The algebraic λ-calculus, 12 years later: a conservativity proof at last

Lionel Vaux Auclair

Institut de Mathématiques de Marseille, Université d'Aix-Marseille, France

Journées Scalp 2019, Lyon, 2019-10-18

Non-determinism in the λ-calculus

$$M, N, \ldots ::= x \mid \lambda x.M \mid M\,N \mid M + N$$

$$\lambda x.M\,N \to_\beta M\,[N/x]$$

and

$$M + N \to_+ M \quad (\text{or } N)$$

Non-determinism in the λ-calculus, contextually

$$M, N, \ldots ::= x \mid \lambda x.M \mid M\,N \mid M + N$$

$$\lambda x.M\,N \to_\beta M\,[N/x]$$

up to:

$$M + N\,P = M\,P + N\,P \qquad\qquad \lambda x.\,(M + N) = \lambda x.M + \lambda x.N$$

somehow implicitly call-by-name

Quantitative non-determinism in the λ-calculus

$$M, N, \ldots ::= x \mid \lambda x.M \mid M\,N \mid M + N \mid 0$$

$$\lambda x.M\,N \to_\beta M\,[N/x]$$

up to:

$$M + N\,P = M\,P + N\,P \qquad \lambda x.\,(M + N) = \lambda x.M + \lambda x.N$$

$$0\,P = 0 \qquad \lambda x.0 = 0$$

somehow implicitly call-by-name

Quantitative non-determinism in the λ-calculus

$$M, N, \ldots ::= x \mid \lambda x.M \mid M\,N \mid M + N \mid 0 \mid a.M \quad (a \in \mathbf{A}, \text{ some semiring})$$

$$\lambda x.M\,N \to_\beta M\,[N/x]$$

up to:

$$M + N\,P = M\,P + N\,P \qquad \lambda x.\,(M + N) = \lambda x.M + \lambda x.N$$

$$0\,P = 0 \qquad \lambda x.0 = 0$$

$$a.M\,P = a.M\,P \qquad \lambda x.\,(a.M) = a.\lambda x.M$$

somehow implicitly call-by-name

Quantitative non-determinism in the λ-calculus

$$M, N, \ldots ::= x \mid \lambda x.M \mid M\,N \mid M + N \mid 0 \mid a.M \quad (a \in \mathbf{A}, \text{ some semiring})$$

$$\lambda x.M\,N \to_\beta M\,[N/x]$$

up to:

$$\begin{aligned} M + N\,P &= M\,P + N\,P \\ 0\,P &= 0 \\ a.M\,P &= a.M\,P \end{aligned} \qquad \begin{aligned} \lambda x.\,(M + N) &= \lambda x.M + \lambda x.N \\ \lambda x.0 &= 0 \\ \lambda x.\,(a.M) &= a.\lambda x.M \end{aligned}$$

up to module equations:

$$M + N = N + M \qquad 0.M = 0 \qquad a.M + b.M = (a + b).M \qquad \ldots$$

Quantitative non-determinism in the λ-calculus

$$M, N, \ldots ::= x \mid \lambda x.M \mid M\,N \mid M + N \mid 0 \mid a.M \quad (a \in \mathbf{A}, \text{ some semiring})$$

$$\lambda x.M\,N \to_\beta M\,[N/x]$$

up to:

$$M + N\,P = M\,P + N\,P \qquad \lambda x.\,(M + N) = \lambda x.M + \lambda x.N$$
$$0\,P = 0 \qquad \lambda x.0 = 0$$
$$a.M\,P = a.M\,P \qquad \lambda x.\,(a.M) = a.\lambda x.M$$

up to module equations:

$$M + N = N + M \qquad 0.M = 0 \qquad a.M + b.M = (a + b).M \qquad \ldots$$

A common framework for non-determinism, probabilistic distributions, quantum stuff, ...

