# On linear combinations of $\lambda$-terms

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**Abstract.** We define an extension of $\lambda$-calculus with linear combinations, endowing the set of terms with a structure of $\mathsf{R}$-module, where $\mathsf{R}$ is a fixed set of scalars. Terms are moreover subject to identities similar to usual pointwise definition of linear combinations of functions with values in a vector space. We then extend $\beta$-reduction on those algebraic $\lambda$-terms as follows: $at + u$ reduces to $at' + u$ as soon as term $t$ reduces to $t'$ and $a$ is a non-zero scalar. We prove that reduction is confluent.
Under the assumption that the set $\mathsf{R}$ of scalars is positive (*i.e.* a sum of scalars is zero iff all of them are zero), we show that this algebraic $\lambda$-calculus is a conservative extension of ordinary $\lambda$-calculus. On the other hand, we show that if $\mathsf{R}$ admits negative elements, then every term reduces to every other term. We investigate the causes of that collapse, and discuss some possible fixes.

*Preliminary definitions and notations.* Recall that a rig (also known as "semi-ring with zero and unit") is the same as a ring, without the condition that every element admits an opposite for addition. Let $\mathsf{R}$ be a rig. We write $\mathsf{R}^\bullet$ for $\mathsf{R} \setminus \{0\}$. We denote by letters $a, b, c$ the elements of $\mathsf{R}$, and say that $\mathsf{R}$ is $\mathsf{positive}$ if, for all $a, b \in \mathsf{R}$, $a + b = 0$ implies $a = 0$ and $b = 0$. An example of positive rig is $\mathbf{N}$, the set of natural numbers, with usual addition and multiplication.

If $i, j \in \mathbf{N}$, we write $[i; j]$ for the set $\{k \in \mathbf{N}; i \leq k \leq j\}$. Also, we write application of $\lambda$-terms à la Krivine: $(s)\, t$ denotes the application of term $s$ to term $t$.

## 1 Introduction

Sums of terms arise naturally in the study of differentiation in $\lambda$-calculus [ER03] or $\lambda\mu$-calculus [Vau06]. In this setting, non-deterministic choice provides a possible computational interpretation of sum. In differential $\lambda$-calculus, however, a more general pattern is introduced: the set of terms is endowed with a structure of $\mathsf{R}$-module, where $\mathsf{R}$ is a commutative rig, and one can form linear combinations of terms. Moreover, in the same way as functions with values in a vector space also form a vector space with operations defined pointwise, we have the following two equalities on terms:

$$\lambda x \left( \sum_{i=1}^{n} a_i s_i \right) = \sum_{i=1}^{n} a_i \lambda x\, s_i \tag{1}$$

and

$$\left(\sum_{i=1}^{n} a_i s_i\right) u = \sum_{i=1}^{n} a_i \,(s_i)\, u \tag{2}$$

for all linear combination $\sum_{i=1}^{n} a_i s_i$ of terms. This mimics the quantitative semantics of $\lambda$-calculus in finiteness spaces [Ehr05]: types are interpreted by particular vector spaces or, more generally, modules, and terms are mapped to analytic functions defined by power series on these spaces.

Apart from the notion of differentiation, one important feature of the above-mentioned works is the way $\beta$-reduction is extended to such linear combinations of terms. Among terms, some are considered simple: they contain no sum in linear position, so that nor (1) nor (2) applies; hence they are intrinsically not sums. These form a basis of the $\mathsf{R}$-module of terms. Reduction $\rightarrow$ is then the least contextual relation such that: if $s$ is a simple term, then

$$(\lambda x\, s)\, t \rightarrow s\,[t/x] \tag{3}$$

and, if $a \in \mathsf{R}^{\bullet}$ is a non-zero scalar,

$$s \rightarrow s' \text{ implies } as + t \rightarrow as' + t. \tag{4}$$

The condition $a \neq 0$ in that last case ensures that $\rightarrow$ actually reduces something, so that reduction is not trivially reflexive.

