Under consideration for publication in Math. Struct. in Comp. Science

The Algebraic Lambda-Calculus

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Received 2 May 2008; Revised 23 May 2009

We introduce an extension of the pure lambda-calculus by endowing the set of terms with a structure of vector space, or more generally of module, over a fixed set of scalars. Terms are moreover subject to identities similar to usual pointwise definition of linear combinations of functions with values in a vector space. We then study a natural extension of beta-reduction in this setting: we prove it is confluent, then discuss consistency and conservativity over the ordinary lambda-calculus. We also provide normalization results for a simple type system.

1. Introduction

Preliminary Definitions and Notations. Recall that a *rig* (or "semiring with zero and unity") is the same thing as a unital ring, without the condition that every element admits an additive inverse. Let $\mathsf{R} = (\mathsf{R}, +, 0, \times, 1)$ be a rig: $(\mathsf{R}, +, 0)$ is a commutative monoid, $(\mathsf{R}, \times, 1)$ is a monoid, $\times$ is distributive over $+$ and 0 is absorbing for $\times$. We write $\mathsf{R}^\bullet$ for $\mathsf{R} \setminus \{0\}$. We denote by letters a, b, c the elements of R, and say that R is *positive* if, for all $a, b \in \mathsf{R}$, $a + b = 0$ implies $a = 0$ and $b = 0$. An example of positive rig is $\mathbf{N}$, the set of natural numbers, with usual operations.

A module over rig R, or R-*module*, is defined in the same way as a unital module over a ring, again without the condition that every element admits an additive inverse. For all set $\mathcal{X}$, the set of formal finite linear combinations of elements of $\mathcal{X}$ with coefficients in R is the free R-module over $\mathcal{X}$, which we denote by $\mathsf{R}\langle\mathcal{X}\rangle$.

Linearity in the λ-Calculus. Girard's linear logic (Gir87), by decomposing intuitionistic implication, made the computational concept of linearity prominent, while relating it with the usual algebraic notion. A program is said to be linear if it uses its argument exactly once. This vague idea can be made more precise, by defining which subterms of a term u are in *linear position* in u:

— in a term which is only a variable x, that occurrence of variable is in linear position;

† Partially supported by the French ANR project *Curry-Howard for concurrency* (CHOCO).

— in an abstraction $u = \lambda x\, s$, the subterms in linear position in u are those of the abstracted subterm s, and u itself;
— in an application $u = (s)\, t$, the subterms in linear position in u are those of the function subterm s, and u itself.

In particular, application is linear in the function but not in the argument. This is to be related with head reduction and memory management: those subterms that are in linear position are evaluated exactly once in the head reduction, they are not copied nor discarded.

Algebraic linearity is generally thought of as commutation with sums. It is well known that the space of all functions from some set to some fixed R-module is itself an R-module, with operations on functions defined pointwise: for instance, the sum of two functions is defined by $(f + g)(x) = f(x) + g(x)$. In (Ehr01) and (Ehr05), Ehrhard introduced denotational models of linear logic where formulas are interpreted as particular vector spaces or modules and proofs corresponding to λ-terms are interpreted as analytic functions defined by power series on these spaces: this is the basic idea of Girard's quantitative semantics (Gir88). This not only guided the study of differentiation in λ-calculus by Ehrhard and Regnier in (ER03), but also offered serious grounding to endow the set of terms with a structure of vector space, or of R-module, where R is a rig: one can form linear combinations of terms, subject to the following two identities:

$$\lambda x \left(\sum_{i=1}^{n} a_i s_i \right) = \sum_{i=1}^{n} a_i \lambda x\, s_i \tag{1}$$

and

$$\left(\sum_{i=1}^{n} a_i s_i \right) u = \sum_{i=1}^{n} a_i (s_i)\, u \tag{2}$$

for all linear combination $\sum_{i=1}^{n} a_i s_i$ of terms. We recover the fact that application is linear in the function and not in the argument, in accordance with the computational notion of linearity.

Reducing Linear Combinations of λ-terms. Apart from differentiation, one important feature of the calculus of (ER03) is the way β-reduction is extended to such linear combinations of terms. Among terms, some are considered simple: they contain no sum in linear position, so that nor (1) nor (2) applies; hence they are intrinsically not sums. These form a basis of the R-module of terms. Reduction $\to$ is then the least contextual relation such that: if s is a simple term, then

$$(\lambda x\, s)\, t \to s\, [t/x] \tag{3}$$

and, if $a \in \mathsf{R}^\bullet$ is a non-zero scalar,

$$s \to s' \text{ implies } as + t \to as' + t \ . \tag{4}$$

Since every ordinary λ-term can be viewed as a simple term, (3) extends usual β-reduction. The requirement that s is simple in (3) and (4), together with the condition

$a \neq 0$ in (4), ensure $\to$ actually reduces something, so that reduction is not trivially reflexive.

Although the previous definition might seem contorted, it is technically efficient. For instance, it is particularly well suited for proving confluence via usual Tait–Martin-Löf technique: introduce a parallel version $\rightrightarrows$ of $\to$ such that $\to \subseteq \rightrightarrows \subseteq \to^*$, and prove that $\rightrightarrows$ enjoys the diamond property. Here $\rightrightarrows$ is reflexive and has the following behaviour on linear combinations of terms:

$$\sum_{i=1}^{n} a_i s_i \rightrightarrows \sum_{i=1}^{n} a_i s'_i \text{ as soon as, for all } i,\ s_i \rightrightarrows s'_i \text{ and } s_i \text{ is simple.} \tag{5}$$

Assuming $s \rightrightarrows s' \rightrightarrows s''$ are simple terms, we have $s + s' \rightrightarrows 2s'$ and $s + s' \rightrightarrows s + s''$: then (5) allows to close that pair of reductions by $2s' \rightrightarrows s' + s''$ and $s + s'' \rightrightarrows s' + s''$. This would not hold if we had forced the s_i's in (5) to be distinct simple terms — that condition would amount to reduce each element of the base of simple terms, in parallel, which may seem a natural choice at first.

Collapse. In (Vau07a), however, the author proved that the above higher-order rewriting of linear combinations collapses as soon as the rig of scalars admits negative elements: if $-1 \in \mathsf{R}$ (so that $1 + (-1) = 0$), then for all terms s and t, $s \to^* t$. This should not be a surprise, since in that case the system involves both negative numbers and potential infinity through arbitrary fixed points. Indeed, take Θ a fixpoint operator of the λ-calculus, such that $(\Theta)\, s \to^* (s)\, (\Theta)\, s$ for all λ-term s. Write ∞_s for $(\Theta)\, \lambda x\, (s + x)$; then $\infty_s \to^* s + \infty_s$, hence ∞_s stands for an infinite amount of s. We get:

$$s = s + \infty_s - \infty_s + \infty_t - \infty_t \to^* s - s + t = t\ .$$

Also, if one can consider fractions of scalars, strong normalizability holds only for normal terms: assume $s \to s'$ and R contains dyadic rationals; then

$$s \ = \ \frac{1}{2}s + \frac{1}{2}s \ \to \ \frac{1}{2}s + \frac{1}{2}s' \ \to \ \frac{1}{4}s + \frac{3}{4}s' \ \to \ \cdots$$

Both these failures indicate that much care is needed when dealing with linear combinations of λ-terms: these make the identity of terms very intricate, much more so than plain α-equivalence, so that its interaction with higher-order rewriting becomes tricky. As a result, although the problem about normalizability was well noted in (ER03), the collapse of reduction in presence of negative coefficients eluded the authors of that paper. In the present contribution, we give a syntactic framework for the study of linear combinations of terms, which aims to be more rigorous and formal than that developed in (ER03) or (Vau07a): in particular, we put much care in developping an explicit implementation of the R-module of terms. Also, we do not consider differentiation nor classical control operators, and only focus on the algebraic structure of terms and the interaction between coefficients and reduction. We call the obtained system the *algebraic λ-calculus*.

Contributions. In section 2, we formalize the definition of the R-module of terms, validating identities (1) and (2), and we introduce the key notion of canonical forms. We also compare this presentation to that of (ER03): terms *à la* Ehrhard–Regnier are just canonical forms of terms in our setting. This is an important part of the present work, which we hope sheds new light on the structure the R-module of terms. In section 3 we define reduction, using rule (4) in the case of a sum, and discuss conservativity w.r.t. ordinary β-reduction. Section 4 presents a Curry-style simple type system for the algebraic λ-calcul. We prove subject reduction holds iff the rig of scalars is positive. In section 5, we discuss necessary conditions for strong normalization of typed terms to hold; we refine these to sufficient conditions and generalize the proof of strong normalization of differential λ-calculus by Ehrhard and Regnier (ER03). We conclude by discussing possible other approaches in section 6.

About Previous Works. Most of the results of this paper were already present in (Vau07a) or even (ER03), sometimes in a weaker form. In those two previous works, however, the focus was on differentiation and the presence of linear combinations of terms and their effects on reduction were considered of marginal interest. As we stated before, this may in particular explain why some of the problems we insist on in this paper were put aside in (ER03). The material of sections 2 and 3 was the subject of the RTA'07 conference extended abstract (Vau07b). Although a very brief outline of a preliminary version of section 5 was given in that last paper, the normalization results of the present article are completely new, in that they strictly generalize those of (Vau07a).

2. Linear Combinations of Terms

In this section, we introduce the set of terms of the algebraic λ-calculus in several steps. First we give a grammar of terms, on which we define α-equivalence and substitution as in Krivine's (Kri90). Then we define a notion of algebraic equality on these terms: this is given by an equivalence relation $\triangleq$ on terms such that the associated quotient set is an R-module, moreover validating identities (1) and (2). The elements of this quotient set are the objects of the algebraic λ-calculus. We then introduce canonical forms of terms as distinguished elements of $\triangleq$-equivalence classes. We show this construction encompasses the abstract presentation by Ehrhard and Regnier in (ER03), based on an increasing sequence of quotients.

2.1. *Raw Terms*

Let be given a denumerable set $\mathcal{V}$ of variables. We use letters among x, y, z to denote variables.

Definition 2.1. The language L^0_R of the *raw terms* of the algebraic λ-calculus over R (denoted by capital letters L, M, N) is given by the following grammar:

$$M, N, \ldots ::= x \mid \lambda x\, M \mid (M)\, N \mid \mathbf{0} \mid aM \mid M + N \ .$$

Definition 2.2. We define free variables of terms as follows:
— variable x is free in term y if $x = y$;
— variable x is free in $\lambda y\, M$ if $x \neq y$ and x is free in M;

— variable x is free in $(M)\,N$ if x is free in M or in N;
— variable x is free in aM if x is free in M;
— variable x is free in term $M + N$ if x is free in M or in N.

In particular, no variable occurs free in term $\mathbf{0}$. Notice however that, by the previous definition, aM might have free variables even if $a = 0$: as far as raw terms are concerned, $0M$ is not the same as $\mathbf{0}$.

From this definition of free variables, we derive α-equivalence (denoted by $\sim$) as in (Kri90). We will always consider raw terms up-to α-equivalence. More formally:

Definition 2.3. The set L_R of the raw terms of the algebraic λ-calculus over R is the quotient set $\mathsf{L}^0_\mathsf{R}/{\sim}$.

Again, we derive the definition of substitution following that in (Kri90). We write $M\,[N/x]$ for the (capture-avoiding) substitution of N for x in M. More generally, if $x_1, \ldots, x_n$ are distinct variables and $N_1, \ldots, N_n$ are terms, we write $M\,[N_1, \ldots, N_n/x_1, \ldots, x_n]$ for the simultaneous capture avoiding substitution of each N_i for each x_i in M. We obtain the following variants of definitions and properties from (Kri90).

Proposition 2.4. For all terms $M, N_1, \ldots, N_n, L_1, \ldots, L_p$ and all distinct variables $x_1, \ldots, x_n,\ y_1, \ldots, y_p$,

$$\begin{array}{ll} & M\,[N_1, \ldots, N_n/x_1, \ldots, x_n]\,[L_1, \ldots, L_p/y_1, \ldots, y_p] \\ \sim & M\,[N'_1, \ldots, N'_n, L_1, \ldots, L_p/x_1, \ldots, x_n, y_1, \ldots, y_p] \end{array}$$

where $N'_i = N_i\,[L_1, \ldots, L_p/y_1, \ldots, y_p]$.

Definition 2.5. A binary relation r on raw terms is said to be *contextual* if it satisfies the following conditions:

— $x \mathrel{\mathrm{r}} x$;
— $\lambda x\, M \mathrel{\mathrm{r}} \lambda x\, M'$ as soon as $M \mathrel{\mathrm{r}} M'$;
— $(M)\,N \mathrel{\mathrm{r}} (M')\,N'$ as soon as $M \mathrel{\mathrm{r}} M'$ and $N \mathrel{\mathrm{r}} N'$;
— $\mathbf{0} \mathrel{\mathrm{r}} \mathbf{0}$;
— $aM \mathrel{\mathrm{r}} aM'$ as soon as $M \mathrel{\mathrm{r}} M'$;
— $M + N \mathrel{\mathrm{r}} M' + N'$ as soon as $M \mathrel{\mathrm{r}} M'$ and $N \mathrel{\mathrm{r}} N'$.

This notion of contextual relation is the analogue of a λ-compatible relation in (Kri90). In particular, a binary relation r is contextual iff it is reflexive and:

— $\lambda x\, M \mathrel{\mathrm{r}} \lambda x\, M'$ as soon as $M \mathrel{\mathrm{r}} M'$;
— $(M)\,N \mathrel{\mathrm{r}} (M')\,N'$ as soon as $M \mathrel{\mathrm{r}} M'$ and $N \mathrel{\mathrm{r}} N'$;
— $aM \mathrel{\mathrm{r}} aM'$ as soon as $M \mathrel{\mathrm{r}} M'$;
— $M + N \mathrel{\mathrm{r}} M' + N'$ as soon as $M \mathrel{\mathrm{r}} M'$ and $N \mathrel{\mathrm{r}} N'$.

Proposition 2.6. If r is a contextual relation, then $M\,[N/x] \mathrel{\mathrm{r}} M\,[N'/x]$ as soon as $N \mathrel{\mathrm{r}} N'$.

Again, this result is only an obvious variant of that of (Kri90).

2.2. *The Module of Terms*

We introduce the actual algebraic content of the calculus by defining an equivalence relation $\triangleq$ encompassing usual identities between linear combinations, together with (1) and (2).