Some issues

Reflexivity

$M = M + 0.N \to_\beta M + 0.N' = M$ $\rightsquigarrow$ no normal form

Some issues

Reflexivity

$M = M + 0.N \to_\beta M + 0.N' = M \leadsto$ no normal form

Solutions:

- Arrighi–Dowek, RTA'08: orient module equations (except for AC), e.g.,

$$0.M \to 0 \qquad a.M + b.M \to (a+b).M \qquad \ldots$$

  - Automatically conservative: reductions from λ-terms are β-reductions.
  - Confluence in this case?
- Ehrhard–Regnier, TCS, 2003: a two-layered syntax with base terms *vs* linear combinations.
  - Confluence is easy.
  - Conservativity in this case?

Some issues

Reflexivity

$M = M + 0.N \to_\beta M + 0.N' = M \rightsquigarrow$ no normal form

Solutions:

- $\cdots$
- Ehrhard–Regnier, TCS, 2003: a two-layered syntax with base terms *vs* linear combinations.
  - Confluence is easy.
  - Conservativity in this case?

$\infty - \infty = \cdots$

Condider $\infty_M := \mathsf{Fix}\,\lambda x.\,(M + x)$ so that $\infty_M \simeq_\beta M + \infty_M$.
Then $0 = \infty_M - \infty_M \simeq_\beta M + \infty_M - \infty_M = M$ *for any term* M!

Some issues

Reflexivity

$M = M + 0.N \to_\beta M + 0.N' = M \rightsquigarrow$ no normal form

Solutions:

- $\cdots$
- Ehrhard–Regnier, TCS, 2003: a two-layered syntax with base terms *vs* linear combinations.
  - Confluence is easy.
  - Conservativity in this case?

$\infty - \infty = \cdots$

Condider $\infty_M := \mathsf{Fix}\,\lambda x.\,(M + x)$ so that $\infty_M \simeq_\beta M + \infty_M$.
Then $0 = \infty_M - \infty_M \simeq_\beta M + \infty_M - \infty_M = M$ *for any term* M!

Solutions:

- Arrighi–Dowek, RTA'08: algebraic rewriting on closed normal forms.
- Typing (Church-style): we have models (Ehrhard, MSCS, 2005, *etc.*).
- V., RTA'07: consider positive coefficients only.

Some issues

Reflexivity

$M = M + 0.N \to_\beta M + 0.N' = M \rightsquigarrow$ no normal form

Solutions:

- $\cdots$
- Ehrhard–Regnier, TCS, 2003: a two-layered syntax with base terms *vs* linear combinations.
  - Confluence is easy.
  - Conservativity in this case?

$\infty - \infty = \cdots$

Condider $\infty_M := \mathsf{Fix}\, \lambda x.\, (M + x)$ so that $\infty_M \simeq_\beta M + \infty_M$.
Then $0 = \infty_M - \infty_M \simeq_\beta M + \infty_M - \infty_M = M$ *for any term* M!

Solutions:

- Arrighi–Dowek, RTA'08: algebraic rewriting on closed normal forms.
- Typing (Church-style): we have models (Ehrhard, MSCS, 2005, *etc.*).
- V., RTA'07: consider positive coefficients only.

Plenty of other nicely weird (or weirdly nice) stuff...

The algebraic λ-calculus (V., RTA'07)

simple terms: $\Lambda_{\mathbf{A}} \ni s, t, \ldots ::= x \mid \lambda x.s \mid s\,T$

terms: $\mathbf{A}\left[\Lambda_{\mathbf{A}}\right] \ni S, T, \ldots ::= \sum_{i=1}^{n} a_i.s_i$

$$\lambda x.S := \sum_{s \in \mathsf{support}(S)} S_s.\lambda x.s \qquad S\,T := \sum_{s \in \mathsf{support}(S)} S_s.s\,T$$

The algebraic λ-calculus (V., RTA'07)

simple terms: $\Lambda_{\mathbf{A}} \ni s, t, \ldots ::= x \mid \lambda x.s \mid s\,T$

terms: $\mathbf{A}[\Lambda_{\mathbf{A}}] \ni S, T, \ldots ::= \sum_{i=1}^{n} a_i.s_i$

$$\lambda x.S := \sum_{s \in \mathsf{support}(S)} S_s.\lambda x.s \qquad S\,T := \sum_{s \in \mathsf{support}(S)} S_s.s\,T$$