The previous definition is both natural in presence of coefficients, and technically efficient. For instance, it is particularly well suited for proving confluence via usual Tait-Martin Löf technique: introduce a parallel version $\Rightarrow$ of $\rightarrow$ such that $\rightarrow \subset \Rightarrow \subset \rightarrow^*$, and prove that $\Rightarrow$ has the diamond property. Here $\Rightarrow$ is defined on sums as follows:

$$\sum_{i=1}^{n} a_i s_i \Rightarrow \sum_{i=1}^{n} a_i s'_i \text{ as soon as, } \forall i \in [1;n],\ s_i \text{ is simple and } s_i \Rightarrow s'_i. \tag{5}$$

This variant of (4) allows to close the following diagram:

$$\begin{array}{ccc} s + s' & \Rightarrow & 2s' \\ \Downarrow & & \Downarrow \\ s + s'' & \Rightarrow & s' + s'' \end{array}$$

assuming $s \Rightarrow s' \Rightarrow s''$ are simple terms. This would not hold if we had forced the $s_i$'s in (5) to be distinct simple terms — that condition would amount to reduce each element of the base of simple terms, in parallel, which may seem a more natural choice at first.

In [Vau06], however, the author proved that this reduction behaves strangely as soon as the rig of scalars admits negative elements: if $-1 \in \mathsf{R}$ (so that $1 + (-1) = 0$), then for all terms $s$ and $t$, $s \rightarrow^* t$. Also, it does not accommodate well with normalization: assume $s \rightarrow s'$ and $\mathsf{R}$ contains positive rational numbers; then

$$s \;=\; \frac{1}{2}s + \frac{1}{2}s \;\rightarrow\; \frac{1}{2}s + \frac{1}{2}s' \;\rightarrow\; \frac{1}{4}s + \frac{3}{4}s' \;\rightarrow\; \dots$$

which forbids strong normalization.

*Contributions.* In this paper, we give a framework for the study of terms with linear combinations, which aims to be more precise and formal than that developed in [ER03] or [Vau06]. Also, we do not consider differentiation nor classical control operators, and only focus on the algebraic structure of terms and the interaction between coefficients and reduction. We call the obtained system algebraic $\lambda$-calculus.

In section 2, we formalize the definition of the $\mathsf{R}$-module of terms; in particular, we implement identities (1) and (2), orienting then from left to right, and identify terms up to equality of the canonical forms thus obtained. This definition is elementary enough that it should be easily implemented in a logical system such as [Coq]. In section 3 we define reduction, using rule (4) in the case of a sum, and discuss conservativity w.r.t. usual $\beta$-reduction. In section 4, we briefly review sufficient conditions for normalization, postponing a full proof of strong normalization until Appendix A. Last, we discuss possible other approaches and further work in section 5.

Most of the results of this paper were already present in [Vau06], and some can be traced back to [ER03]. In those two previous works, however, the focus was on differentiation and the presence of linear combinations of terms and their effects on reduction were considered of marginal interest. This may in particular explain why some of the problems we insist on in this paper eluded [ER03].

## 2 Linear combinations of terms

In this section, we introduce the set of terms of algebraic $\lambda$-calculus in several steps. First we give a grammar of terms, on which we define $\alpha$-equivalence and substitution as in Krivine's [Kri90]. Then we define canonical forms of terms; this endows the set of terms with a structure of module, by identifying terms up to equality of canonical forms.

### 2.1 Raw terms

Let be given a denumerable set $\mathsf{V}$ of variables. We use letters among $x, y, z$ to denote variables.

**Definition 1.** *The set $\mathrm{L}_{\mathsf{R}}$ of raw terms of algebraic $\lambda$-calculus over $\mathsf{R}$ is inductively defined by the following rules:*

- *any variable $x$ is a term,* i.e. $\mathsf{V} \subset \mathrm{L}_{\mathsf{R}}$*;*
- *if $x \in \mathsf{V}$ and $\sigma \in \mathrm{L}_{\mathsf{R}}$, then $\lambda x\, \sigma \in \mathrm{L}_{\mathsf{R}}$;*
- *if $\sigma, \tau \in \mathrm{L}_{\mathsf{R}}$ then $(\sigma)\, \tau \in \mathrm{L}_{\mathsf{R}}$;*
- $\mathbf{0} \in \mathrm{L}_{\mathsf{R}}$*;*
- *if $a \in \mathsf{R}$ and $\sigma \in \mathrm{L}_{\mathsf{R}}$ then $a\sigma \in \mathrm{L}_{\mathsf{R}}$;*
- *if $\sigma$ and $\tau \in \mathrm{L}_{\mathsf{R}}$, then $\sigma + \tau \in \mathrm{L}_{\mathsf{R}}$.*

