilistic/quantum/...)

- A language for morphisms in cartesian closed categories of
  (non necessarily linear) maps between *(particular)* vector spaces

The algebraic λ-calculus (V., RTA 2007)

$\Lambda_{\mathbf{A}} \ni M, N, \ldots ::= x \mid \lambda x.M \mid M\,N \mid M + N \mid \mathbf{0} \mid a.M \quad (a \in \mathbf{A}$, some semiring$)$

$$(\lambda x.M)\,N \rightarrow_{\mathbf{A}} M[N/x]$$

$$\begin{aligned} (M+N)\,P &= M\,P + N\,P & \lambda x.(M+N) &= \lambda x.M + \lambda x.N \\ \mathbf{0}\,P &= \mathbf{0} & \lambda x.\mathbf{0} &= \mathbf{0} \\ (a.M)\,P &= a.M\,P & \lambda x.(a.M) &= a.\lambda x.M \end{aligned}$$

+ module equations + contextuality:

$$M \rightarrow_{\mathbf{A}} M' \quad \Longrightarrow \quad a.M + N \rightarrow_{\mathbf{A}} a.M' + N \qquad (a \neq 0)$$

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Theorem (Ehrhard–Regnier, TCS, 2003)

This reduction is confluent.

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- The differential λ-calculus (Ehrhard–Regnier, TCS, 2003) without differentiation

A museum of horrors

$$\begin{array}{lll} \infty_M & := & \mathsf{Fix}(\lambda x.(M + x)) \\ & \to^*_{\mathbf{A}} & M + \infty_M \\ & \to^*_{\mathbf{A}} & nM + \infty_M \end{array}$$

$$\begin{array}{lll} M & = & \frac{1}{2}M + \frac{1}{2}M \\ & \to_{\mathbf{A}} & \frac{1}{2}M + \frac{1}{2}M' \end{array}$$

$$\mathbf{0} = M - M \to_{\mathbf{A}} M' - M$$

$$\mathbf{0} = \infty_M - \infty_M \simeq_{\mathbf{A}} M$$

A museum of horrors

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no normal forms

up to vector space equations, β-equality is unsound!

$$\mathbf{0} = \infty_M - \infty_M \simeq_{\mathbf{A}} M$$

Possible fixes

- Algebraic rewriting on closed normal forms (Arrighi–Dowek, RTA'08).
- Remove the identity $\mathbf{0} = 0.M$ (Valiron, DCM 2010).
- Typing, Church-style
  (we have models: Ehrhard, MSCS, 2005, Valiron, MSCS, 2013, *etc.*).
- V., RTA'07: consider positive coefficients only
  (then confluence implies consistency).

Conservativity in the positive case

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A first conservativity proof (Ehrhard–Regnier, 2003)

Theorem (TeReSe, Exercise 1.3.21.(iii))

If an abstract rewrite system $(A, \to)$ *is a sub-ARS of* $(A', \to')$ *and* $\to'$ *is confluent then* $(A', \simeq')$ *is conservative over* $(A, \simeq)$*.*

Since $\to_{\mathbf{A}}$ extends $\to_\beta$, and $\to_{\mathbf{A}}$ is confluent, $\to_{\mathbf{A}}$ must be conservative.

A first conservativity non-proof (Ehrhard–Regnier, 2003)

Theorem (TeReSe, Exercise 1.3.21.(iii))

If an abstract rewrite system $(A, \to)$ is a sub-ARS of $(A', \to')$ and $\to'$ is confluent then $(A', \simeq')$ is conservative over $(A, \simeq)$.

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But that would work when $-1 \in \mathbf{A}$ too!

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Definition

$(A, \to)$ *is a sub-ARS of* $(A', \to')$ *if:*

- $A \subseteq A'$ *and* $\to \subseteq \to'$,
- *if* $a \to' a'$ *with* $a \in A$ *then* $a \to a'$.

$\to_{\mathbf{A}}$ does not extend $\to_\beta$ in this sense!

Even with positive coefficients: $M \to_{\mathbf{A}} \frac{1}{2}M' + \frac{1}{2}M$

Conservativity of iterated reduction is sufficient

Let $\mathsf{F}(S)$ denote the full parallel reduct of S (fire all redexes simultaneously).

Lemma

If $S \to^n_{\mathbf{A}} S'$ then $S' \to^*_{\mathbf{A}} \mathsf{F}^n(S)$.

TODO

If $M, N \in \Lambda$ and $M \to^*_{\mathbf{A}} N$ then $M \to^*_{\beta} N$.

Theorem

If $M, N \in \Lambda$ and $M \simeq_{\mathbf{A}} N$ then $M \simeq_{\beta} N$.

Proof.