$$\frac{}{(\lambda x.s)\ T \to_{\beta_{\mathbf{A}}} s\,[T/x]}$$

$$\frac{s \to_{\beta_{\mathbf{A}}} S'}{\lambda x.s \to_{\beta_{\mathbf{A}}} \lambda x.S'} \qquad \frac{s \to_{\beta_{\mathbf{A}}} S'}{s\,T \to_{\beta_{\mathbf{A}}} S'\,T} \qquad \frac{T \mathrel{\widetilde{\to}}_{\beta_{\mathbf{A}}} T'}{s\,T \to_{\beta_{\mathbf{A}}} s\,T'} \qquad \frac{s \to_{\beta_{\mathbf{A}}} S'}{s\,T \to_{\beta_{\mathbf{A}}} S'\,T}$$

$$\frac{s \to_{\beta_{\mathbf{A}}} S' \qquad a \neq 0}{a.s + T \mathrel{\widetilde{\to}}_{\beta_{\mathbf{A}}} a.S' + T}$$

The algebraic λ-calculus (V., RTA'07)

aka the differential λ-calculus without the differential

simple terms: $\Lambda_{\mathbf{A}} \ni s, t, \ldots ::= x \mid \lambda x.s \mid s\, T$

terms: $\mathbf{A}\left[\Lambda_{\mathbf{A}}\right] \ni S, T, \ldots ::= \sum_{i=1}^{n} a_i.s_i$

$$\lambda x.S := \sum_{s \in \mathsf{support}(S)} S_s.\lambda x.s \qquad S\, T := \sum_{s \in \mathsf{support}(S)} S_s.s\, T$$

$$\frac{}{(\lambda x.s)\ T \to_{\beta_{\mathbf{A}}} s\,[T/x]}$$

$$\frac{s \to_{\beta_{\mathbf{A}}} S'}{\lambda x.s \to_{\beta_{\mathbf{A}}} \lambda x.S'} \qquad \frac{s \to_{\beta_{\mathbf{A}}} S'}{s\, T \to_{\beta_{\mathbf{A}}} S'\, T} \qquad \frac{T \widetilde{\to}_{\beta_{\mathbf{A}}} T'}{s\, T \to_{\beta_{\mathbf{A}}} s\, T'} \qquad \frac{s \to_{\beta_{\mathbf{A}}} S'}{s\, T \to_{\beta_{\mathbf{A}}} S'\, T}$$

$$\frac{s \to_{\beta_{\mathbf{A}}} S' \qquad a \neq 0}{a.s + T \widetilde{\to}_{\beta_{\mathbf{A}}} a.S' + T}$$

The algebraic λ-calculus (V., RTA'07)

aka the differential λ-calculus without the differential

simple terms: $\Lambda_{\mathbf{A}} \ni s, t, \ldots ::= x \mid \lambda x.s \mid s\,T$

terms: $\mathbf{A}\,[\Lambda_{\mathbf{A}}] \ni S, T, \ldots ::= \sum_{i=1}^{n} a_i.s_i$

$$\lambda x.S := \sum_{s \in \mathsf{support}(S)} S_s.\lambda x.s \qquad S\,T := \sum_{s \in \mathsf{support}(S)} S_s.s\,T$$

$$\frac{}{(\lambda x.s)\ T \to_{\beta_{\mathbf{A}}} s\,[T/x]}$$

$$\frac{s \to_{\beta_{\mathbf{A}}} S'}{\lambda x.s \to_{\beta_{\mathbf{A}}} \lambda x.S'} \qquad \frac{s \to_{\beta_{\mathbf{A}}} S'}{s\,T \to_{\beta_{\mathbf{A}}} S'\,T} \qquad \frac{T \widetilde{\to}_{\beta_{\mathbf{A}}} T'}{s\,T \to_{\beta_{\mathbf{A}}} s\,T'} \qquad \frac{s \to_{\beta_{\mathbf{A}}} S'}{s\,T \to_{\beta_{\mathbf{A}}} S'\,T}$$

$$\frac{s \to_{\beta_{\mathbf{A}}} S' \qquad a \neq 0}{a.s + T \widetilde{\to}_{\beta_{\mathbf{A}}} a.S' + T}$$

Theorem (Ehrhard–Regnier, TCS, 2003)

$\widetilde{\to}_{\beta_{\mathbf{A}}}$ *is confluent*

Conservativity

Theorem

If $\mathbf{A}$ *is positive, then the algebraic* λ*-calculus is a conservative extension of the ordinary* λ*-calculus:*

$$\text{if } M, N \in \Lambda \text{ and } M \simeq_{\beta_{\mathbf{A}}} N \text{ then } M \simeq_{\beta} N.$$

Conservativity

We say $\mathbf{A}$ is *positive* if: $a + b = 0$ implies $a = b = 0$.