**Definition 2.** *We define free variables of terms as follows:*

- *variable $x$ is free in term $y$ if $x = y$;*

– *variable $x$ is free in $\lambda y\, \sigma$ if $x \neq y$ and $x$ is free in $\sigma$;*
– *variable $x$ is free in $(\sigma)\, \tau$ if $x$ is free in $\sigma$ or in $\tau$;*
– *no variable is free in* $\mathbf{0}$;
– *variable $x$ is free in $a\sigma$ if $x$ is free in $\sigma$;*
– *variable $x$ is free in term $\sigma + \tau$ if $x$ is free in $\sigma$ or in $\tau$.*

From this definition of free variables, we derive $\alpha$-equivalence and substitution as in [Kri90]. We write $\sigma \sim \tau$ when $\sigma$ is $\alpha$-convertible to $\tau$, and we write $\sigma\, [\tau/x]$ for the (capture-avoiding) substitution of $\tau$ for $x$ in $\sigma$. More generally, if $x_1, \ldots, x_n$ are distinct variables and $\tau_1, \ldots, \tau_n$ are terms, we write $\sigma\, [\tau_1, \ldots, \tau_n/x_1, \ldots, x_n]$ for the simultaneous substitution of each $\tau_i$ for each $x_i$ in $\sigma$. Recall the following definitions and properties from [Kri90].

**Proposition 1.** *For all terms $\sigma, \tau_1, \ldots, \tau_n, \upsilon_1, \ldots, \upsilon_p$ and all distinct variables $x_1, \ldots, x_n, y_1, \ldots, y_p$,*

$$\begin{aligned} &\sigma\, [\tau_1, \ldots, \tau_n/x_1, \ldots, x_n]\, [\upsilon_1, \ldots, \upsilon_p/y_1, \ldots, y_p] \\ \sim\, &\sigma\, [\upsilon_1, \ldots, \upsilon_p, \tau'_1, \ldots, \tau'_n/y_1, \ldots, y_p, x_1, \ldots, x_n] \end{aligned}$$

*where $\tau'_i = \tau_i\, [\upsilon_1, \ldots, \upsilon_p/y_1, \ldots, y_p]$.*

**Definition 3.** *A binary relation* $\mathrm{r}$ *on raw terms is said* contextual *if it satisfies the following conditions:*

– $x\ \mathrm{r}\ x$;
– $\lambda x\, \sigma\ \mathrm{r}\ \lambda x\, \sigma'$ *as soon as* $\sigma\ \mathrm{r}\ \sigma'$;
– $(\sigma)\, \tau\ \mathrm{r}\ (\sigma')\, \tau'$ *as soon as* $\sigma\ \mathrm{r}\ \sigma'$ *and* $\tau\ \mathrm{r}\ \tau'$;
– $\mathbf{0}\ \mathrm{r}\ \mathbf{0}$;
– $a\sigma\ \mathrm{r}\ a\sigma'$ *as soon as* $\sigma\ \mathrm{r}\ \sigma'$;
– $\sigma + \tau\ \mathrm{r}\ \sigma' + \tau'$ *as soon as* $\sigma\ \mathrm{r}\ \sigma'$ *and* $\tau\ \mathrm{r}\ \tau'$.

**Proposition 2.** *If* $\mathrm{r}$ *is a contextual relation, then* $\sigma\, [\tau/x]\ \mathrm{r}\ \sigma\, [\tau'/x]$ *as soon as* $\tau\ \mathrm{r}\ \tau'$.

### 2.2 Permutative equality

If $\sigma_1, \ldots, \sigma_n \in \mathrm{L}_{\mathsf{R}}$, then we write $\sigma_1 + \ldots + \sigma_n$ for $\sigma_1 + (\ldots + \sigma_n)$. If, moreover, $a_1, \ldots, a_n \in \mathsf{R}$ then we write $\sum_{i=1}^{n} a_i\sigma_i$ for the term $a_1\sigma_1 + \ldots + a_n\sigma_n + \mathbf{0}$.