Definition 2.7. *Algebraic equality* $\triangleq$ is defined on raw terms as the least contextual equivalence relation such that the following identities hold:
— axioms of commutative monoid:

$$\mathbf{0} + M \triangleq M \tag{6a}$$
$$(M + N) + L \triangleq M + (N + L) \tag{6b}$$
$$M + N \triangleq N + M \tag{6c}$$

— axioms of module over rig R:

$$a(M + N) \triangleq aM + aN \tag{7a}$$
$$aM + bM \triangleq (a + b)M \tag{7b}$$
$$a(bM) \triangleq (ab)M \tag{7c}$$
$$1M \triangleq M \tag{7d}$$
$$0M \triangleq \mathbf{0} \tag{7e}$$
$$a\mathbf{0} \triangleq \mathbf{0} \tag{7f}$$

— linearity in the λ-calculus:

$$\lambda x\, \mathbf{0} \triangleq \mathbf{0} \tag{8a}$$
$$\lambda x\, (aM) \triangleq a\, (\lambda x\, M) \tag{8b}$$
$$\lambda x\, (M + N) \triangleq \lambda x\, M + \lambda x\, N \tag{8c}$$
$$(\mathbf{0})\, L \triangleq \mathbf{0} \tag{8d}$$
$$(aM)\, L \triangleq a\, ((M)\, L) \tag{8e}$$
$$(M + N)\, L \triangleq (M)\, L + (N)\, L \tag{8f}$$

We call *algebraic λ-terms* the elements of $\mathsf{L}_\mathsf{R}/\triangleq$, i.e. the $\triangleq$-classes of raw terms. If $M \in \mathsf{L}_\mathsf{R}$, we write $\underline{M}$ for its $\triangleq$-class.

Notice that identity (7f) could be removed, as it is derived from (7e) and (7c). Identities (8a) through (8c) subsume (1) and identities (8d) through (8f) subsume (2). Then the quotient set $\mathsf{L}_\mathsf{R}/\triangleq$ is an R-module validating (1) and (2).

Definition 2.8. For all $M_1, \ldots, M_n \in \mathsf{L}_\mathsf{R}$, we write $M_1 + \cdots + M_n$ or even $\sum_{i=1}^n M_i$ for the term $M_1 + (\cdots + M_n)$ (or $\mathbf{0}$ if $n = 0$).

One might think of a raw term $M \in \mathsf{L}_\mathsf{R}$ as a *writing* of its $\triangleq$-class, which is an element of the R-module $\mathsf{L}_\mathsf{R}/\triangleq$. Among raw terms, some should be distinguished as canonical writings. More precisely, we want to make the following statement meaningful: every term $M \in \mathsf{L}_\mathsf{R}$ can be uniquely written as $M \triangleq \sum_{i=1}^n a_i s_i$ where the s_i's are pairwise distinct *base elements* and the a_i's are non zero.

A good candidate for such a canonical base is obtained as follows:

— all the identities in groups of equations (6) (7) and (8), except (6c), can be oriented from left to right to form a rewrite system;
— raw terms which are normal in this rewrite system, and are of the shape x, $\lambda x\, M$ or $(M)\, N$, can be considered as base elements (they are not sums);
— every $M \in \mathsf{L_R}$ has a normal form in this system, which can be written as a linear combination of base terms.

Notice however that a normal form in this system need not be canonical: consider, e.g., $x + y + x$. The problem is of course that we left out commutativity: adding (6c) would break the very notion of normal form. Rewriting up to commutativity, or up to associativity and commutativity, is a notable trend in rewriting theory, with well-established litterature: let's just cite (PS81). Even closer to our subject, Arrighi and Dowek proposed in (AD05) an associative–commutative rewrite system implementing a computational notion of vector space, which is very close to what we have just outlined.

In the current setting, however, our focus is on precising the syntax of the algebraic λ-calculus: we are only interested in the definition of canonical forms and base elements. Hence we do not fully reproduce such a rewrite-theoretic development. We rather extend our notion of equality of terms *a minima*, so that the order of summands in $\sum_{i=1}^{n} M_i$ no longer matters. As far as syntax is concerned, this is quite benign. Moreover, the reduction of the algebraic λ-calculus, to be defined in section 3, is introduced as a relation on $\mathsf{L_R}/\triangleq$: associativity and commutativity will be dissolved in $\triangleq$.

Definition 2.9. *Permutative equality* $\equiv\ \subseteq \mathsf{L_R} \times \mathsf{L_R}$ is the least contextual equivalence relation such that, $\sum_{i=1}^{n} M_i \equiv \sum_{i=1}^{n} M_{f(i)}$ holds, for all $M_1, \dots, M_n \in \mathsf{L_R}$ and all permutation f of $\{1, \dots, n\}$.

Since free variables of a sum do not depend on the order of the summands, $\equiv$ preserves free variables.

Definition 2.10. We write Λ_R for the quotient set $\mathsf{L_R}/\equiv$, and we call *permutative terms* the elements of Λ_R.

Proposition 2.11. Substitution is well defined on Λ_R: if $M, M' \in \mathsf{L_R}$ are such that $M \equiv M'$ and, for all $i \in \{1, \dots, n\}$, $N_i, N_i{}' \in \mathsf{L_R}$ are such that $N_i \equiv N_i'$, then $M\,[N_1, \dots, N_n/x_1, \dots, x_n] \equiv M'\,[N_1', \dots, N_n'/x_1, \dots, x_n]$ for all pairwise distinct variables $x_1, \dots, x_n$.

Except when stated otherwise, we will use the same notation for a raw term M and its $\equiv$ class, and use them interchangeably. This is harmless in general: the properties we consider are all invariant under $\equiv$ and we define functions on Λ_R by induction on raw terms, compatibility with $\equiv$ being obvious.

Notice already that algebraic equality subsumes permutative equality on raw terms, so that $\triangleq$ is well defined on Λ_R and $(\mathsf{L_R}/\triangleq) = (\Lambda_\mathsf{R}/\triangleq)$.

2.3. *Canonical Forms*

We can now define canonical forms of terms as particular permutative terms such that every class in $\Lambda_{\mathsf{R}}/\triangleq$ contains exactly one canonical element.

Definition 2.12. We define the set $\mathsf{C}_{\mathsf{R}} \subset \Lambda_{\mathsf{R}}$ of *canonical terms* (denoted by capital letters S, T, U, V, W) and the set $\mathsf{B}_{\mathsf{R}} \subset \mathsf{C}_{\mathsf{R}}$ of *base terms* (denoted by small letters s, t, u, v, w) by mutual induction as follows:

— any variable x is a base term;
— let $x \in \mathcal{V}$ and s a base term, then $\lambda x\, s$ is a base term;
— let s a base term and T a canonical term, then $(s)\,T$ is a base term;
— let $a_1, \dots, a_n \in \mathsf{R}^\bullet$ and $s_1, \dots, s_n$ pairwise distinct base terms, then $\sum_{i=1}^n a_i s_i$ is a canonical term.

The reader should easily get the intuition that for all canonical terms $S, T \in \mathsf{C}_{\mathsf{R}}$, $S \triangleq T$ iff $S = T$ (a formal proof of this result follows, as a corollary of Theorem 2.17). Mapping s to the "singleton" $1s$ defines an injection from base terms into canonical terms.

Definition 2.13. We define the *height* of base terms and canonical terms by mutual induction:

— $\mathrm{h}(x) = 1$;
— $\mathrm{h}(\lambda x\, s) = 1 + \mathrm{h}(s)$;
— $\mathrm{h}((s)\,T) = 1 + \max(\mathrm{h}(s), \mathrm{h}(T))$;
— $\mathrm{h}(\sum_{i=1}^n a_i s_i) = \max_{1 \le i \le n}(\mathrm{h}(s_i))$ (which is 0 iff $n = 0$).

Definition 2.14. Let $M = \sum_{i=1}^n a_i s_i \in \Lambda_{\mathsf{R}}$ be a linear combination of base terms, not necessarily canonical. For all base term s, we call *coefficient of s in M* the scalar $\sum_{1 \le i \le n,\, s_i = s} a_i$ (the sum of those a_i's such that $s_i = s$), which we denote by $M_{(s)}$. Then we define $\mathsf{cansum}\,(M) \in \mathsf{C}_{\mathsf{R}}$ by:

$$\mathsf{cansum}\,(M) = \sum_{j=1}^p M_{(t_j)} t_j$$

where $\{t_1, \dots, t_p\}$ is the set of those s_i's with a non-zero coefficient in M.

We now define a function mapping terms in Λ_{R} to their canonical forms.

Definition 2.15. Canonization of terms $\mathsf{can} : \Lambda_{\mathsf{R}} \longrightarrow \mathsf{C}_{\mathsf{R}}$ is given by

— $\mathsf{can}\,(x) = 1x$;
— if $\mathsf{can}\,(M) = \sum_{i=1}^n a_i s_i$ then $\mathsf{can}\,(\lambda x\, M) = \sum_{i=1}^n a_i\,(\lambda x\, s_i)$;
— if $\mathsf{can}\,(M) = \sum_{i=1}^n a_i s_i$ and $\mathsf{can}\,(N) = T$ then $\mathsf{can}\,((M)\,N) = \sum_{i=1}^n a_i (s_i)\,T$;
— $\mathsf{can}\,(\mathbf{0}) = \mathbf{0}$;
— if $\mathsf{can}\,(M) = \sum_{i=1}^n a_i s_i$ then $\mathsf{can}\,(aM) = \mathsf{cansum}\,(\sum_{i=1}^n (a a_i) s_i)$;
— if $\mathsf{can}\,(M) = \sum_{i=1}^n a_i s_i$ and $\mathsf{can}\,(N) = \sum_{i=n+1}^{n+p} a_i s_i$ then

$$\mathsf{can}\,(M + N) = \mathsf{cansum}\left(\sum_{i=1}^{n+p} a_i s_i\right).$$

Notice that in the penultimate case (definition of $\mathsf{can}\,(aM)$) the only effect of the application of cansum is to prune all the summands $(aa_i)s_i$ such that $aa_i = 0$.

Lemma 2.16. Canonization enjoys the following properties.

(i) Variables free in $\mathsf{can}\,(M)$ are also free in M. The converse does not hold in general.
(ii) For all base term s, $\mathsf{can}\,(s) = 1s$.
(iii) For all canonical term S, $\mathsf{can}\,(S) = S$.
(iv) For all term $M \in \Lambda_{\mathsf{R}}$, $\mathsf{can}\,(\mathsf{can}\,(M)) = \mathsf{can}\,(M)$.
(v) For all $M, N_1, \ldots, N_n \in \Lambda_{\mathsf{R}}$ and all variables $x_1, \ldots, x_n$ not free in any of these terms, $\mathsf{can}\,(M\,[N_1, \ldots, N_n/x_1, \ldots, x_n]) = \mathsf{can}\,(\mathsf{can}\,(M)\,[\mathsf{can}\,(N_1)\,, \ldots, \mathsf{can}\,(N_n)\,/x_1, \ldots, x_n])$.

Proof. Fact (i) is straigthforward from the previous definition. Facts (ii) and (iii) are proved by mutual induction on the definitions of base terms and canonical terms; fact (iv) follows from (iii). Fact (v) is proved by induction on M: all inductive steps follow directly from the definitions of canonization and substitution. □

Theorem 2.17. Algebraic equality is equality of canonical forms: for all $M, N \in \Lambda_{\mathsf{R}}$ $M \triangleq N$ iff $\mathsf{can}\,(M) = \mathsf{can}\,(N)$.

Proof. For all $M, N \in \Lambda_{\mathsf{R}}$, we write $M \triangleq' N$ iff $\mathsf{can}\,(M) = \mathsf{can}\,(N)$. It should be clear that $\triangleq'$ is an equivalence relation. It is contextual because the definition of canonization is by induction on permutative terms. It moreover validates equations (6a) through (8f): just apply can to both members of each equation and conclude. By the definition of $\triangleq$, we get $\triangleq \subseteq \triangleq'$. Conversely, one can easily check that $\mathsf{can}\,(M) \triangleq M$ for all $M \in \Lambda_{\mathsf{R}}$: this is the whole point of the definition of canonization. Hence the reverse inclusion: if $M \triangleq' N$, then $M \triangleq \mathsf{can}\,(M) = \mathsf{can}\,(N) \triangleq N$. □

Corollary 2.18. For all $S, T \in \mathsf{C}_{\mathsf{R}}$, $S \triangleq T$ iff $S = T$.

Proof. This a direct consequence of the previous theorem and fact (iii) of Lemma 2.16. □

Corollary 2.19. Substitution is well defined on $\Lambda_{\mathsf{R}}/{\triangleq}$: if $M, M' \in \Lambda_{\mathsf{R}}$ are such that $M \triangleq M'$ and, for all $i \in \{1, \ldots, n\}$, $N_i, N_i{}' \in \Lambda_{\mathsf{R}}$ are such that $N_i \triangleq N'_i$, then $M\,[N_1, \ldots, N_n/x_1, \ldots, x_n] \triangleq M'\,[N'_1, \ldots, N'_n/x_1, \ldots, x_n]$ for all pairwise distinct variables $x_1, \ldots, x_n$.

Proof. First apply Theorem 2.17 to the hypotheses and conclusion: we must prove

$$\mathsf{can}\,(M\,[N_1, \ldots, N_n/x_1, \ldots, x_n]) = \mathsf{can}\,(M'\,[N'_1, \ldots, N'_n/x_1, \ldots, x_n])$$

knowing that $\mathsf{can}\,(M) = \mathsf{can}\,(M')$ and, for all $i \in \{1, \ldots, n\}$, $\mathsf{can}\,(N_i) = \mathsf{can}\,(N'_i)$. We conclude by fact (v) of Lemma 2.16. □

Corollary 2.20. We can define an R-module structure on C_{R} as follows:

$$\begin{array}{rccc} \text{zero:} & & \mathbf{0} \in \mathsf{C}_{\mathsf{R}} & \\ \text{sum:} & (S,T) \in \mathsf{C}_{\mathsf{R}} \times \mathsf{C}_{\mathsf{R}} & \mapsto & \mathsf{can}\,(S+T) \in \mathsf{C}_{\mathsf{R}} \\ \text{scalar multiplication:} & (a,S) \in \mathsf{R} \times \mathsf{C}_{\mathsf{R}} & \mapsto & \mathsf{can}\,(aS) \in \mathsf{C}_{\mathsf{R}}; \end{array}$$

so that can is an isomorphism of R-modules from $\Lambda_\mathsf{R}/\triangleq$ to C_R.