Theorem

If $\mathbf{A}$ *is positive, then the algebraic* λ*-calculus is a conservative extension of the ordinary* λ*-calculus:*

$$\textit{if } M, N \in \Lambda \textit{ and } M \simeq_{\beta_{\mathbf{A}}} N \textit{ then } M \simeq_{\beta} N.$$

A first conservativity proof

Theorem (TeReSe, Exercise 1.3.21.(iii))

If an abstract rewrite system $(A, \to)$ is an extension of $(A', \to')$ and $\to'$ is confluent then $(A', \to')$ is conservative over $(A, \to)$.

Since $\widetilde{\to}_{\beta_{\mathbf{A}}}$ extends $\to_\beta$, and $\widetilde{\to}_{\beta_{\mathbf{A}}}$ is confluent, $\widetilde{\to}_{\beta_{\mathbf{A}}}$ must be conservative (Ehrhard–Regnier, TCS, 2003).

A first conservativity proof

Theorem (TeReSe, Exercise 1.3.21.(iii))

If an abstract rewrite system $(A, \to)$ is an extension of $(A', \to')$ and $\to'$ is confluent then $(A', \to')$ is conservative over $(A, \to)$.

Since $\widetilde{\to}_{\beta_{\mathbf{A}}}$ extends $\to_\beta$, and $\widetilde{\to}_{\beta_{\mathbf{A}}}$ is confluent, $\widetilde{\to}_{\beta_{\mathbf{A}}}$ must be conservative (Ehrhard–Regnier, TCS, 2003).
But wait…

A first conservativity proof

Theorem (TeReSe, Exercise 1.3.21.(iii))

If an abstract rewrite system $(A, \to)$ *is an extension of* $(A', \to')$ *and* $\to'$ *is confluent then* $(A', \to')$ *is conservative over* $(A, \to)$.

Since $\widetilde{\to}_{\beta_{\mathbf{A}}}$ extends $\to_\beta$, and $\widetilde{\to}_{\beta_{\mathbf{A}}}$ is confluent, $\widetilde{\to}_{\beta_{\mathbf{A}}}$ must be conservative (Ehrhard–Regnier, TCS, 2003).
But wait... that would work when $-1 \in \mathbf{A}$ too.

A first conservativity non-proof

Theorem (TeReSe, Exercise 1.3.21.(iii))

If an abstract rewrite system $(A, \to)$ *is an extension of* $(A', \to')$ *and* $\to'$ *is confluent then* $(A', \to')$ *is conservative over* $(A, \to)$*.*

Since $\widetilde{\to}_{\beta_{\mathbf{A}}}$ extends $\to_\beta$, and $\widetilde{\to}_{\beta_{\mathbf{A}}}$ is confluent, $\widetilde{\to}_{\beta_{\mathbf{A}}}$ must be conservative (Ehrhard–Regnier, TCS, 2003).
But wait… that would work when $-1 \in \mathbf{A}$ too.

Definition

$(A, \to)$ *extends* $(A', \to')$ *if:*

- $A \subseteq A'$,
- $\to = \to' \cap (A \times A)$
- A *is closed under* $\to'$*, i.e.* $a \in A$ *and* $a \to' a'$ *implies* $a' \in A$*.*

$\widetilde{\to}_{\beta_{\mathbf{A}}}$ does not extend $\to_\beta$ in this sense!

Another conservativity proof (V., RTA'07)

$$\frac{}{x \lhd x} \qquad \frac{M \lhd s}{\lambda x.M \lhd \lambda x.s} \qquad \frac{M \lhd s \qquad N \lhd T}{M\,N \lhd s\,T} \qquad \frac{M \lhd s \in \mathsf{support}\,(S)}{M \lhd S}$$

Lemma

Assume $\mathbf{A}$ *is positive. If* $S \stackrel{\sim}{\rightarrow}_{\beta_{\mathbf{A}}} S'$ *and* $M' \lhd S'$ *then there exists* $M \lhd S$ *with* $M \rightarrow_\beta M'$.