Linear combinations $\sum_{i=1}^{n} a_i\sigma_i$ should be thought of as multisets of couples, *i.e.* we identify $\sum_{i=1}^{n} a_i\sigma_i$ with all $\sum_{i=1}^{n} a_{f(i)}\sigma_{f(i)}$ where $f$ is any permutation of $[1; n]$. This is more formally stated in the following definition.

**Definition 4.** Permutative equality $\equiv\, \subseteq \mathrm{L}_{\mathsf{R}} \times \mathrm{L}_{\mathsf{R}}$ *is the least contextual equivalence relation on raw terms such that:*

– $\sigma \equiv \tau$ *as soon as* $\sigma \sim \tau$;
– $\sigma + \tau \equiv \tau + \sigma$ *for all* $\sigma, \tau \in \mathrm{L}_{\mathsf{R}}$;
– $(\sigma + \tau) + \upsilon \equiv \sigma + \tau + \upsilon$ *for all* $\sigma, \tau, \upsilon \in \mathrm{L}_{\mathsf{R}}$.

Notice that $\sum_{i=1}^{n} a_i \sigma_i \equiv \sum_{j=1}^{p} b_j \tau_j$ iff $n = p$ and, for all $j \in [1;n]$, $b_j = a_{f(j)}$, with $f$ some fixed permutation of $[1;n]$. Also, since free variables of a sum do not depend on the order of the summands, $\equiv$ preserves free variables.

Permutative equality is the basic equality intended on terms. It states that we consider terms up to $\alpha$-equality and that we form linear combinations up to associativity and commutativity of sum. We write $\Lambda_{\mathsf{R}}$ for the set of terms with equality $\equiv$. This means that as long we consider $\sigma$ and $\tau$ as terms in $\Lambda_{\mathsf{R}}$, we say they are equal if $\sigma \equiv \tau$. A function defined on $\Lambda_{\mathsf{R}}$ is a function with domain $\mathrm{L}_{\mathsf{R}}$ which is invariant by $\equiv$.

**Proposition 3.** *Substitution is defined on $\Lambda_{\mathsf{R}}$: if $\sigma \equiv \sigma'$ and $\tau_i \equiv \tau_i'$, for all $i \in [1;n]$, then $\sigma\,[\tau_1, \ldots, \tau_n/x_1, \ldots, x_n] \equiv \sigma'\,[\tau_1', \ldots, \tau_n'/x_1, \ldots, x_n]$ for all distinct variables $x_1, \ldots, x_n$.*

In the following, if $\sigma$ and $\tau \in \Lambda_{\mathsf{R}}$, we define

$$\delta^{\equiv}_{\sigma,\tau} = \begin{cases} 1 \text{ if } \sigma \equiv \tau \\ 0 \text{ otherwise} \end{cases}$$

### 2.3 The R-module of terms

In this subsection, we introduce the algebraic content of the calculus: we endow the set of terms with a structure of R-module, enjoying usual identities between linear combinations together with (1) and (2). For that purpose, we define canonical forms of terms, so that equality of terms (in the abovementioned algebraic sense) amounts to permutative equality of canonical forms.

**Definition 5.** *Atomic terms and canonical terms are defined as follows:*

- *any variable $x$ is an atomic term;*
- *if $x \in \mathsf{V}$ and $s$ is an atomic term, then $\lambda x\, s$ is an atomic term;*
- *if $s$ is an atomic term and $T$ is a canonical term, then $(s)\,T$ is an atomic term;*
- *if $a_1, \ldots, a_n \in \mathsf{R}^\bullet$ and $s_1, \ldots, s_n$ are $n$ pairwise distinct ($\not\equiv$) atomic terms, then $\sum_{i=1}^{n} a_i s_i$ is a canonical term.*

We consider atomic and canonical terms up to permutative equality. We write $\mathrm{A}_{\mathsf{R}}$ for the set of atomic terms and $\mathrm{C}_{\mathsf{R}}$ for the set of canonical terms, both endowed with $\equiv$ as identity relation. One defines an injection from atomic terms into canonical terms, mapping $s$ to the "singleton" $1s + \mathbf{0}$. In the following, we write $\boldsymbol{\sigma}$ for term $1\sigma + \mathbf{0}$.