Proof. By Theorem 2.17, can is well defined on $\Lambda_\mathsf{R}/\triangleq$, and is injective. It is surjective by fact (iii) of Lemma 2.16. The R-module structure of C_R then follows from that of $\Lambda_\mathsf{R}/\triangleq$. □

By this isomorphism, and $\triangleq$ being contextual, the quotient structure of algebraic terms is subsumed by the mutually inductive structure of base terms and canonical terms. If $\mathcal{C}$ is a set of canonical terms, we write $\underline{\mathcal{C}} = \{\underline{S};\ S \in \mathcal{C}\}$; then $(\Lambda_\mathsf{R}/\triangleq) = \underline{\mathsf{C}_\mathsf{R}}$. When we prove properties on algebraic terms, we can thus use induction on base terms and canonical terms: we then check that the corresponding property on algebraic terms follows through can, which is in general obvious. We will abuse terminology by claiming our proof is by induction on algebraic terms. Also, we will often define functions on $\Lambda_\mathsf{R}/\triangleq$ by induction on base terms and canonical terms: the actual function is obtained by composition with can. For instance, we define the height of algebraic terms by: $\mathrm{h}(\underline{M}) = \mathrm{h}(\mathsf{can}\,(\underline{M}))$.

2.4. *Abstract presentation*

Our presentation of the R-module of terms differs from that by Ehrhard and Regnier in (ER03), in that we introduce explicitly two dinstinct levels of syntax: permutative terms on the one hand (Λ_R) and algebraic terms ($\Lambda_\mathsf{R}/\triangleq$) on the other hand.

One can see the R-module of canonical terms from Corollary 2.20 as a concrete presentation of the one adopted by Ehrhard and Regnier: define an increasing sequence $(\mathsf{R}\,\langle\Delta_\mathsf{R}(k)\rangle)_{k\geq 0}$ of free R-modules generated by simple terms of bounded height.

Definition 2.21. We define the set $\Delta_\mathsf{R}(k)$ of *simple terms of height at most k*, by induction on k: let $\Delta_\mathsf{R}(0) = \emptyset$; we define the elements of $\Delta_\mathsf{R}(k+1)$ from those of $\Delta_\mathsf{R}(k)$ by the following clauses:

— if $\sigma \in \Delta_\mathsf{R}(k)$ then $\sigma \in \Delta_\mathsf{R}(k+1)$;
— if $x \in \mathcal{V}$ then $x \in \Delta_\mathsf{R}(k+1)$;
— if $\sigma \in \Delta_\mathsf{R}(k)$ then $\lambda x\,\sigma \in \Delta_\mathsf{R}(k+1)$;
— if $\sigma \in \Delta_\mathsf{R}(k)$ and $\tau \in \mathsf{R}\,\langle\Delta_\mathsf{R}(k)\rangle$ then $(\sigma)\,\tau \in \Delta_\mathsf{R}(k+1)$.

Then we define the set of all *simple terms* as $\Delta_\mathsf{R} = \bigcup_k \Delta_\mathsf{R}(k)$ and the set of *terms* $\mathsf{R}\,\langle\Delta_\mathsf{R}\rangle = \bigcup_k \mathsf{R}\,\langle\Delta_\mathsf{R}(k)\rangle$.

Notice that, although it is not made clear in the original paper, two quotient constructions are interleaved at each height: α-equivalence and the free R-module construction. In our opinion, this makes for a very intricate notion of equality on terms, so that the status of prominent and well-established notions in the setting of the ordinary λ-calculus becomes less immediate: for instance, what is a free occurrence of variable in a term, how do we define properly α-conversion on $\mathsf{R}\,\langle\Delta_\mathsf{R}\rangle$, what are the subterms of a term? Of course, these questions can be given satisfactory answers: we only claim that the simplicity of the definition is only apparent.

As expected, $\mathsf{R}\,\langle\Delta_\mathsf{R}\rangle$ and $(\Lambda_\mathsf{R}/\triangleq)$ are actually the same R-module of algebraic terms: define $\mathsf{B}_\mathsf{R}(k)$ (resp. $\mathsf{C}_\mathsf{R}(k)$) as the set of base terms (resp. canonical terms) of height at most k; then, clearly, $\Delta_{\mathsf{R}}(k)$ is $\underline{\mathsf{B}_{\mathsf{R}}(k)}$ and $\mathsf{R}\langle\Delta_{\mathsf{R}}(k)\rangle$ is $\underline{\mathsf{C}_{\mathsf{R}}(k)}$. Hence $\Delta_{\mathsf{R}} = \underline{\mathsf{B}_{\mathsf{R}}}$ and $\mathsf{R}\langle\Delta_{\mathsf{R}}\rangle = \underline{\mathsf{C}_{\mathsf{R}}} = (\Lambda_{\mathsf{R}}/\triangleq)$. This is one important contribution of the present paper: bring new light on the structure of $\mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$, by deliberately introducing α-equivalence and permutative equality separately from equality of linear combinations (i.e. algebraic equality). Also, this makes prominent the fact that the reduction of the algebraic λ-calculus is defined up to $\triangleq$ (see next section).

So, from now on, we formally identify Δ_{R} with $\underline{\mathsf{B}_{\mathsf{R}}}$ and $\mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$ with $\underline{\mathsf{C}_{\mathsf{R}}}$ by replacing Definition 2.21 with the following one:

Definition 2.22. We define *simple terms* as the $\triangleq$-classes of base terms. We write Δ_{R} for the set of simple terms and $\mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$ for the set of algebraic terms, which we may just call *terms*.

When we write a simple term (resp. a term) as $\underline{s}$, $\underline{t}$, $\underline{u}$, $\underline{v}$ or $\underline{w}$, (resp. $\underline{S}$, $\underline{T}$, $\underline{U}$, $\underline{V}$ or $\underline{W}$), it is implicit that s, t, u, v, or w is a base term (resp. S, T, U, V, or W is a canonical term). When we make no such assumption, we write $\underline{L}$, $\underline{M}$ or $\underline{N}$ or use greek letters σ, τ, ρ. We will often use the notations $\lambda x\,\sigma$, $(\sigma)\,\tau$, $a\sigma$, $\sigma+\tau$ with the obvious sense: these are well defined by contextuality of $\triangleq$.

Definition 2.23. For all $\underline{S} \in \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$ and $\underline{s} \in \Delta_{\mathsf{R}}$, we define the coefficient of $\underline{s}$ in $\underline{S}$ by $\underline{S}_{(\underline{s})} = S_{(s)}$. We then define the support of $\underline{S}$ as the set of all simple terms with a non-zero coefficient in $\underline{S}$:

$$\mathsf{Supp}\,(\underline{S}) = \left\{\underline{s} \in \Delta_{\mathsf{R}};\ \underline{S}_{(\underline{s})} \neq 0\right\}.$$

If $\mathcal{S}$ is a set of simple terms, we write $\mathsf{R}\langle\mathcal{S}\rangle$ for the set of linear combinations of elements of $\mathcal{S}$, i.e.

$$\mathsf{R}\langle\mathcal{S}\rangle = \left\{\underline{\sum_{i=1}^{n} a_i s_i};\ \forall i \in \{1,\dots,n\},\ \underline{s_i} \in \mathcal{S}, a_i \in \mathsf{R}\right\}$$

or, equivalently, $\mathsf{R}\langle\mathcal{S}\rangle = \{\sigma \in \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle;\ \mathsf{Supp}\,(\sigma) \subseteq \mathcal{S}\}$.

3. Reductions

In this section, we define reduction using (3) and (4) as key reduction rules: this captures the definition of reduction in (ER03), minus differentiation, in the setting of the algebraic λ-calculus.

3.1. *Reduction and Linear Combinations of Terms*

We call *relation from simple terms to terms* any subset of $\Delta_{\mathsf{R}} \times \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$, and we call *relation from terms to terms* any subset of $\mathsf{R}\langle\Delta_{\mathsf{R}}\rangle \times \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$. Given a relation r from simple terms to terms we define two new relations $\bar{\mathrm{r}}$ and $\tilde{\mathrm{r}}$ from terms to terms by:

— $\sigma\ \bar{\mathrm{r}}\ \sigma'$ if $\sigma = \underline{\sum_{i=1}^{n} a_i s_i}$ and $\sigma' = \underline{\sum_{i=1}^{n} a_i S'_i}$ where, for all $i \in \{1,\dots,n\}$, $\underline{s_i}\ \mathrm{r}\ \underline{S'_i}$;
— $\sigma\ \tilde{\mathrm{r}}\ \sigma'$ if $\sigma = \underline{as+T}$ and $\sigma' = \underline{aS'+T}$ where $a \neq 0$ and $\underline{s}\ \mathrm{r}\ \underline{S'}$.

Clearly, $\widetilde{\mathrm{r}} \subseteq \overline{\mathrm{r}}$. It is important that, in the above definitions we do not require $\sum_{i=1}^{n} a_i s_i$ nor $as+T$ to be canonical terms: $\widetilde{\mathrm{r}}$ matches equation (4), while $\overline{\mathrm{r}}$ matches (5). We will use these constructions in the definitions of one-step β-reduction $\to$ and parallel reduction $\rightrightarrows$: we introduce these as relations from simple terms to terms, so that the actual reduction relations on terms are obtained as $\widetilde{\to}$ and $\overline{\rightrightarrows}$ respectively.

Notice that we cannot define reduction by induction on terms: if there are $a, b \in \mathsf{R}^\bullet$ such that $a+b=0$ then $\underline{\mathbf{0}} = a\sigma + b\sigma$ for all $\sigma \in \mathsf{R}\langle\Delta_\mathsf{R}\rangle$; hence, by rule (4), $\underline{\mathbf{0}}$ may reduce. Following (ER03), we rather define simple term reduction $\to$ by induction on the depth of the fired redex.

Definition 3.1. We define an increasing sequence of relations from simple terms to terms by the following statements. Let $\to_0$ be the empty relation $\emptyset \subseteq \Delta_\mathsf{R} \times \mathsf{R}\langle\Delta_\mathsf{R}\rangle$. Assume $\to_k$ is defined. Then we set $\sigma \to_{k+1} \sigma'$ as soon as one of the following holds:

— $\sigma = \underline{\lambda x\, s}$ and $\sigma = \underline{\lambda x\, S'}$ with $\underline{s} \to_k \underline{S'}$;
— $\sigma = \underline{(s)\, T}$ and $\sigma' = \underline{(S')\, T}$ with $\underline{s} \to_k \underline{S'}$, or $\sigma' = \underline{(s)\, T'}$ with $\underline{T} \,\widetilde{\to_k}\, \underline{T'}$;
— $\sigma = \underline{(\lambda x\, s)\, T}$ and $\sigma' = \underline{s\,[T/x]}$.

Let $\to \,=\, \bigcup_{k\in\mathbf{N}} \to_k$. We call *one-step reduction* or simply *reduction*, the relation $\widetilde{\to}$.

Lemma 3.2. We have $\widetilde{\to} = \bigcup_{k\in\mathbf{N}} \widetilde{\to_k}$.

Proof. This is a consequence of the more general following properties of $\widetilde{\cdot}$: if (r_n) is an increasing sequence of relations from simple terms to terms, then $(\widetilde{\mathrm{r}_n})$ is also increasing (monotony) and $\widetilde{\bigcup_n \mathrm{r}_n} = \bigcup_n \widetilde{\mathrm{r}_n}$ (ω-continuity). □

Lemma 3.3. If $\sigma \in \Delta_\mathsf{R}$ and $\sigma' \in \mathsf{R}\langle\Delta_\mathsf{R}\rangle$, then $\sigma \to \sigma'$ iff one of the following holds:

(i) $\sigma = \lambda x\, \tau$ and $\sigma = \lambda x\, \tau'$ with $\tau \to \tau'$;
(ii) $\sigma = (\tau)\, \rho$ and $\sigma' = (\tau')\, \rho$ with $\tau \to \tau'$, or $\sigma' = (\tau)\, \rho'$ with $\rho \,\widetilde{\to}\, \rho'$;
(iii) $\sigma = (\lambda x\, \tau)\, \rho$ and $\sigma' = \tau\,[\rho/x]$;

where, in each case $\tau \in \Delta_\mathsf{R}$.

Proof. If (i) or the first case of (ii) holds, then it holds at some depth k, hence $\sigma \to_{k+1} \sigma'$. If the second case of (ii) holds, then by Lemma 3.2, we get $\rho \,\widetilde{\to_k}\, \rho'$ for some k, hence $\sigma \to_{k+1} \sigma'$. If (iii) holds then $\sigma \to_1 \sigma'$. Conversely, if $\sigma \to \sigma'$ then there is k such that $\sigma \to_k \sigma'$ and one of (i) (ii) or (iii) holds by the definition of $\to_k$ (and Lemma 3.2 in the second case of (ii)). □

Let $\widetilde{\to}^*$ be the reflexive and transitive closure of $\widetilde{\to}$.

Lemma 3.4. Let $\sigma, \sigma', \tau \in \mathsf{R}\langle\Delta_\mathsf{R}\rangle$ with $\sigma \,\widetilde{\to}\, \sigma'$. Then for all $\tau \in \mathsf{R}\langle\Delta_\mathsf{R}\rangle$ and all $a \in \mathsf{R}$ we have: $\lambda x\, \sigma \,\widetilde{\to}\, \lambda x\, \sigma'$, $(\sigma)\, \tau \,\widetilde{\to}\, (\sigma')\, \tau$, $(\tau)\, \sigma \,\widetilde{\to}^*\, (\tau)\, \sigma'$, $\sigma + \tau \,\widetilde{\to}\, \sigma' + \tau$ and $a\sigma \,\widetilde{\to}^*\, a\sigma'$.

Proof. Write $\sigma = \underline{S} = \underline{bu + V}$ and $\sigma' = \underline{S'} = \underline{bU' + V}$ with $b \neq 0$ and and $\underline{u} \to \underline{U'}$, and write $\tau = \underline{T} = \underline{\sum_{i=1}^{n} a_i t_i}$. Then by Lemma 3.3, $\underline{\lambda x\, u} \to \underline{\lambda x\, U'}$ and $\underline{(u)\, T} \to \underline{(U')\, T}$, hence

$$\lambda x\, \sigma = \underline{b\lambda x\, u + \lambda x\, V} \,\widetilde{\to}\, \underline{b\lambda x\, U' + \lambda x\, V} = \lambda x\, \sigma'$$

and

$$(\sigma)\,\tau = \underline{b(u)\,T + (V)\,T} \mathrel{\widetilde{\to}} \underline{b(U')\,T + (V)\,T} = (\sigma')\,\tau.$$

Also, for each i, $\underline{(t_i)\,S} \to \underline{(t_i)\,S'}$: then, in n $\widetilde{\to}$-steps, $(\tau)\,\sigma = \underline{\sum_{i=1}^n a_i(t_i)\,S}$ reduces to $(\tau)\,\sigma' = \underline{\sum_{i=1}^n a_i(t_i)\,S'}$. For sum: $\sigma + \tau = \underline{bu + V + T} \mathrel{\widetilde{\to}} \underline{bU' + V + T} = \sigma' + \tau$. If $ab = 0$ then $\underline{abu} = \underline{abU'} = \underline{\mathbf{0}}$, hence $a\sigma = a\sigma'$; otherwise $a\sigma = \underline{abu + aV} \mathrel{\widetilde{\to}} \underline{abU' + aV} = a\sigma'$. □

Lemma 3.5. The relation $\widetilde{\to}^*$ is contextual.