Proof. Idea: since $\mathbf{A}$ is positive, M' necessarily comes from subterms of S', obtained by reducing subterms of S. □

Another conservativity non-proof (V., RTA'07)

$$\frac{}{x \lhd x} \qquad \frac{M \lhd s}{\lambda x.M \lhd \lambda x.s} \qquad \frac{M \lhd s \qquad N \lhd T}{M\,N \lhd s\,T} \qquad \frac{M \lhd s \in \mathsf{support}\,(S)}{M \lhd S}$$

Lemma

Assume $\mathbf{A}$ *is positive. If* $S \widetilde{\rightarrow}_{\beta_{\mathbf{A}}} S'$ *and* $M' \lhd S'$ *then there exists* $M \lhd S$ *with* $M \rightarrow_{\beta} M'$*.*

Proof. Idea: since $\mathbf{A}$ is positive, M' necessarily comes from subterms of S', obtained by reducing subterms of S. □

Wrong!

$$\Delta\ (M+N) \rightarrow_{\beta_{\mathbf{A}}} (M+N)\ (M+N) \rhd M\,N.$$

A partial conservativity proof (V., CSL'17)

Let $\mathsf{F}(S)$ denote the full parallel reduct of S (fire all redexes simultaneously).

Lemma

If $S \to^n_{\beta_{\mathbf{A}}} S'$ then $S' \to^*_{\beta_{\mathbf{A}}} \mathsf{F}^n(S)$.

Theorem

Assume $\mathbf{A}$ is positive. If $M \simeq_{\beta_{\mathbf{A}}} N$ and M is β-normalizable then $N \to^*_{\beta} \mathsf{NF}(M)$.

Proof.

- By confluence, $N \to^*_{\beta_{\mathbf{A}}} \mathsf{NF}(M)$.
- By the previous Lemma, $\mathsf{NF}(M) \to^*_{\beta_{\mathbf{A}}} \mathsf{F}^*(N)$.
- By positivity, $\mathsf{NF}(M) = L^*(N)$.
- We always have $N \to^*_{\beta} L(N)$.

□

Yet another partial conservativity proof (V., CSL'17)

Let $\Theta(S)$ denote the Taylor expansion of S.

Lemma

If $M \in \Lambda$, $\Theta(M)$ is normalizable and $\mathsf{NF}(\Theta(M)) = \Theta(\mathsf{BT}(M))$.

Lemma

Assume $\mathbf{A}$ is positive. If $S \to_{\beta_{\mathbf{A}}} S'$ then $\Theta(S) \to \Theta(S')$.

Lemma

If τ is normalizable and $\tau \to \tau'$ then τ' is normalizable and $\mathsf{NF}(\tau) = \mathsf{NF}(\tau')$.

Corollary

Assume $\mathbf{A}$ is positive. If $M \simeq_{\beta_{\mathbf{A}}} N$ then $\mathsf{BT}(M) = \mathsf{BT}(N)$.

A simple and (hopefully) correct proof

J.w.w. Emma Kerinec (now at Paris 13).

Idea

Define a variant $\prec$ of $\triangleleft$, such that $M \prec S \widetilde{\to}_{\beta_{\mathbf{A}}} S'$ implies $M \prec S'$.

- Interestingly, the definition focusses on β-expansion rather than $\widetilde{\to}_{\beta_{\mathbf{A}}}$.
- The technique rings some bells…

A relation between λ-terms and algebraic λ-terms

$$\frac{M \to_\beta^* x}{M \prec x}$$

$$\frac{M \to_\beta^* \lambda x.N \qquad N \prec s}{M \prec \lambda x.s}$$

$$\frac{M \to_\beta^* N\,P \qquad N \prec s \qquad P \mathrel{\widehat{\precsim}} T}{M \prec s\,T}$$

$$\frac{M \prec s \text{ for each } s \in \mathsf{support}\,(S)}{M \mathrel{\widehat{\precsim}} S}$$