**Definition 6.** *Let $\sigma = \sum_{i=1}^{n} a_i s_i$ be a linear combination of atomic terms. For all atomic term $s$, we call coefficient of $s$ in $\sigma$ the scalar*

$$\sum_{i=1}^{n} \delta^{\equiv}_{s,s_i} a_i.$$

*Then we define*

$$\mathsf{cansum}(\sigma) = \sum_{j=1}^{p} b_j t_j$$

*where $\{t_1, \ldots, t_p\}$ is the set (modulo $\equiv$) of those $s_i$ with a non-zero coefficient in $\sigma$ and, for all $j \in [1;p]$, $b_j$ is the coefficient of $t_j$ in $\sigma$.*

Hence, if $\sigma$ is a linear combination of atomic terms, then $\mathsf{cansum}(\sigma)$ is a canonical term.

**Definition 7.** *Canonization of terms* $\mathsf{can} : \Lambda_{\mathsf{R}} \longrightarrow \mathrm{C}_{\mathsf{R}}$ *is given by*

- $\mathsf{can}(x) = \boldsymbol{x}$;
- *if* $\mathsf{can}(\sigma) = \sum_{i=1}^{n} a_i s_i$ *then* $\mathsf{can}(\lambda x\, \sigma) = \sum_{i=1}^{n} a_i \lambda x\, s_i$;
- *if* $\mathsf{can}(\sigma) = \sum_{i=1}^{n} a_i s_i$ *and* $\mathsf{can}(\tau) = T$ *then* $\mathsf{can}((\sigma)\, \tau) = \sum_{i=1}^{n} a_i\, (s_i)\, T$;
- $\mathsf{can}(\mathbf{0}) = \mathbf{0}$;
- *if* $\mathsf{can}(\sigma) = \sum_{i=1}^{n} a_i s_i$ *then* $\mathsf{can}(a\sigma) = \mathsf{cansum}\left(\sum_{i=1}^{n} (a a_i) s_i\right)$;
- *if* $\mathsf{can}(\sigma) = \sum_{i=1}^{n} a_i s_i$ *and* $\mathsf{can}(\tau) = \sum_{i=n+1}^{n+p} a_i s_i$ *then*
$$\mathsf{can}(\sigma + \tau) = \mathsf{cansum}\left(\sum_{i=1}^{n+p} a_i s_i\right).$$

It is easily checked that this definition is invariant by $\equiv$.

**Proposition 4.** *Canonization enjoys the following properties.*

*(i)* *Variables free in* $\mathsf{can}(\sigma)$ *are also free in* $\sigma$. *The converse does not hold in general.*
*(ii)* *If* $s$ *is an atomic term, then* $\mathsf{can}(s) \equiv \boldsymbol{s}$.
*(iii)* *If* $S$ *is a canonical term, then* $\mathsf{can}(S) \equiv S$; *hence* $\mathsf{can}(\mathsf{can}(\sigma)) \equiv \mathsf{can}(\sigma)$ *for all term* $\sigma$.
*(iv)* *For all terms* $\sigma$ *and* $\tau$ *and all variable* $x$,

$$\mathsf{can}(\sigma\, [\tau/x]) \equiv \mathsf{can}(\mathsf{can}(\sigma)\, [\mathsf{can}(\tau)/x]).$$

**Definition 8.** *Algebraic equality is permutative equality of canonical forms:* $\sigma \overset{\triangledown}{=} \tau$ *if* $\mathsf{can}(\sigma) \equiv \mathsf{can}(\tau)$.

Although it does not preserve free variables, algebraic equality is a contextual equivalence relation. Restricted to canonical terms, it is the same as $\equiv$.