Proof. This is a straightforward consequence of Lemma 3.4 using reflexivity and transitivity. □

3.2. *Confluence*

We prove the confluence of $\widetilde{\to}$ by usual Tait–Martin-Löf technique: introduce a parallel extension of reduction (in which redexes can be fired simultaneously) and prove it enjoys the diamond property (i.e. strong confluence).

3.2.1. *Parallel reduction*

Definition 3.6. We define an increasing sequence of relations from simple terms to terms by the following statements. Let $\rightrightarrows_0$ be the identity relation on Δ_{R}, extended as a relation from simple terms to terms. Assume $\rightrightarrows_k$ is defined. Then we set $\sigma \rightrightarrows_{k+1} \sigma'$ as soon as one of the following holds:

— $\sigma = \underline{\lambda x\, s}$ and $\sigma' = \underline{\lambda x\, S'}$ with $\underline{s} \rightrightarrows_k \underline{S'}$;
— $\sigma = \underline{(s)\,T}$ and $\sigma' = \underline{(S')\,T'}$ with $\underline{s} \rightrightarrows_k \underline{S'}$ and $\underline{T} \mathrel{\overline{\rightrightarrows_k}} \underline{T'}$;
— $\sigma = \underline{(\lambda x\, s)\,T}$ and $\sigma' = \underline{S'\,[T'/x]}$ with $\underline{s} \rightrightarrows_k \underline{S'}$ and $\underline{T} \mathrel{\overline{\rightrightarrows_k}} \underline{T'}$.

Let $\rightrightarrows = \bigcup_{k\in\mathbf{N}} \rightrightarrows_k$. We call *parallel reduction* the relation $\overline{\rightrightarrows}$.

Lemma 3.7. We have $\overline{\rightrightarrows} = \bigcup_{k\in\mathbf{N}} \overline{\rightrightarrows_k}$.

Proof. Similarly to Lemma 3.2: $\overline{\,\cdot\,}$ is monotone and ω-continuous. □

Lemma 3.8. If $\sigma \in \Delta_{\mathsf{R}}$ and $\sigma' \in \mathsf{R}\,\langle\Delta_{\mathsf{R}}\rangle$, then $\sigma \rightrightarrows \sigma'$ iff one of the following holds:

(i) $\sigma = \lambda x\, \tau$ and $\sigma = \lambda x\, \tau'$ with $\tau \rightrightarrows \tau'$;
(ii) $\sigma = (\tau)\,\rho$ and $\sigma' = (\tau')\,\rho'$ with $\tau \rightrightarrows \tau'$ and $\rho \mathrel{\overline{\rightrightarrows}} \rho'$;
(iii) $\sigma = (\lambda x\, \tau)\,\rho$ and $\sigma' = \tau'\,[\rho'/x]$ with $\tau \rightrightarrows \tau'$ and $\rho \mathrel{\overline{\rightrightarrows}} \rho'$;

where, in each case $\tau \in \Delta_{\mathsf{R}}$.

Proof. Like in Lemma 3.3, this is just rephrasing the definition of $\rightrightarrows$, using Lemma 3.7 where $\overline{\rightrightarrows}$ is involved. □

Lemma 3.9. Relation $\overline{\rightrightarrows}$ is contextual.

Proof. The proof is very similar to that of Lemma 3.4, using Lemma 3.8 and the definition of $\overline{\rightrightarrows}$. □

Lemma 3.10. $(\lambda x\, \sigma)\,\tau \mathrel{\overline{\rightrightarrows}} \sigma'\,[\tau'/x]$ as soon as $\sigma \mathrel{\overline{\rightrightarrows}} \sigma'$ and $\tau \mathrel{\overline{\rightrightarrows}} \tau'$.

Proof. This is a straightforward consequence of Lemmas 3.8 and 3.9. □

Lemma 3.11. The following strict inclusions hold: $\widetilde{\to} \subset \overline{\rightrightarrows} \subset \widetilde{\to}^*$.

Proof. The fact that $\widetilde{\to} \subseteq \overline{\rightrightarrows}$ is straightforward from the definitions. The fact that $\rightrightarrows_k \subseteq \widetilde{\to}^*$ and $\overline{\rightrightarrows_k} \subseteq \widetilde{\to}^*$ is easily proved by induction on k, hence $\overline{\rightrightarrows} \subseteq \widetilde{\to}^*$. Inclusions are strict: write $I = \lambda x\, x$, then $\underline{(I)\,(I)\,I} \;\overline{\rightrightarrows}\; \underline{I}$ but $\underline{(I)\,(I)\,I} \;\widetilde{\not\to}\; \underline{I}$, and $\underline{((I)\,I)\,I} \;\widetilde{\to}^*\; \underline{I}$ but $\underline{((I)\,I)\,I} \;\overline{\not\rightrightarrows}\; \underline{I}$. □

3.2.2. *Reductions and Substitution.* The main property of parallel reduction is the following one, which fails for one-step reduction.

Lemma 3.12. Let x be a variable and $\sigma, \tau, \sigma', \tau'$ be terms. If $\sigma \;\overline{\rightrightarrows}\; \sigma'$ and $\tau \;\overline{\rightrightarrows}\; \tau'$ then

$$\sigma\,[\tau/x] \;\overline{\rightrightarrows}\; \sigma'\,[\tau'/x] \;\; .$$

Proof. We prove by induction on k that if $\sigma \;\overline{\rightrightarrows_k}\; \sigma'$ and $\tau \;\overline{\rightrightarrows}\; \tau'$ then $\sigma\,[\tau/x] \;\overline{\rightrightarrows}\; \sigma'\,[\tau'/x]$. If $k = 0$ then $\sigma' = \sigma$; then by Lemma 3.9 and Proposition 2.6, we have $\sigma\,[\tau/x] \;\overline{\rightrightarrows}\; \sigma\,[\tau'/x] = \sigma'\,[\tau'/x]$. Suppose the result holds for some k, then we extend it to $k+1$ by inspecting the possible cases for reduction $\sigma \;\overline{\rightrightarrows_{k+1}}\; \sigma'$. We first address the case in which σ is simple and $\sigma \rightrightarrows_{k+1} \sigma'$. Then one of the following statements applies (we write $\tau = \underline{T}$ and $\tau' = \underline{T'}$):

— $\sigma = \underline{\lambda y\, u}$ with $y \neq x$ and y not free in T, and $\sigma' = \underline{\lambda y\, U'}$ with $\underline{u} \rightrightarrows_k \underline{U'}$; hence, by the induction hypothesis, $\underline{u\,[T/x]} \;\overline{\rightrightarrows}\; \underline{U'\,[T'/x]}$ and we get

$$\sigma\,[\tau/x] = \underline{\lambda y\,(u\,[T/x])} \;\overline{\rightrightarrows}\; \underline{\lambda y\,(U'\,[T'/x])} = \sigma'\,[\tau'/x]$$

by Lemma 3.9;

— $\sigma = \underline{(u)\,V}$ and $\sigma' = \underline{(U')\,V'}$ with $\underline{u} \rightrightarrows_k \underline{U'}$ and $\underline{V} \;\overline{\rightrightarrows_k}\; \underline{V'}$; hence, by the induction hypothesis, $\underline{u\,[T/x]} \;\overline{\rightrightarrows}\; \underline{U'\,[T'/x]}$ and $\underline{V\,[T/x]} \;\overline{\rightrightarrows}\; \underline{V'\,[T'/x]}$, and we get

$$\sigma\,[\tau/x] = \underline{(u\,[T/x])\,V\,[T/x]} \;\overline{\rightrightarrows}\; \underline{(U'\,[T'/x])\,V'\,[T'/x]} = \sigma'\,[\tau'/x]$$

by Lemma 3.9;

— $\sigma = \underline{(\lambda y\, u)\,V}$ and $\sigma' = \underline{U'\,[V'/y]}$ with $\underline{u} \rightrightarrows_k \underline{U'}$, $\underline{V} \;\overline{\rightrightarrows_k}\; \underline{V'}$, $x \neq y$ and y not free in T; hence, by the induction hypothesis, $\underline{u\,[T/x]} \;\overline{\rightrightarrows}\; \underline{U'\,[T'/x]}$ and $\underline{V\,[T/x]} \;\overline{\rightrightarrows}\; \underline{V'\,[T'/x]}$, and we get

$$\sigma\,[\tau/x] = \underline{(\lambda y\, u\,[T/x])\,V\,[T/x]} \;\overline{\rightrightarrows}\; \underline{(U'\,[T'/x])\,[V'\,[T'/x]/y]} = \sigma'\,[\tau'/x]$$

by Lemma 3.10.

Now assume $\sigma \;\overline{\rightrightarrows_{k+1}}\; \sigma'$. By definition, this amounts to the following: $\sigma = \underline{\sum_{i=1}^n a_i s_i}$ and $\sigma = \underline{\sum_{i=1}^n a_i S'_i}$, with $\underline{s_i} \rightrightarrows_{k+1} \underline{S'_i}$ for all i. We have just shown that we then have $\underline{s_i\,[T/x]} \;\overline{\rightrightarrows}\; \underline{S'_i\,[T'/x]}$ for all i. We conclude by Lemma 3.9. □

From Lemmas 3.11 and 3.12, we can derive a very similar result for $\widetilde{\to}^*$:

Corollary 3.13. Let x be a variable and $\sigma, \tau, \sigma', \tau'$ be terms. If $\sigma \mathrel{\widetilde{\to}^*} \sigma'$ and $\tau \mathrel{\widetilde{\to}^*} \tau'$ then

$$\sigma\,[\tau/x] \mathrel{\widetilde{\to}^*} \sigma'\,[\tau'/x] \ .$$

3.2.3. *Church-Rosser.* We finish the proof of confluence by showing that the $\overline{\rightrightarrows}$-reducts of a fixed term σ all $\overline{\rightrightarrows}$-reduce to one of them (obtained by firing all redexes of σ, simultaneously).

Definition 3.14. We define inductively on term σ its full parallel reduct $\sigma\!\downarrow$ by:

$$\begin{aligned}
\underline{x}\!\downarrow &= \underline{x} \\
\underline{\lambda x\, s}\!\downarrow &= \lambda x\ \underline{s}\!\downarrow \\
\underline{(\lambda x\, s)\, T}\Big\downarrow &= (\underline{s}\!\downarrow)\,[\underline{T}\!\downarrow /x] \\
\underline{(s)\, T}\Big\downarrow &= (\underline{s}\!\downarrow)\ \underline{T}\!\downarrow \text{ if } s \text{ is a variable or an application} \\
\underline{\sum_{i=1}^{n} a_i s_i}\Bigg\downarrow &= \sum_{i=1}^{n} a_i\ \underline{s_i}\!\downarrow .
\end{aligned}$$

Lemma 3.15. If σ and σ' are such that $\sigma \mathrel{\overline{\rightrightarrows}} \sigma'$, then $\sigma' \mathrel{\overline{\rightrightarrows}} \sigma\!\downarrow$.

Proof. One simply proves by induction on k that if $\sigma \rightrightarrows_k \sigma'$ or $\sigma \mathrel{\overline{\rightrightarrows_k}} \sigma'$ then $\sigma' \mathrel{\overline{\rightrightarrows}} \sigma\!\downarrow$, using Lemma 3.9 in general, and Lemma 3.10 in the case of a redex. □

Theorem 3.16. Relation $\overline{\rightrightarrows}$ is strongly confluent. Hence, relation $\widetilde{\to}$ enjoys the Church-Rosser property.

Proof. Strong confluence of $\overline{\rightrightarrows}$ is a straightforward corollary of Lemma 3.15. It implies confluence of $\widetilde{\to}$ by Lemma 3.11. □

3.2.4. *Trivia.* There is a case in which confluence is much easier to establish: if 1 admits an opposite $-1 \in \mathsf{R}$. In this case, assume $\sigma \mathrel{\widetilde{\to}^*} \sigma'$. Since $\widetilde{\to}^*$ is contextual, $\sigma' = \sigma' + (-1)\sigma + \sigma \mathrel{\widetilde{\to}^*} \sigma' + (-1)\sigma' + \sigma = \sigma$. Hence $\widetilde{\to}^*$ is symmetric, which obviously implies Church-Rosser. But this has little meaning: in the next section, we show that reduction becomes trivial as soon as $-1 \in \mathsf{R}$.

3.3. *Conservativity*

Every ordinary λ-term is also a raw term of the algebraic λ-calculus, whose $\triangleq$-class is simple. Let Λ denote the set of all λ-terms and $\to_\Lambda$ denote the usual β-reduction of the λ-calculus: it is then clear that, for all $s, s' \in \Lambda$, $s \to_\Lambda s'$ implies $\underline{s} \to \underline{s'}$. Denote by $\leftrightarrow$ the reflexive, symmetric and transitive closure of $\widetilde{\to}$ and $\leftrightarrow_\Lambda$ the usual β-equivalence of the λ-calculus.

Lemma 3.17. The algebraic λ-calculus preserves the equalities of the λ-calculus, i.e. for all λ-terms s and t, $s \leftrightarrow_\Lambda t$ implies $\underline{s} \leftrightarrow \underline{t}$.

Proof. This is a straightforward consequence of the confluence of $\to_\Lambda$ and the fact that $\to_\Lambda \subset \widetilde{\to}$. □

One may wonder if the reverse also holds, i.e. if equivalence classes of λ-terms in the algebraic λ-calculus are the same as in the ordinary λ-calculus. If R is $\mathbf{N}$, then $\widetilde{\to}$-reductions from λ-terms are exactly $\to_\Lambda$-reductions (restricted to λ-terms, $\triangleq$ then only amounts to α-conversion), and the result holds by the same argument as in Lemma 3.17. In the general case, however, a λ-term does not necessarily reduce to another λ-term, hence the proof is not as easy.

3.3.1. *The Positive Case.* Recall that a rig R is said to be positive if, for all $a, b \in \mathsf{R}$, $a + b = 0$ implies $a = b = 0$. In that case, we prove that: for all $s, s' \in \Lambda$, $\underline{s} \leftrightarrow \underline{s'}$ implies $s \leftrightarrow_\Lambda s'$ (Theorem 3.24).