Admissible rules (=lemmas)

$$\frac{M \to_\beta^* \lambda x.N \qquad N \mathrel{\widehat{\precsim}} S}{M \mathrel{\widehat{\precsim}} \lambda x.S}$$

$$\frac{M \to_\beta^* N\,P \qquad N \mathrel{\widehat{\precsim}} S \qquad P \mathrel{\widehat{\precsim}} T}{M \mathrel{\widehat{\precsim}} S\,T}$$

Here it goes

Lemma

If $M \in \Lambda$ *then* $M \prec M$.

Proof. Easy induction on M. $\square$

Here it goes

Lemma

If $M \in \Lambda$ *then* $M \prec M$.

Lemma

If $M \prec N \in \Lambda$ *then* $M \to_\beta^* N$.

Proof. Easy induction on the derivation of $M \prec N$. □

Here it goes

Lemma

If $M \in \Lambda$ *then* $M \prec M$.

Lemma

If $M \prec N \in \Lambda$ *then* $M \to_\beta^* N$.

Lemma

If $M \to_\beta M' \mathrel{\widehat{\precsim}} S$ *then* $M \mathrel{\widehat{\precsim}} S$.

Proof. Easy induction on the derivation of $M' \mathrel{\widehat{\precsim}} S$. □

Here it goes

Lemma

If $M \in \Lambda$ *then* $M \prec M$.

Lemma

If $M \prec N \in \Lambda$ *then* $M \to_\beta^* N$.

Lemma

If $M \to_\beta M' \mathrel{\hat{\prec}} S$ *then* $M \mathrel{\hat{\prec}} S$.

Lemma

If $M \mathrel{\hat{\prec}} S$ *and* $N \mathrel{\hat{\prec}} T$ *then* $M\,[N/x] \mathrel{\hat{\prec}} S\,[T/x]$.

Proof. Easy induction on the derivation of $M \mathrel{\hat{\prec}} S$, using the previous Lemma in the variable case. □

Here it goes

Lemma

If $M \in \Lambda$ *then* $M \prec M$.

Lemma

If $M \prec N \in \Lambda$ *then* $M \to_\beta^* N$.

Lemma

If $M \to_\beta M' \mathrel{\widehat{\precsim}} S$ *then* $M \mathrel{\widehat{\precsim}} S$.

Lemma

If $M \mathrel{\widehat{\precsim}} S$ *and* $N \mathrel{\widehat{\precsim}} T$ *then* $M[N/x] \mathrel{\widehat{\precsim}} S[T/x]$.

Lemma

If $M \mathrel{\widehat{\precsim}} S \mathrel{\widetilde{\rightarrow}_{\tilde{\beta}}} S'$ *then* $M \mathrel{\widehat{\precsim}} S'$.

Proof. Easy induction on the derivation of $S \mathrel{\widetilde{\rightarrow}_{\tilde{\beta}}} S'$, using the previous Lemma in the redex case. □

Here it goes

Lemma

If $M \in \Lambda$ *then* $M \prec M$.

Lemma

If $M \prec N \in \Lambda$ *then* $M \to_\beta^* N$.

Lemma

If $M \to_\beta M' \mathrel{\widehat{\precsim}} S$ *then* $M \mathrel{\widehat{\precsim}} S$.

Lemma

If $M \mathrel{\widehat{\precsim}} S$ *and* $N \mathrel{\widehat{\precsim}} T$ *then* $M[N/x] \mathrel{\widehat{\precsim}} S[T/x]$.

Lemma

If $M \mathrel{\widehat{\precsim}} S \mathrel{\widetilde{\to}_{\tilde{\beta}}} S'$ *then* $M \mathrel{\widehat{\precsim}} S'$.

Theorem

If $M \mathrel{\widetilde{\to}_{\tilde{\beta}}^*} M'$ *then* $M \to_\beta^* M'$.

Conclusions

- Of course, this is the well known */please help us find the reference/* technique.

Conclusions

- Of course, this is the well known */please help us find the reference/* technique.
- Maybe this can be applied elsewhere (e.g., for the conservativity of reduction on Taylor expansion).