**Definition 9.** *A* simple term *is a term* $\sigma$ *such that* $\mathsf{can}(\sigma) = \boldsymbol{s}$ *with* $s$ *atomic, or equivalently such that there exists an atomic term* $s$ *with* $s \overset{\triangledown}{=} \sigma$. *We write* $\Delta_{\mathsf{R}}$ *for the set of simple terms, with equality* $\overset{\triangledown}{=}$ *and* $\mathsf{R}\langle \Delta_{\mathsf{R}} \rangle$ *for the set of all terms, with equality* $\overset{\triangledown}{=}$, *which is the free* $\mathsf{R}$*-module generated by* $(\Delta_{\mathsf{R}}, \overset{\triangledown}{=})$.

**Lemma 1.** *If terms* $\sigma$, $\sigma'$, $\tau$ *and* $\tau'$ *are such that* $\sigma \overset{\triangledown}{=} \sigma'$ *and* $\tau \overset{\triangledown}{=} \tau'$, *then, for all variable* $x$, $\sigma\, [\tau/x] \overset{\triangledown}{=} \sigma'\, [\tau'/x]$.

*Proof.* This is a straightforward application of Proposition 4, (iv).

Hence, algebraic equality is compatible with substitution, *i.e.* substitution is defined on $\mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$. We now extend the notion of coefficient as follows:

**Definition 10.** *Let $\sigma$ be a simple term and $s$ an atomic term such that $\sigma \overset{\triangledown}{=} s$. We define the coefficient of $\sigma$ in $\tau$, denoted by $\tau_{(\sigma)}$, as the coefficient of $s$ in $\mathsf{can}(\tau)$.*

We call support of $\sigma$ the set of those simple terms with a non-zero coefficient in $\sigma$:

$$\mathrm{Supp}(\sigma) = \{\tau \in \Delta_{\mathsf{R}};\ \sigma_{(\tau)} \neq 0\}.$$

**Definition 11.** *If $\mathcal{X}$ is a set (modulo $\overset{\triangledown}{=}$) of simple terms, we write $\mathsf{R}\langle\mathcal{X}\rangle$ for the set of linear combinations of elements of $\mathcal{X}$,* i.e.

$$\mathsf{R}\langle\mathcal{X}\rangle = \left\{\sigma \overset{\triangledown}{=} \sum_{i=1}^{n} a_i\sigma_i;\ \forall i \in [1;n],\ \sigma_i \in \mathcal{X}\right\}$$

*or, equivalently,*

$$\mathsf{R}\langle\mathcal{X}\rangle = \{\sigma \in \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle;\ \mathrm{Supp}(\sigma) \subseteq \mathcal{X}\}\,.$$

One may introduce a convergent rewrite system (modulo associativity and commutativity of sum [PS81]) $R$ on terms in $\Lambda_{\mathsf{R}}$, such that $\sigma \overset{\triangledown}{=} \tau$ iff $\mathrm{NF}_R(\sigma) \equiv \mathrm{NF}_R(\tau)$, where $\mathrm{NF}_R$ stands for "normal form in $R$". This is left as an easy exercise for the reader (beware that we do not ask $\mathrm{NF}_R(\sigma) = \mathsf{can}(\sigma)$). Apart from the lack of space, we do not use this approach simply because such a rewrite system $R$ would not be part of the reduction rules we introduce thereafter: we will define reduction of terms as (roughly) $\beta$-reduction up to $\overset{\triangledown}{=}$, and not as the union of $\beta$-reduction and canonization.

## 3 Reductions

In this section, we define reduction using (3) and (4) as key reduction rules: this captures the definition of reduction in [ER03], minus differentiation, in the setting of algebraic $\lambda$-calculus.

### 3.1 Reduction and linear combinations of terms

We call algebraic relation from simple terms to terms any subset of $\Delta_{\mathsf{R}} \times \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$ (which is invariant under $\overset{\triangledown}{=}$). If $\mathrm{r}$ is an algebraic relation from simple terms to terms, $\sigma\ \mathrm{r}\ \sigma'$ iff there are $t \in \mathrm{A}_{\mathsf{R}}$ and $T' \in \mathrm{C}_{\mathsf{R}}$ such that $\sigma \overset{\triangledown}{=} t$, $\sigma' \overset{\triangledown}{=} T'$ and $t\ \mathrm{r}\ T'$.