Definition 3.18. We define $\Lambda : \mathsf{R}\langle\Delta_\mathsf{R}\rangle \longrightarrow \mathcal{P}(\Lambda)$ by induction on terms:

$$\begin{aligned}
\Lambda(\underline{x}) &= \{x\} \\
\Lambda(\underline{\lambda x\, s}) &= \{\lambda x\, u;\ u \in \Lambda(\underline{s})\} \\
\Lambda\left(\underline{(s)\, T}\right) &= \{(u)\, v;\ u \in \Lambda(\underline{s}) \text{ and } v \in \Lambda(\underline{T})\} \\
\Lambda\left(\underline{\sum_{i=1}^{n} a_i s_i}\right) &= \bigcup_{i=1}^{n} \Lambda(\underline{s_i}).
\end{aligned}$$

The crucial point in that definition is that the sum $\sum_{i=1}^{n} a_i s_i$ being canonical entails that, for all i, $a_i \neq 0$.

Proposition 3.19. If $s \in \Lambda$, then $\Lambda(\underline{s}) = \{s\}$.

Lemma 3.20. If R is positive and terms $\sigma \in \mathsf{R}\langle\Delta_\mathsf{R}\rangle$ and $\sigma' \in \mathsf{R}\langle\Delta_\mathsf{R}\rangle$ are such that $\sigma \widetilde{\to} \sigma'$, then for all $s' \in \Lambda(\sigma')$, either $s' \in \Lambda(\sigma)$ or there exists $s \in \Lambda(\sigma)$ such that $s \to_\Lambda s'$.

Proof. The proof is by induction on the depth of the reduction $\sigma \widetilde{\to} \sigma'$, i.e. the least k such that $\sigma \widetilde{\to}_k \sigma'$. All induction steps are straightforward, except for the extension from $\to_k$ to $\widetilde{\to}_k$: assume $\sigma = \underline{at + U}$ and $\sigma = \underline{aT' + U}$ with $a \neq 0$ and $\underline{t} \to_k \underline{T'}$. By definition, $\Lambda(\sigma') = \Lambda(\underline{aT' + U}) \subseteq \Lambda(\underline{T'}) \cup \Lambda(\underline{U})$. Since R is positive, the coefficient of t in $\mathsf{can}(at + U)$ is non-zero: hence $\Lambda(\sigma) = \Lambda(\underline{at + U}) = \Lambda(\underline{t}) \cup \Lambda(\underline{U})$. Now assume $v' \in \Lambda(\sigma')$: either $v' \in \Lambda(\underline{U}) \subseteq \Lambda(\sigma)$; or $v' \in \Lambda(\underline{T'})$, and then, by the induction hypothesis, either $v' \in \Lambda(\underline{t}) \subseteq \Lambda(\sigma)$ or there exists $v \in \Lambda(\underline{t}) \subseteq \Lambda(\sigma)$ such that $v \to_\Lambda v'$. □

Corollary 3.21. If R is positive and $s \in \Lambda$ and $\sigma \in \mathsf{R}\langle\Delta_\mathsf{R}\rangle$ are such that $\underline{s} \widetilde{\to}^* \sigma$, then for all $t \in \Lambda(\sigma)$, $s \to_\Lambda^* t$.

Lemma 3.22. If σ and $\sigma' \in \mathsf{R}\langle\Delta_\mathsf{R}\rangle$ are such that $\sigma \overline{\overline{\to}} \sigma'$ then $\sigma{\downarrow} \overline{\overline{\to}} \sigma'{\downarrow}$.

Proof. The proof is easy and very close to that of Lemma 3.15. □

We define iterated full reduction by $\sigma{\downarrow}^0 = \sigma$ and $\sigma{\downarrow}^{n+1} = (\sigma{\downarrow}^n){\downarrow}$.

Lemma 3.23. If $\sigma \overline{\rightrightarrows}^n \tau$ then $\tau \widetilde{\to}^* \sigma{\downarrow}^n$.

Proof. The proof is by induction on n. If $n = 0$, $\sigma = \tau = \sigma{\downarrow}^0$ and this is reflexivity of $\widetilde{\to}^*$. Assume the result holds at rank n. If $\sigma \overline{\rightrightarrows}^n \tau \overline{\rightrightarrows} \tau'$, then, by the induction hypothesis, $\tau \widetilde{\to}^* \sigma{\downarrow}^n$. Since $\widetilde{\to}^*$ is also the transitive closure of $\overline{\rightrightarrows}$, Lemma 3.22 entails $\tau{\downarrow} \widetilde{\to}^* \sigma{\downarrow}^{n+1}$. By Lemma 3.15, we have $\tau' \overline{\rightrightarrows} \tau{\downarrow}$, hence $\tau' \widetilde{\to}^* \sigma{\downarrow}^{n+1}$. □

Theorem 3.24. If R is positive and $s, t \in \Lambda$ are such that $s \leftrightarrow t$ then $s \leftrightarrow_\Lambda t$.

Proof. Assume $s, t \in \Lambda$ and $\underline{s} \leftrightarrow \underline{t}$. By the Church-Rosser property of $\widetilde{\to}$ (Theorem 3.16), there exists $\sigma \in \mathsf{R}\langle\Delta_\mathsf{R}\rangle$ such that $\underline{s} \widetilde{\to}^* \sigma$ and $\underline{t} \widetilde{\to}^* \sigma$. By Lemma 3.23, there exists some $n \in \mathbf{N}$ such that $\sigma \widetilde{\to}^* \tau = \underline{s}{\downarrow}^n$. Notice that for all $\underline{w} \in \underline{\Lambda}$, $\underline{w}{\downarrow} \in \underline{\Lambda}$; hence $\tau \in \underline{\Lambda}$ and we write $\tau = \underline{v}$ with $v \in \Lambda$. We have $\underline{s} \widetilde{\to}^* \underline{v}$ and $\underline{t} \widetilde{\to}^* \underline{v}$: by positivity of R and Corollary 3.21, we obtain that, for all $v' \in \Lambda(\underline{v})$, there are $s' \in \Lambda(\underline{s})$ and $t' \in \Lambda(\underline{t})$ such that $s' \to_\Lambda^* v'$ and $t' \to_\Lambda^* v'$. By Proposition 3.19, $\Lambda(\underline{s}) = \{s\}$, $\Lambda(\underline{t}) = \{t\}$ and $\Lambda(\underline{v}) = \{v\}$, hence the conclusion. □

3.3.2. *Collapse.* If R is not positive, we show that reductional equality collapses: $\leftrightarrow$ identifies terms which bear absolutely no relationship with each other.

Lemma 3.25. Assume, there are $a, b \in \mathsf{R}^\bullet$ such that $a + b = 0$, then for all term σ, $\underline{\mathbf{0}} \widetilde{\to}^* a\sigma \widetilde{\to}^* \underline{\mathbf{0}}$.

Proof. Take Θ a fixed point combinator of the λ-calculus, such that $(\Theta)\, s \to_\Lambda^* (s)\,(\Theta)\, s$ for all λ-term s. Write ∞_σ for $(\Theta)\, \lambda x\, (\sigma + \underline{x})$; then $\infty_\sigma \widetilde{\to}^* \sigma + \infty_\sigma$. We get:

$$\underline{\mathbf{0}} = a\infty_\sigma + b\infty_\sigma \widetilde{\to}^* a\sigma + a\infty_\sigma + b\infty_\sigma = a\sigma$$

and

$$a\sigma = a\sigma + a\infty_\sigma + b\infty_\sigma \widetilde{\to}^* a\sigma + a\infty_\sigma + b\sigma + b\infty_\sigma = \underline{\mathbf{0}}.$$

□

Corollary 3.26. If R is such that 1 has an opposite, i.e. $-1 \in \mathsf{R}$ with $1 + (-1) = 0$, then for all terms σ and τ, $\sigma \widetilde{\to}^* \tau$.

4. Simple Type System

Raw terms may be given implicative propositional types in a natural way. Assume we have a denumerable set of basic types $\phi, \psi, \ldots$; we build types from basic types using intuitionistic arrow: if A and B are types, then so is $A \to B$. Typing rules are given in Figure 1. Notice that scalar coefficients have no influence on typing. In particular, we make no assumption on the actual structure of R.

Proposition 4.1. Typing in the algebraic λ-calculus enjoys the following properties:

(i) If $\Gamma \vdash M : A$ then free variables of M are declared in Γ.

$$\frac{}{\Gamma, x : A \vdash x : A}\ (\text{Ax})$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x\, M : A \to B}\ (\text{Abs}) \qquad \frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash (M)\, N : B}\ (\text{App})$$

$$\frac{}{\Gamma \vdash \mathbf{0} : A}\ (\text{Zero}) \qquad \frac{\Gamma \vdash M : A}{\Gamma \vdash aM : A}\ (\text{Scal}) \qquad \frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : A}{\Gamma \vdash M + N : A}\ (\text{Add})$$

Figure 1. Typing rules for the algebraic λ-calculus.

(ii) If $\Gamma \vdash M : A$ then, for all Γ' whose domain is disjoint from that of Γ, we have $\Gamma, \Gamma' \vdash M : A$.
(iii) If $M \equiv M'$ then $\Gamma \vdash M : A$ iff $\Gamma \vdash M' : A$.
(iv) For all canonical term S, $\Gamma \vdash S : A$ if and only if, for all $u \in \Lambda\left(\underline{S}\right)$, $\Gamma \vdash u : A$.
(v) For all raw term M, if $\Gamma \vdash M : A$ then $\Gamma \vdash \mathsf{can}\left(M\right) : A$.

The converse of that last proposition does not hold: for instance, for all raw term M, $\mathsf{can}\left(0M\right) = \mathbf{0}$ can be given any type in any context whereas $0M$ satisfies the same typing judgements as M. Hence such a straightforward notion of typing is not compatible with algebraic equality $\triangleq$.

Definition 4.2. The term σ is ***weakly typable*** of type A in context Γ if $\Gamma \vdash \mathsf{can}\left(\sigma\right) : A$ is derivable. We write $\Gamma \vdash_{\mathsf{R}} \sigma : A$ for $\Gamma \vdash \mathsf{can}\left(\sigma\right) : A$.

Proposition 4.3. For all $\sigma \in \mathsf{R}\left\langle \Delta_{\mathsf{R}} \right\rangle$, $\Gamma \vdash_{\mathsf{R}} \sigma : A$ iff $\Gamma \vdash_{\mathsf{R}} \underline{s} : A$, for all $\underline{s} \in \mathsf{Supp}\left(\sigma\right)$.

Now we show that subject reduction holds for weak typing, as soon as R is positive (Lemma 4.6).

Lemma 4.4. Let $\sigma, \tau \in \Lambda_{\mathsf{R}}$. If $\Gamma, x : A \vdash_{\mathsf{R}} \sigma : B$ and $\Gamma \vdash_{\mathsf{R}} \tau : A$ then $\Gamma \vdash_{\mathsf{R}} \sigma\left[\tau/x\right] : B$.

Proof. One proves by induction on the derivation of $\Gamma, x : A \vdash M : B$ that if moreover $\Gamma \vdash N : A$ then $\Gamma \vdash M\left[N/x\right] : B$. The result follows by taking $M = \mathsf{can}\left(\sigma\right)$ and $N = \mathsf{can}\left(\tau\right)$, using fact (v) of Lemma 2.16. □

Lemma 4.5. For all $\sigma, \tau \in \mathsf{R}\left\langle \Delta_{\mathsf{R}} \right\rangle$ and all $a \in \mathsf{R}$, $\mathsf{Supp}\left(\sigma + \tau\right) \subseteq \mathsf{Supp}\left(\sigma\right) \cup \mathsf{Supp}\left(\tau\right)$ and $\mathsf{Supp}\left(a\sigma\right) \subseteq \mathsf{Supp}\left(\sigma\right)$. If R is positive, we moreover have: $\mathsf{Supp}\left(\sigma + \tau\right) = \mathsf{Supp}\left(\sigma\right) \cup \mathsf{Supp}\left(\tau\right)$.

Proof. For all $\underline{s} \in \Delta_{\mathsf{R}}$, we have $(\sigma + \tau)_{(\underline{s})} = \sigma_{(\underline{s})} + \tau_{(\underline{s})}$ and $(a\sigma)_{(\underline{s})} = a\sigma_{(\underline{s})}$. By the definition of $\mathsf{Supp}\left(\sigma + \tau\right)$ and $\mathsf{Supp}\left(a\sigma\right)$, we get the above inclusions. If R is positive, $(\sigma + \tau)_{(\underline{s})} \neq 0$ as soon as $\sigma_{(\underline{s})} \neq 0$ or $\tau_{(\underline{s})} \neq 0$, hence $\mathsf{Supp}\left(\sigma + \tau\right) = \mathsf{Supp}\left(\sigma\right) \cup \mathsf{Supp}\left(\tau\right)$. □

Notice that we do not necessarily have $\mathsf{Supp}\left(a\sigma\right) = \mathsf{Supp}\left(\sigma\right)$ when $a \neq 0$ and R is positive: see Lemma 5.3 for a sufficient condition.

Lemma 4.6. Assume R is positive. If $\sigma \stackrel{\sim}{\to} \sigma'$ and $\Gamma \vdash_{\mathsf{R}} \sigma : A$ then $\Gamma \vdash_{\mathsf{R}} \sigma' : A$.

Proof. We prove by induction on base terms and canonical terms that if either $\Gamma \vdash s : A$ and $\underline{s} \to \sigma'$, or $\Gamma \vdash S : A$ and $\underline{S} \mathrel{\widetilde{\to}} \sigma'$, then $\Gamma \vdash \sigma' : A$. For base terms, we check that all possible cases for reduction $\underline{s} \to \sigma'$ preserve weak typing, which is straightforward by induction hypotheses (using Lemma 4.4 in the case of a redex). Now assume $\Gamma \vdash S : A$ and write $\underline{S} = \underline{at + U}$ and $\sigma' = \underline{aT' + U}$, with $a \neq 0$ and $\underline{t} \to \underline{T'}$. By Lemma 4.5, $\mathsf{Supp}\,(\underline{S}) = \{\underline{t}\} \cup \mathsf{Supp}\,(\underline{U})$ (this is where we use the positivity condition). By Proposition 4.3, $\Gamma \vdash t : A$ and $\Gamma \vdash U : A$. By the induction hypothesis on base term t, we get $\Gamma \vdash T' : A$. By Lemma 4.5 again, $\mathsf{Supp}\,(\sigma') \subseteq \mathsf{Supp}\,(\underline{T'}) \cup \mathsf{Supp}\,(\underline{U})$, and we get $\Gamma \vdash_{\mathsf{R}} \sigma' : A$ by Proposition 4.3. □

5. On Normalization Properties

Unsurprisingly, if R is not positive, there is no normal term: assume there are $a, b \in \mathsf{R}^{\bullet}$ such that $a + b = 0$ and let $\sigma \in \Delta_{\mathsf{R}}$ and $\sigma' \in \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$ be such that $\sigma \to \sigma'$; then for all $\tau \in \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$, $\tau = a\sigma + b\sigma + \tau$ and then $\tau \mathrel{\widetilde{\to}} a\sigma' + b\sigma + \tau$. Hence every term τ reduces.