- By confluence, there is P such that $M \to^*_{\mathbf{A}} P$ and $N \to^*_{\mathbf{A}} P$.
- By the Lemma, $P \to^*_{\mathbf{A}} \mathsf{F}^*(N)$ hence $M \to^*_{\mathbf{A}} \mathsf{F}^*(N)$.
- We obtain $M \to^*_{\beta} \mathsf{F}^*(N)$ using TODO.

□

Another conservativity proof (V., RTA'07)

Idea

Extract $N \in \Lambda$ from S such that $M \to_{\mathbf{A}}^* S$: then $M \to_\beta^* N$.

$$\frac{}{x \triangleleft x} \qquad \frac{M \triangleleft S}{\lambda x.M \triangleleft \lambda x.S} \qquad \frac{M \triangleleft S \qquad N \triangleleft T}{M\,N \triangleleft S\,T} \qquad \frac{M \triangleleft S \qquad (a \neq 0)}{M \triangleleft a.S + T}$$

Lemma

Assume $\mathbf{A}$ *is positive. If* $S \to_{\mathbf{A}} S'$ *and* $M' \triangleleft S'$ *then there exists* $M \triangleleft S$ *with* $M \to_\beta M'$.

Proof. Since $\mathbf{A}$ is positive, M' necessarily comes from subterms of S', obtained by reducing subterms of S. □

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Lemma

~~*Assume* $\mathbf{A}$ *is positive. If* $S \rightarrow_{\mathbf{A}} S'$ *and* $M' \lhd S'$ *then there exists* $M \lhd S$ *with* $M \rightarrow_{\beta} M'$.~~

Wrong!

$$S = \Delta\,(M+N) \rightarrow_{\mathbf{A}} (M+N)\,(M+N) \rhd M\,N$$

but we only have $S \rhd \Delta\,M$ and $S \rhd \Delta\,N$

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$$S = \Delta\,(M + N) \to_{\mathbf{A}} (M + N)\,(M + N) \triangleright M\,N$$

but we only have $S \triangleright \Delta\,M$ and $S \triangleright \Delta\,N$

Note that there is no $P \in \Lambda$ *s.t.* $P \to_{\mathbf{A}}^{*} S = \Delta\,(M + N)$.

A *mashup* of β-reductions

Goal

Define $\vdash \subset \Lambda \times \Lambda_{\mathbf{A}}$ such that

$$\Lambda \ni M \to_{\mathbf{A}}^{*} S \;\Rightarrow\; M \vdash S \quad \text{and} \quad M \vdash N \in \Lambda \;\Rightarrow\; M \to_{\beta}^{*} N$$

Mashup

Paste together β-reduction sequences, then continue below constructors.

$$\frac{M \to_{\beta}^{*} x}{M \vdash x} \qquad \frac{M \to_{\beta}^{*} \lambda x.N \qquad N \vdash S}{M \vdash \lambda x.S} \qquad \frac{M \to_{\beta}^{*} N\,P \qquad N \vdash S \qquad P \vdash T}{M \vdash S\,T}$$

$$\frac{}{M \vdash \mathbf{0}} \qquad \frac{M \vdash S \qquad M \vdash T}{M \vdash S + T} \qquad \frac{M \vdash S}{M \vdash a.S}$$

Here it goes

Lemma

If $M \in \Lambda$ *then* $M \vdash M$.

Lemma

If $M \vdash N \in \Lambda$ *then* $M \to_\beta^* N$.

Lemma

If $M \to_\beta M' \vdash S$ *then* $M \vdash S$.

Proof. Easy inductions. □

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Lemma

If $M \to_\beta M' \vdash S$ *then* $M \vdash S$.

Lemma

If $M \vdash S$ *and* $N \vdash T$ *then* $M[N/x] \vdash S[T/x]$.

Proof. Easy induction on the derivation of $M \vdash S$, using the previous Lemma in the variable case. □

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If $M \in \Lambda$ *then* $M \vdash M$.

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Lemma

If $M \vdash S \to_{\mathbf{A}} S'$ *then* $M \vdash S'$.

Proof. Easy induction on S, using the previous Lemma in the redex case. □

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Lemma

If $M \vdash S \to_{\mathbf{A}} S'$ *then* $M \vdash S'$.

Theorem

If $M, M' \in \Lambda$ *and* $M \to_{\mathbf{A}}^* M'$ *then* $M \to_\beta^* M'$.

Proof. $M \vdash M$ and $M \to_{\mathbf{A}}^* M'$ hence $M \vdash M'$ and then $M \to_\beta^* M'$. □

Conclusions

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$$M \vdash S \triangleright N \in \Lambda \;\Rightarrow\; M \to_{\beta}^{*} N$$

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- In fact we have:

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- This can be applied elsewhere: e.g., for the conservativity of reduction of resource terms through the Taylor expansion of λ-terms (Rémy Cerda's talk at TLLA 2023).