Moreover, even if R is positive, it may be the case that the only normalizable terms are normal terms. Indeed, assume R is the set $\mathbf{Q}^+$ of non-negative rational numbers (which is a positive rig), and $\sigma \to \sigma'$; then there is an infinite sequence of reductions from σ:

$$\sigma = \frac{1}{2}\sigma + \frac{1}{2}\sigma \mathrel{\widetilde{\to}} \frac{1}{2}\sigma + \frac{1}{2}\sigma' \mathrel{\widetilde{\to}} \frac{1}{4}\sigma + \frac{3}{4}\sigma' \mathrel{\widetilde{\to}} \cdots \mathrel{\widetilde{\to}} \frac{1}{2^n}\sigma + \frac{2^n - 1}{2^n}\sigma' \mathrel{\widetilde{\to}} \cdots$$

In order to establish the strong normalization of typed terms, we will therefore assume that R is *finitely splitting* in the following sense: for all $a \in \mathsf{R}$,

$$\{(a_1, \ldots, a_n) \in (\mathsf{R}^{\bullet})^n\,;\ n \in \mathbf{N} \text{ and } a = a_1 + \cdots + a_n\}$$

is finite. We can then define the *width* function

$$\mathsf{w}\,(a) = \max\{n \in \mathbf{N};\ \exists (a_1, \ldots, a_n) \in (\mathsf{R}^{\bullet})^n \text{ s.t. } a = a_1 + \cdots + a_n\}\,.$$

The width function relates the additive structure of R with that of $\mathbf{N}$ as shown by the following lemma:

Lemma 5.1. If R is finitely splitting, then it is positive. Moreover, for all $a, b \in \mathsf{R}$, $\mathsf{w}\,(a) = 0$ iff $a = 0$ and $\mathsf{w}\,(a + b) \geq \mathsf{w}\,(a) + \mathsf{w}\,(b)$.

Proof. Assume R is finitely splitting. Since 0 is neutral for addition in R, the empty sequence is the only element of $\{(a_1, \ldots, a_n) \in (\mathsf{R}^{\bullet})^n\,;\ n \in \mathbf{N} \text{ and } a_1 + \cdots + a_n = 0\}$. Hence $\mathsf{w}\,(0) = 0$ and R is positive. If $a \neq 0$ then $\mathsf{w}\,(a) >= 1$. Hence $\mathsf{w}\,(a) = 0$ implies $a = 0$. Now let $a, b \in \mathsf{R}$. We can write $a = a_1 + \cdots + a_{\mathsf{w}(a)}$ and $b = b_1 + \cdots + b_{\mathsf{w}(b)}$, where the a_i's and the b_j's are non zero. Then $a + b = a_1 + \cdots + a_{\mathsf{w}(a)} + b_1 + \cdots + b_{\mathsf{w}(b)}$ hence $\mathsf{w}\,(a + b) \geq \mathsf{w}\,(a) + \mathsf{w}\,(b)$. □

One essential point of this section is to show that the finite splitting condition efficiently prevents those tricky situations we have just evidenced in $\mathbf{Q}^+$. We are led to prove that strongly normalizing terms are exactly the linear combinations of strongly normalizing simple terms.

The finite splitting property is actually not sufficient for that purpose. Take, for instance, $\mathsf{R} = \mathbf{N} \times \mathbf{N}$, with operations defined pointwise: $(p, q) + (p', q') = (p + p', q + q')$ and $(p, q)(p', q') = (pp', qq')$. It is easily checked that this defines a finitely splitting rig, with $\mathsf{w}\,(p, q) = p + q$. Now write $a = (1, 0)$ and $b = (0, 1)$: we have $\mathsf{w}\,(a) = \mathsf{w}\,(b) = 1$, $a + b = (1, 1) = 1_{\mathsf{R}}$ and $ab = (0, 0) = 0_{\mathsf{R}}$. Then, if we write $\delta = \lambda x\,(x)\,x$, we notice that the only $\widetilde{\rightarrow}$-reduct of term $\underline{a(\delta)\,b\delta}$ is $\underline{\mathbf{0}}$, which is normal, whereas the simple term $\underline{(\delta)\,b\delta}$ has no normal form.

We will therefore require R to be finitely splitting *and* to satisfy the following *integral domain* property: for all $a, b \in \mathsf{R}$, if $ab = 0$ then either $a = 0$ or $b = 0$. In that case, we obtain the following four lemmas.

Lemma 5.2. For all $a, b \in \mathsf{R}$, $\mathsf{w}\,(ab) \geq \mathsf{w}\,(a)\,\mathsf{w}\,(b)$. In particular, $\mathsf{w}\,(1) = 1$.

Proof. Write $a = a_1 + \cdots + a_{\mathsf{w}(a)}$ and $b = b_1 + \cdots + b_{\mathsf{w}(b)}$, where the a_i's and the b_j's are non zero. Then, developping $ab = (a_1 + \cdots + a_{\mathsf{w}(a)})(b_1 + \cdots + b_{\mathsf{w}(b)})$ we obtain $\mathsf{w}\,(a)\,\mathsf{w}\,(b)$ summands, which are all non zero by the integral domain property of R. □

Lemma 5.3. If $\sigma = a\tau + \rho$ with $a \neq 0$ then $\mathsf{Supp}\,(\sigma) = \mathsf{Supp}\,(\tau) \cup \mathsf{Supp}\,(\rho)$.

Proof. By Lemma 4.5, all that remains to be shown is that $\mathsf{Supp}\,(a\tau) = \mathsf{Supp}\,(\tau)$: this follows directly from the integral domain property of R. □

Lemma 5.4. For all σ, σ' such that $\sigma \mathrel{\widetilde{\rightarrow}} \sigma'$, $a\sigma + \tau \mathrel{\widetilde{\rightarrow}} a\sigma' + \tau$ also holds as soon as $a \neq 0$.

Proof. Again, this is a direct consequence of the integral domain property of R. □

Lemma 5.5. For all $\sigma \in \Delta_{\mathsf{R}}$ and all $\sigma' \in \mathsf{R}\,\langle\Delta_{\mathsf{R}}\rangle$, $\sigma \mathrel{\widetilde{\rightarrow}} \sigma'$ iff $\sigma \rightarrow \sigma'$.

Proof. By Lemma 5.3 and the fact that $\mathsf{Supp}\,(\sigma) = \{\sigma\}$, if we write $\sigma = \underline{as + T}$ with $a \neq 0$, then $\underline{s} = \sigma$ and there is $b \in \mathsf{R}$ such that $\underline{T} = b\sigma$. Necessarily, we have $a + b = 1$, which by Lemma 5.2 implies $a = 1$ and $b = 0$. Hence the result by definition of $\widetilde{\rightarrow}$. □

In subsection 5.1, we prove that, under these conditions, $\sigma \in \mathsf{R}\,\langle\Delta_{\mathsf{R}}\rangle$ is strongly normalizing iff every simple term in $\mathsf{Supp}\,(\sigma)$ is strongly normalizing. We then develop the proof of strong normalization of simply typed terms, in subsections 5.2 through 5.4, following Krivine's version of Tait's reducibility method (Kri90). From this, we derive a weak normalization result with the only assumption that R is positive, in subsection 5.5.

Examples. Obviously, the rig $\mathbf{N}$ is finitely splitting with $\mathsf{w}\,(n) = n$ for all $n \in \mathbf{N}$, and has no zero divisor. One more interesting instance is the rig of all polynomials over variables $\xi_1, \ldots, \xi_n$ with non-negative integer coefficients, denoted by $\mathsf{P}_n = \mathbf{N}[\xi_1, \ldots, \xi_n]$: for all $P \in \mathsf{P}_n$, $\mathsf{w}\,(P) = P(1, \ldots, 1)$. Such a rig of polynomials is involved in the weak normalization scheme we develop in section 5.5. All other examples we know of are given by variants of P_n, for instance:

— any rig $\mathsf{R}[\xi_1, \ldots, \xi_n]$, where R is itself an integral finitely splitting rig;

— any similar rig of polynomials, with the restriction that the ξ_i's do not commute: $\xi_i\xi_j \neq \xi_j\xi_i$ when $i \neq j$ (this is a rig which satisfies our conditions, yet is not commutative for multiplication);
— any similar rig of polynomials, relaxed in that the ξ_i's are supposed to be idempotent: $\xi_i\xi_i = \xi_i$ for all i.

5.1. *Scalars and normalization*

From now on, we assume R is finitely splitting and integral. Under these conditions, we prove a term is strongly normalizing iff it is a linear combination of strongly normalizing simple terms (Theorem 5.11).

Lemma 5.6. Let $\sigma \in \mathsf{R}\langle\Delta_\mathsf{R}\rangle$. There are only finitely many terms σ' such that $\sigma \widetilde{\rightarrow} \sigma'$.

Proof. The proof is by induction on $\mathrm{h}(\sigma)$. If $\mathrm{h}(\sigma) = 0$ then $\sigma = \underline{\mathbf{0}}$ and the property holds trivially by Lemma 4.5. Assume that the property holds for all σ such that $\mathrm{h}(\sigma) \leq k$. Let $\sigma \in \mathsf{R}\langle\Delta_\mathsf{R}\rangle$ be such that $\mathrm{h}(\sigma) = k+1$. For each term $\sigma' \in \mathsf{R}\langle\Delta_\mathsf{R}\rangle$ such that $\sigma \widetilde{\rightarrow} \sigma'$, there are $\underline{t} \in \Delta_\mathsf{R}$, $\underline{T'}, \underline{U} \in \mathsf{R}\langle\Delta_\mathsf{R}\rangle$ and $a \in \mathsf{R}^\bullet$ such that $\sigma = \underline{at + U}$, $\sigma' = \underline{aT' + U}$ and $\underline{t} \rightarrow \underline{T'}$. By Lemma 4.5, $\underline{t} \in \mathsf{Supp}(\sigma)$: there are finitely many such simple terms. Moreover, due to the finite splitting condition on R, for each such $\underline{t}$ there exist finitely many $a \in \mathsf{R}^\bullet$ and $\underline{U} \in \mathsf{R}\langle\Delta_\mathsf{R}\rangle$ such that $\sigma = \underline{at + U}$. A simple inspection of the definition of $\rightarrow$ shows that, by inductive hypothesis applied to subterms of $\underline{t}$ (i.e. $\triangleq$-classes of subterms of t, all of height at most k), $\underline{t}$ $\rightarrow$-reduces to finitely many terms, which are all the possible choices for $\underline{T'}$. □

König's lemma thus justifies the following definition:

Definition 5.7. If σ is a strongly normalizing term, we denote by $|\sigma|$ the length of the longest sequence of $\widetilde{\rightarrow}$-reductions from σ to its normal form. We denote by N_R the set of strongly normalizing simple terms and $\mathsf{N}_\mathsf{R}(n) = \{\sigma \in \mathsf{N}_\mathsf{R} \text{ s.t. } |\sigma| \leq n\}$.

Then $\mathsf{R}\langle\mathsf{N}_\mathsf{R}\rangle$ is the set of linear combinations of strongly normalizing simple terms:

$$\mathsf{R}\langle\mathsf{N}_\mathsf{R}\rangle = \{\sigma \in \mathsf{R}\langle\Delta_\mathsf{R}\rangle\,;\ \mathsf{Supp}(\sigma) \subseteq \mathsf{N}_\mathsf{R}\}\,.$$

In the following, we prove that $\mathsf{R}\langle\mathsf{N}_\mathsf{R}\rangle$ is exactly the set of all strongly normalizing terms. We first show the easiest inclusion.

Lemma 5.8. The support of every strongly normalizing term is a finite subset of N_R. More precisely, if σ is strongly normalizing, then $\mathsf{Supp}(\sigma) \subset \mathsf{N}_\mathsf{R}(|\sigma|)$.

Proof. By Lemma 5.4, from a sequence of reductions from $\tau \in \mathsf{Supp}(\sigma)$, we can derive a sequence of reductions from σ of the same length. □

We now establish the reverse inclusion: the terms in $\mathsf{R}\langle\mathsf{N}_\mathsf{R}\rangle$ are strongly normalizing. The proof boils down to the following idea: to each $\sigma \in \mathsf{R}\langle\mathsf{N}_\mathsf{R}\rangle$, we associate a finite multiset $\|\sigma\|$ of natural numbers so that if $\sigma \widetilde{\rightarrow} \sigma'$ then $\|\sigma\| > \|\sigma'\|$, where $>$ denotes the *multiset order* (which is a well-order).

First we fix notations for multisets. We write $\mathcal{M}_{\mathrm{fin}}(\mathbf{N})$ for the set of finite multisets of natural numbers. If $p_1, \ldots, p_n \in \mathbf{N}$, we write $[p_1, \ldots, p_n] \in \mathcal{M}_{\mathrm{fin}}(\mathbf{N})$ for the multiset containing exactly $p_1, \ldots, p_n$, taking repetitions into account. If $\mu, \nu \in \mathcal{M}_{\mathrm{fin}}(\mathbf{N})$, $\mu + \nu$ denotes the disjoint union of μ and ν, and if $k \in \mathbf{N}$, $k\mu$ denotes the multiset $\sum_{i=1}^{k} \mu$. Now assume $\mu = [p_1, \ldots, p_m]$ and $\nu = [q_1, \ldots, q_n]$, with $p_1 \leq \cdots \leq p_m$ and $q_1 \leq \cdots \leq q_n$, we recall that $\mu < \nu$ for the multiset order iff one of the following holds:

— $m = 0$ and $n > 0$;
— $mn \neq 0$ and $p_m < q_n$;
— $mn \neq 0$, $p_m = q_n$ and $[p_1, \ldots, p_{m-1}] < [q_1, \ldots, q_{n-1}]$.

This strict order is the transitive closure of the following relation: $\mu \prec \mu'$ iff $\mu = \nu + [p_1, \ldots, p_m]$ and $\mu' = \nu + [q]$ where, for all i, $p_i < q$. The well-foundedness of the multiset order amounts to the fact that there is no infinite descending chain for $\prec$.

Definition 5.9. For all $\tau \in \Delta_{\mathsf{R}}$ and $\sigma \in \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$, we write $\mathsf{w}_\tau(\sigma)$ for the width of the coefficient of τ in σ: $\mathsf{w}_\tau(\sigma) = \mathsf{w}\left(\sigma_{(\tau)}\right)$. If moreover $\sigma \in \mathsf{R}\langle\mathsf{N}_{\mathsf{R}}\rangle$, we write

$$\|\sigma\| = \sum_{\tau \in \mathsf{Supp}(\sigma)} \mathsf{w}_\tau(\sigma)\,[\,|\tau|\,]\,.$$

For instance, if σ is a strongly normalizing simple term, $\|\sigma\| = \mathsf{w}(1)\,[\,|\sigma|\,] = [\,|\sigma|\,]$.

Lemma 5.10. Let $\sigma \in \mathsf{R}\langle\mathsf{N}_{\mathsf{R}}\rangle$ and let σ' be such that $\sigma \to \sigma'$. Then $\sigma' \in \mathsf{R}\langle\mathsf{N}_{\mathsf{R}}\rangle$ and $\|\sigma'\| < \|\sigma\|$.

Proof. Write $\sigma = \underline{as + T}$ and $\sigma' = \underline{aS' + T}$ with $\underline{s} \to \underline{S'}$. Since $\sigma \in \mathsf{R}\langle\mathsf{N}_{\mathsf{R}}\rangle$, Lemma 5.3 entails $\underline{s} \in \mathsf{N}_{\mathsf{R}}$: write $|\underline{s}| = p + 1$. Clearly, $\underline{S'}$ is strongly normalizing and $|\underline{S'}| \leq p$. By Lemma 5.8, $\mathsf{Supp}(\underline{S'}) \subset \mathsf{N}_{\mathsf{R}}(p)$. Then Lemma 5.3 implies $\mathsf{Supp}(\sigma') = \mathsf{Supp}(\underline{S'}) \cup \mathsf{Supp}(\underline{T}) \subset \mathsf{N}_{\mathsf{R}}$. Hence $\|\sigma'\|$ is well defined.

We now prove that $\|\sigma'\| \prec \|\sigma\|$. The following two facts provide a sufficient condition:

(i) For all $q > |\underline{s}|$, the multiplicity of q in $\|\sigma'\|$ is the same as in $\|\sigma\|$.
(ii) The multiplicity of $|\underline{s}|$ in $\|\sigma'\|$ is stricty less than in $\|\sigma\|$.

Fact (i) boils down to the following equation

$$\sum_{\underline{t} \in \mathsf{N}_{\mathsf{R}}(q)} \mathsf{w}_{\underline{t}}(\sigma) = \sum_{\underline{t} \in \mathsf{N}_{\mathsf{R}}(q)} \mathsf{w}_{\underline{t}}(\sigma')$$

for all $q > |\underline{s}|$. It is then sufficient to show that, for $q > |\underline{s}|$ and for all $\underline{t} \in \mathsf{N}_{\mathsf{R}}(q)$, $\mathsf{w}_{\underline{t}}(\sigma') = \mathsf{w}_{\underline{t}}(\sigma)$. Since $\mathsf{Supp}(\underline{S'}) \subset \mathsf{N}_{\mathsf{R}}(p)$ and $p < q$, we deduce that $\underline{S'}_{(\underline{t})} = 0$ and $\sigma'_{(\underline{t})} = \underline{T}_{(\underline{t})} = \sigma_{(\underline{t})}$ and we conclude.

Similarly, to prove fact (ii), we must show that

$$\sum_{\underline{t} \in \mathsf{N}_{\mathsf{R}}(|\underline{s}|)} \mathsf{w}_{\underline{t}}(\sigma) > \sum_{\underline{t} \in \mathsf{N}_{\mathsf{R}}(|\underline{s}|)} \mathsf{w}_{\underline{t}}(\sigma')$$

Let $\underline{t} \in \mathsf{N}_{\mathsf{R}}(|\underline{s}|)$. With the same argument as above, $\underline{S'}_{(\underline{t})} = 0$ and then $\sigma'_{(\underline{t})} = \underline{T}_{(\underline{t})}$. If $t \neq s$, we thus have $\sigma'_{(\underline{t})} = \sigma_{(\underline{t})}$, hence $\mathsf{w}_{\underline{t}}(\sigma') = \mathsf{w}_{\underline{t}}(\sigma)$. Moreover, by Lemma 5.1, $\mathsf{w}_{\underline{s}}(\sigma) = \mathsf{w}\left(a + \underline{T}_{(\underline{s})}\right) \geq \mathsf{w}(a) + \mathsf{w}_{\underline{s}}(\underline{T})$ and $\mathsf{w}(a) > 0$. Since $\underline{T}_{(\underline{s})} = \sigma'_{(\underline{s})}$, we obtain $\mathsf{w}_{\underline{s}}(\sigma) > \mathsf{w}_{\underline{s}}(\sigma')$ □

We can now state the final theorem of this subsection:

Theorem 5.11. The set of all strongly normalizing terms is $\mathsf{R}\langle \mathsf{N_R}\rangle$.

Proof. One inclusion is Lemma 5.8. The other one follows from Lemma 5.10 and the fact that the multiset order is a well-order. □

5.2. *Saturated sets*

We now define a notion of saturation on sets of simple terms, and prove $\mathsf{N_R}$ is saturated. Here the conditions we imposed on R are crucial, since the proof heavily relies on Theorem 5.11.

Definition 5.12. Let $\mathcal{X}$ be a set of simple terms. An $\mathcal{X}$-redex is a simple term of the following shape:

$$\sigma = \underline{(\lambda x\, s)\, T}$$

where $\underline{s} \in \mathcal{X}$ and $\underline{T} \in \mathsf{R}\langle \mathcal{X}\rangle$. We write $\mathrm{Red}(\sigma)$ for the term obtained by firing this redex: $\mathrm{Red}(\sigma) = \underline{s\,[T/x]}$.

Definition 5.13. The set $\mathcal{X}$ is *saturated* if, for all $\mathsf{N_R}$-redex σ and all $\tau_1, \ldots, \tau_n \in \mathsf{R}\langle \mathsf{N_R}\rangle$, $(\mathrm{Red}(\sigma))\,\tau_1 \cdots \tau_n \in \mathsf{R}\langle \mathcal{X}\rangle$ implies $(\sigma)\,\tau_1 \cdots \tau_n \in \mathcal{X}$.

Lemma 5.14. The set $\mathsf{N_R}$ is saturated.

Proof. We have to prove that, for all $\mathsf{N_R}$-redex σ and all $\tau_1, \ldots, \tau_n \in \mathsf{R}\langle \mathsf{N_R}\rangle$, if $(\mathrm{Red}(\sigma))\,\tau_1 \cdots \tau_n \in \mathsf{R}\langle \mathsf{N_R}\rangle$ then $(\sigma)\,\tau_1 \cdots \tau_n \in \mathsf{N_R}$. We write $\sigma = \underline{(\lambda x\, s)\, T_0}$ where $\underline{s} \in \mathsf{N_R}$ and $\underline{T_0} \in \mathsf{R}\langle \mathsf{N_R}\rangle$, and, for each i, write $\tau_i = \underline{T_i}$. With these notations, we are led to prove that, for all $\underline{s} \in \mathsf{N_R}$ and all $\underline{T_0}, \ldots, \underline{T_n} \in \mathsf{R}\langle \mathsf{N_R}\rangle$, if

$$\underline{(s\,[T_0/x])\, T_1 \cdots T_n} \in \mathsf{R}\langle \mathsf{N_R}\rangle\,, \tag{9}$$

then

$$\rho = \underline{(\lambda x\, s)\, T_0 \cdots T_n} \in \mathsf{N_R}.$$

By Theorem 5.11, each $\underline{T_i}$ is strongly normalizing. We prove the result by induction on $|\underline{s}| + \sum_{i=0}^{n} |\underline{T_i}|$. By Lemma 5.5, it is sufficient to show that for all ρ' such that $\rho \to \rho'$, ρ' is strongly normalizing. The reduction $\rho \to \rho'$ can occur at the following positions:

— at the root of the $\mathsf{N_R}$-redex;
— inside $\underline{s}$;
— inside one of the $\underline{T_i}$'s.

Head reduction. In the first case, which is the only possible one if $|\underline{s}| + \sum_{i=0}^{n} |\underline{T_i}| = 0$, $\rho' = (\mathrm{Red}(\sigma))\,\tau_1 \cdots \tau_n$ so hypothesis (9) applies directly.

Reduction in the function. Consider the case in which reduction occurs inside $\underline{s}$. So $\rho' = \underline{(\lambda x\, S')\, T_0 \cdots T_n}$ with $\underline{s} \to \underline{S'}$. Write the canonical term $S' = \sum_{l=1}^{q} a_l s'_l$ and, for all $l \in \{1, \ldots, q\}$, define $\rho'_l = \underline{(\lambda x\, s'_l)\, T_0 \cdots T_n}$ so that $\rho' = \sum_{l=1}^{q} a_l \rho'_l$. It is then sufficient to prove that, for all $l \in \{1, \ldots, q\}$, $\rho'_l \in \mathsf{N_R}$. For all l, $\left|\underline{s'_l}\right| < |\underline{s}|$ and the induction hypothesis applies to the data $\underline{s'_l}, \underline{T_0}, \ldots, \underline{T_n}$. Hence it is sufficient to show that $\underline{(s'_l\,[T_0/x])\,T_1\cdots T_n} \in \mathsf{R}\langle \mathsf{N_R}\rangle$. By hypothesis (9), $(\mathrm{Red}\,(\sigma))\,\tau_1\cdots\tau_n \in \mathsf{R}\langle \mathsf{N_R}\rangle$. Since $\underline{s} \to \underline{S'}$, Corollary 3.13 and Lemma 3.5 imply $(\mathrm{Red}\,(\sigma))\,\tau_1\cdots\tau_n \mathrel{\widetilde{\to}}^* \underline{\sum_{l=1}^{q} a_l(s'_l\,[T_0/x])\,T_1\cdots T_n}$. Hence each $\underline{(s'_l\,[T_0/x])\,T_1\cdots T_n} \in \mathsf{R}\langle \mathsf{N_R}\rangle$ by Lemma 5.3.

Reduction in an argument. Consider the case in which reduction occurs inside $\underline{T_i}$: $\rho' = \underline{(\lambda x\, s)\,T_0\cdots T'_i\cdots T_n}$ with $\underline{T_i} \mathrel{\widetilde{\to}} \underline{T'_i}$. Since $\left|\underline{T'_i}\right| < \left|\underline{T_i}\right|$, the induction hypothesis applies to the data $\underline{s}, \underline{T_0}, \ldots, \underline{T'_i}, \ldots, \underline{T_n}$. Hence it is sufficient to show that (9) holds for that data: $\underline{(s\,[T_0/x])\,T_1\cdots T'_i\cdots T_n} \in \mathsf{R}\langle \mathsf{N_R}\rangle$ — or $\underline{(s\,[T'_0/x])\,T_1\cdots T_n} \in \mathsf{R}\langle \mathsf{N_R}\rangle$ if $i = 0$. We can conclude directly, since this is a $\widetilde{\to}^*$-reduct of $(\mathrm{Red}\,(\sigma))\,\tau_1\cdots\tau_n \in \mathsf{R}\langle \mathsf{N_R}\rangle$ by contextuality of $\widetilde{\to}^*$ — plus Proposition 2.6 if $i = 0$.

□

5.3. *Reducibility*

To each simple type, we associate a saturated subset of $\mathsf{N_R}$ as follows.

Definition 5.15. If $\mathcal{X}$ and $\mathcal{Y}$ are sets of simple terms, one defines $\mathcal{X} \to \mathcal{Y} \subseteq \Delta_\mathsf{R}$ by:

$$\mathcal{X} \to \mathcal{Y} = \{\sigma \in \Delta_\mathsf{R};\ \text{ for all } \tau \in \mathsf{R}\langle \mathcal{X}\rangle, (\sigma)\,\tau \in \mathcal{Y}\}\,.$$

Proposition 5.16. If $\mathcal{X}, \mathcal{X}', \mathcal{Y}, \mathcal{Y}' \subseteq \Delta_\mathsf{R}$ are such that $\mathcal{X} \subseteq \mathcal{X}'$ and $\mathcal{Y}' \subseteq \mathcal{Y}$, then $\mathcal{X}' \to \mathcal{Y}' \subseteq \mathcal{X} \to \mathcal{Y}$.

Lemma 5.17. If $\mathcal{S}$ is a saturated set and $\mathcal{X} \subseteq \mathsf{N_R}$, then $\mathcal{X} \to \mathcal{S}$ is saturated.

Proof. This is straightforward from the definitions of saturation and $\mathcal{X} \to \mathcal{S}$. □

Definition 5.18. We define the interpretation A^* of type A by induction on A:

— $\phi^* = \mathsf{N_R}$ if ϕ is a basic type;
— $(A \to B)^* = A^* \to B^*$.

Definition 5.19. Let $\mathsf{E_R}$ be the set of all simple terms σ of shape $\sigma = (\underline{x})\,\tau_1\cdots\tau_n$, where $\tau_1, \ldots, \tau_n \in \mathsf{R}\langle \mathsf{N_R}\rangle$. These are called *neutral terms*.

Lemma 5.20. The following inclusions hold:

$$\mathsf{E_R} \subseteq (\mathsf{N_R} \to \mathsf{E_R}) \subseteq (\mathsf{E_R} \to \mathsf{N_R}) \subseteq \mathsf{N_R}.$$

Proof. Of course, $\mathsf{E_R} \subseteq \mathsf{N_R}$, hence the central inclusion, by Proposition 5.16. The first inclusion holds by definition of $\mathsf{E_R}$. If $\tau \in \mathsf{E_R} \to \mathsf{N_R}$, let x be any variable, $\underline{x} \in \mathsf{E_R}$ and we have $(\tau)\,\underline{x} \in \mathsf{N_R}$, which implies $\tau \in \mathsf{N_R}$ by Lemma 3.4; hence the last inclusion. □

Corollary 5.21. For all type A, $\mathsf{E_R} \subseteq A^* \subseteq \mathsf{N_R}$.

5.4. *Adequation*

We finish the strong normalization proof: every simply typed term lies in the interpretation of its type. More formally:

Theorem 5.22. Let σ be a term and assume

$$x_1 : A_1, \dots, x_m : A_m \vdash_{\mathsf{R}} \sigma : A$$

is derivable. Let $\sigma_1 \in \mathsf{R}\langle A_1^* \rangle, \dots, \sigma_m \in \mathsf{R}\langle A_m^* \rangle$. Then

$$\sigma[\sigma_1, \dots, \sigma_m / x_1, \dots, x_m] \in \mathsf{R}\langle A^* \rangle .$$

Proof. Write $\tau = \sigma[\sigma_1, \dots, \sigma_m / x_1, \dots, x_m]$. We prove $\tau \in \mathsf{R}\langle A^* \rangle$ by induction on $\mathsf{can}(\sigma)$.

Variable. $\sigma = \underline{x_i}$ for some i and $A = A_i$. Then $\tau = \sigma_i \in \mathsf{R}\langle A_i^* \rangle$ by hypothesis.

Application. $\sigma = \underline{(s)\,T}$ with $x_1 : A_1, \dots, x_m : A_m \vdash s : B \to A$ and $x_1 : A_1, \dots, x_m : A_m \vdash T : B$. By inductive hypothesis,

$$\underline{s}[\sigma_1, \dots, \sigma_m / x_1, \dots, x_m] \in \mathsf{R}\langle (B \to A)^* \rangle$$

and

$$\underline{T}[\sigma_1, \dots, \sigma_m / x_1, \dots, x_m] \in \mathsf{R}\langle B^* \rangle .$$

Hence $\tau \in \mathsf{R}\langle A^* \rangle$ by definition of $B^* \to A^*$.

Abstraction. $\sigma = \underline{\lambda x\, s}$ and $A = B \to C$ with

$$x_1 : A_1, \dots, x_m : A_m, x : B \vdash s : C.$$

We assume x is distinct from every x_i and does not occur free in any $\mathsf{can}(\sigma_i)$. Then $\tau = \underline{\lambda x\, S'}$ with

$$\underline{S'} = \underline{s}[\sigma_1, \dots, \sigma_m / x_1, \dots, x_m].$$

We show that $\tau \in \mathsf{R}\langle (B \to C)^* \rangle$ using the definition of $B^* \to C^*$: let $\underline{T} \in \mathsf{R}\langle B^* \rangle$, we have to prove $\underline{(\lambda x\, S')\, T} \in \mathsf{R}\langle C^* \rangle$. Since C^* is saturated, it is sufficient to show that $\underline{S'[T/x]} \in \mathsf{R}\langle C^* \rangle$. By Proposition 2.4,

$$\underline{S'[T/x]} = \underline{s}[\underline{T}, \sigma_1, \dots, \sigma_m / x, x_1, \dots, x_m]$$

and we conclude by the induction hypothesis applied to $\underline{s}$.

Linear combinations. $\sigma = \underline{\sum_{i=1}^n a_i s_i}$ and $\Gamma \vdash s_i : A$ for all $i \in \{1, \dots, n\}$. Then, by the induction hypothesis, each $\underline{s_i}[\sigma_1, \dots, \sigma_m / x_1, \dots, x_m] \in \mathsf{R}\langle A^* \rangle$ and we conclude.

□

We get the following corollary of Theorem 5.22.

Theorem 5.23. All weakly typable terms are strongly normalizing.

Proof. Let $\sigma \in \mathsf{R}\langle \Delta_{\mathsf{R}} \rangle$ be such that $x_1 : A_1, \dots, x_m : A_m \vdash_{\mathsf{R}} \sigma : A$ is derivable. For all $i \in \{1, \dots, n\}$, since $\mathsf{E}_{\mathsf{R}} \subseteq A_i^*$, $\underline{x_i} \in \mathsf{R}\langle A_i^* \rangle$. Hence $\sigma = \sigma\left[\underline{x_1}, \dots, \underline{x_m} / x_1, \dots, x_m\right] \in \mathsf{R}\langle A^* \rangle$ by Theorem 5.22 and we conclude by Corollary 5.21 and Theorem 5.11. □

5.5. *Weak normalization scheme*

Remember that we forced strong conditions on R in the beginning of this section. One can get rid of this restriction by slightly changing the notion of normal form, as was already noted by Ehrhard and Regnier in (ER03). In the following, we provide a full development of their argument.

Definition 5.24. We define *pre-normal terms* and *pre-neutral terms* by the following inductive statements:

— $\sigma \in \Delta_{\mathsf{R}}$ is a pre-neutral term if $\sigma = \underline{x}$ with $x \in \mathcal{V}$, or $\sigma = \underline{(s)\,T}$, where $\underline{s}$ is a pre-neutral term and $\underline{T}$ is a pre-normal term;
— $\sigma \in \Delta_{\mathsf{R}}$ is a simple pre-normal term if σ is pre-neutral, or $\sigma = \underline{\lambda x\, s}$ where $\underline{s}$ is a simple pre-normal term;
— σ is a pre-normal term if, for all $\underline{s} \in \mathsf{Supp}\,(\sigma)$, $\underline{s}$ is a simple pre-normal term.

Intuitively, pre-normal terms are those terms σ such that $\mathsf{can}\,(\sigma)$ contains no redex. Hence:

Proposition 5.25. If R is positive then pre-normal terms are exactly normal terms (and pre-neutral terms are exactly neutral terms).

A rig of polynomials. Let R be any rig and Ξ be a set of variables in bijection with R: to every $a \in \mathsf{R}$ we associate $\xi_a \in \Xi$ such that $\xi_a = \xi_b$ iff $a = b$, and $\Xi = \{\xi_a;\, a \in \mathsf{R}\}$.

Definition 5.26. Let $\mathsf{P} = \mathbf{N}\,[\Xi]$ be the rig of polynomials with non-negative integer coefficients over variables in Ξ. If $P \in \mathsf{P}$, and $f : \mathsf{R} \longrightarrow \mathsf{R}'$ where R' is any rig, we denote by

$$P\{a \mapsto f(a)\}$$

the valuation of P at f, i.e. the scalar (in R') obtained by replacing each ξ_a in P by $f(a)$, for all $a \in \mathsf{R}$.

Definition 5.27. If $P \in \mathsf{P}$, we denote by $[\![P]\!]$ the value of P in R:

$$[\![P]\!] = P\{a \mapsto a\} \in \mathsf{R}.$$

Lemma 5.28. The rig P is finitely splitting and has no zero divisor.

Proof. The width function is exactly the sum of all coefficients:

$$\mathsf{w}\,(P) = P\{a \mapsto 1\} \in \mathbf{N}.$$

□

Hence Theorem 5.23 applies and we obtain:

Corollary 5.29. All weakly typable terms in $\mathsf{P}\,\langle\Delta_{\mathsf{P}}\rangle$ are strongly normalizing.

We extend the valuation of a term in $\mathsf{P}\,\langle\Delta_{\mathsf{P}}\rangle$ as the term in $\mathsf{R}\,\langle\Delta_{\mathsf{R}}\rangle$ obtained by replacing each polynomial coefficient with its value.

Definition 5.30. We define $\llbracket\cdot\rrbracket : \mathsf{P}\langle\Delta_{\mathsf{P}}\rangle \longrightarrow \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$ by induction on terms:

$$\begin{array}{rcl}
\llbracket\underline{x}\rrbracket & = & x \\
\llbracket\underline{\lambda x\, s}\rrbracket & = & \lambda x\,\llbracket\underline{s}\rrbracket \\
\left\llbracket\underline{(s)\,T}\right\rrbracket & = & (\llbracket\underline{s}\rrbracket)\,\llbracket\underline{T}\rrbracket \\
\left\llbracket\underline{\sum_{i=1}^{n} P_i s_i}\right\rrbracket & = & \sum_{i=1}^{n} \llbracket P_i\rrbracket\,\llbracket\underline{s_i}\rrbracket\,.
\end{array}$$

Proposition 5.31. For all $\sigma \in \mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$, if σ is a pre-normal term, then $\llbracket\sigma\rrbracket \in \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$ is a pre-normal term.

Lemma 5.32. For all $\sigma, \sigma' \in \mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$, if $\sigma \,\widetilde{\rightarrow}\, \sigma'$, then $\llbracket\sigma\rrbracket \,\widetilde{\rightarrow}^{*}\, \llbracket\sigma\rrbracket'$.

Proof. The proof is easy by induction on reduction $\sigma \,\widetilde{\rightarrow}\, \sigma'$. □

Definition 5.33. For all $M \in \Lambda_{\mathsf{R}}$, define $\check{M} \in \Lambda_{\mathsf{P}}$ as the permutative term obtained from M by replacing every coefficient a with the monomial χ_a.

Lemma 5.34. For all $\underline{S} \in \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$, $\underline{S} = \left\llbracket\underline{\check{S}}\right\rrbracket$.

Proof. For all $\underline{s} \in \mathsf{Supp}\,(\underline{S})$, $\underline{S}_{(\underline{s})} = S_{(s)} = \left\llbracket\xi_{S_{(s)}}\right\rrbracket = \left\llbracket\check{S}_{(\check{s})}\right\rrbracket$. □

Lemma 5.35. Let $\underline{S} \in \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$. If $\Gamma \vdash_{\mathsf{R}} \underline{S} : A$ then $\Gamma \vdash \check{S} : A$.

Proof. One easily proves by induction on permutative term M that that if $\Gamma \vdash M : A$ then $\Gamma \vdash \check{M} : A$. □

Theorem 5.36. Let $\sigma \in \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$ be a weakly typable term. Then σ is weakly normalizing in the sense that it reduces to a pre-normal form.

Proof. If σ is weakly typable then, by Lemma 5.35, $\check{\sigma}$ is typable. By Theorem 5.23, $\check{\sigma}$ is strongly normalizing, hence $\check{\sigma} \,\widetilde{\rightarrow}^{*}\, \tau$ where τ is normal. By Proposition 5.25, τ is pre-normal, and so is $\llbracket\tau\rrbracket$ by Proposition 5.31. By Lemma 5.32, $\sigma \,\widetilde{\rightarrow}^{*}\, \llbracket\tau\rrbracket$, hence the conclusion. □

Recall that if R is positive, then every pre-normal form is a normal form; in this case Theorem 5.36 states a genuine weak normalization property.

6. Other Approaches and Related Work

Undeterminate Forms. It is noteworthy that the collapse we described in section 3.3 involves a term ∞_σ such that $\infty_\sigma \,\widetilde{\rightarrow}^{*}\, n\sigma + \infty_\sigma$, for all $n \in \mathbf{N}$: reduction of ∞_σ generates an unbounded amount of σ. This is not a surprise, since the untyped algebraic λ-calculus involves both linear algebra and arbitrary fixed points. The term $\infty_\sigma + (-1)\infty_\sigma$ is then analoguous to the well know indeterminate form $\infty - \infty$ of the affinely extended real number line (that is $\mathbf{R} \cup \{-\infty, \infty\}$, the two-point compactification of $\mathbf{R}$, where the usual operations can be extended only partially). The collapse of reduction in presence of negative scalars follows from the fact that we consider $\infty_\sigma - \infty_\sigma = \underline{\mathbf{0}}$.

Notice that our observations do not depend on equations (1) and (2). As a matter of fact, if there exists $\eta \in \mathsf{R}$ with $1 + \eta = 0$, then any contextual equivalence relation $\cong$ defined on raw terms such that:
— $\cong$ contains β-reduction, i.e. $(\lambda x\, M)\, N \cong M\,[N/x]$ for all $M, N \in \Lambda_{\mathsf{R}}$;
— $\cong$ contains R-module equations (groups of equations (6) and (7));
is unsound. Indeed, we can define $\infty_M \in \Lambda_{\mathsf{R}}$ for all $M \in \Lambda_{\mathsf{R}}$, and then $\infty_M + \eta\infty_M$ is $\cong$-equal to both M and $\mathbf{0}$:

$$\infty_M + \eta\infty_M \cong (1+\eta)\infty_M \cong \mathbf{0}$$

and

$$\begin{array}{rcll} \infty_M + \eta\infty_M & \cong & (M + \infty_M) + \eta\infty_M & \text{by iterated } \beta\text{-reductions} \\ & \cong & M + (\infty_M + \eta\infty_M) & \\ & \cong & M + (1+\eta)\infty_M & \\ & \cong & M + \mathbf{0} & \\ & \cong & M. & \end{array}$$

One seemingly natural variant of one-step reduction is the following one, which we already outlined in our introduction. Rather than (4), extend reduction from simple terms to all terms by:

$$\sigma \mathrel{\widehat{\to}} \sigma' \text{ if } \sigma = \underline{as + T} \text{ and } \sigma' = \underline{aS' + T}, \text{ with } a \neq 0,\ T_{(s)} = 0 \text{ and } \underline{s} \to \underline{S'}. \tag{10}$$

As far as reduction is concerned, this amounts to restrict the syntax to canonical forms of terms. Notice this is not contextual in the sense of definition 2.5. This is still unsound in general, however: one can reproduce the argument of section 3.3.2, replacing $a\infty_\sigma + b\infty_\sigma$ with $a\infty_\sigma + b(\lambda x\, x)\, \infty_\sigma$.

We have already mentioned another technique to deactivate coefficients and tame $\triangleq$ during reduction: replace the coefficients of a term with formal variables, then reduce some steps, last replace the variables with their values. Reduction $\widehat{\to}$ can be seen as a strategy in this setting. In particular, $\widehat{\to}$ is well-behaved as far as normalization is concerned: the trick involving rational coefficients is no longer possible, and (weakly) typed terms are strongly normalizing.

A possible fix to the collapse while retaining the algebraic structure of the calculus might involve typing, in order to ward arbitrary fixed points off. Then one has to introduce some typed notion of reduction: we have seen that typability isn't even preserved under our notion of reduction. This is the subject of current work, in connection with the quantitative semantics of simply typed ordinary λ-calculus in the finiteness spaces of (Ehr05).

Algebraic Rewriting. In (AD08), Arrighi and Dowek introduced the linear algebraic λ-calculus. The background setting is quite unrelated: their work provides a framework for quantum computation; in particular, terms represent linear operators, hence application is bilinear rather than linear in the function only. Notwithstanding this distinction, their approach to λ-calculus with linear combinations of terms contrasts with ours: consider terms up to $\equiv$ rather than some variant of $\triangleq$, and handle the identities between linear combinations, together with analogues of (1) and (2), as reduction rules.

Confronted to problems similar to those we exposed above in presence of negative coefficients, they opted for a completely different solution, far more natural in their setting: restrict those reduction rules involving rewriting of linear combinations to closed terms in normal form. This allows to tame some of the intrinsic potential infinity of the pure λ-calculus, avoiding to consider indeterminate forms. Up to these restrictions, they prove confluence for the whole system.

This opens interesting perspectives for future work, already the subject of a collaboration with Arrighi and Dowek. In particular, it seems a system similar to that of (AD08) can be designed in the setting of the algebraic λ-calculus. Moreover one can see the divergence on the treatment of linearity in both works as the manifestation of the call-by-name (CBN) vs. call-by-value (CBV) duality: Arrighi–Dowek's linear algebraic λ-calculus is intrinsically a CBV system, whereas our algebraic λ-calculus is rooted in the CBN translation of λ-calculus in linear logic (recall that it originates in the presentation the differential λ-calculus of Ehrhard and Regnier). It is a matter of particular interest whether both calculi enjoy the same relationship as is known between the CBN and CBV flavours of pure λ-calculus.

Aknowledgements

I am deeply indebted to René David, who suggested the use of the multiset order in the proof of Theorem 5.11.

I thank the anonymous referee, for her/his careful reading and many useful suggestions and comments, concerning both style and mathematical content.

I also enjoyed enlightening discussions with Pablo Arrighi and Gilles Dowek.

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