Similarly, we call algebraic relation from terms to terms any subset of $\mathsf{R}\langle\Delta_{\mathsf{R}}\rangle \times \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$. Again, such a relation is uniquely defined by its restriction to $\mathrm{C}_{\mathsf{R}} \times \mathrm{C}_{\mathsf{R}}$. Given an algebraic relation $\mathrm{r}$ from simple terms to terms we define two new algebraic relations $\bar{\mathrm{r}}$ and $\widetilde{\mathrm{r}}$ from terms to terms by:

– $\sigma \;\bar{\mathrm{r}}\; \sigma'$ if $\sigma \overset{\triangledown}{=} \sum_{i=1}^{n} a_i s_i$ and $\sigma' \overset{\triangledown}{=} \sum_{i=1}^{n} a_i S'_i$, where for all $i \in [1;n]$, $s_i$ is atomic, $S'_i$ is canonical and $s_i \;\mathrm{r}\; S'_i$;
– $\sigma \;\tilde{\mathrm{r}}\; \sigma'$ if $\sigma \overset{\triangledown}{=} at + U$ and $\sigma' \overset{\triangledown}{=} aT' + U$, where $a \neq 0$, $t$ is atomic, $T'$ and $U$ are canonical and $t \;\mathrm{r}\; T'$.

We cannot define reduction by induction on terms: if there are $a, b \in \mathsf{R}^{\bullet}$ such that $a + b = 0$ then $\mathbf{0} \overset{\triangledown}{=} a\sigma + b\sigma$ for all $\sigma \in \mathsf{R}\langle \Delta_{\mathsf{R}} \rangle$; hence, by rule (4), $\mathbf{0}$ may reduce. We rather define simple term reduction $\to$ by induction on the depth of the fired redex, so that reduction of terms is given by $\widetilde{\to}$.

**Definition 12.** *We define an increasing sequence of algebraic relations from simple terms to terms by the following statements. $\to_0$ is the empty relation. Assume $\to_k$ is defined. Then we set $\sigma \to_{k+1} \sigma'$ as soon as one of the following holds:*

– $\sigma \overset{\triangledown}{=} \lambda x\, s$ *and* $\sigma' \overset{\triangledown}{=} \lambda x\, S'$ *with* $s \to_k S'$*;*
– $\sigma \overset{\triangledown}{=} (s)\, T$ *and* $\sigma' \overset{\triangledown}{=} (S')\, T$ *with* $s \to_k S'$*, or* $\sigma' \overset{\triangledown}{=} (s)\, T'$ *with* $T \,\widetilde{\to_k}\, T'$*;*
– $\sigma \overset{\triangledown}{=} (\lambda x\, s)\, T$ *and* $\sigma' \overset{\triangledown}{=} s\, [T/x]$*.*

Let $\to\; = \bigcup_{k \in \mathbf{N}} \to_k$. We call one-step reduction or simply reduction, the algebraic relation $\widetilde{\to}$.

**Proposition 5.** $\widetilde{\to} = \bigcup_{k \in \mathbf{N}} \widetilde{\to_k}$.

**Lemma 2.** *If $s \in \mathrm{A}_{\mathsf{R}}$ and $S', T, T' \in \mathrm{C}_{\mathsf{R}}$, are such that $s \to S'$ and $T \,\widetilde{\to}\, T'$ then:*

$$\lambda x\, s \to \lambda x\, S'$$
$$(s)\, T \to (S')\, T$$
$$(s)\, T \to (s)\, T' \quad (*)$$

*Proof.* The first two relations are straightforward from the definition of $\to$. The same holds for relation $(*)$, through proposition 5.

Let $\widetilde{\to}^{*}$ be the reflexive and transitive closure of $\widetilde{\to}$.

**Lemma 3.** *The relation $\widetilde{\to}^{*}$ is contextual.*

*Proof.* This results from Lemma 2, using reflexivity, transitivity and the definition of can.

### 3.2 Confluence

We prove confluence of $\widetilde{\to}$ by usual Tait-Martin-Löf technique: introduce a parallel extension of reduction (in which redexes can be fired simultaneously) and prove this enjoys the diamond property (*i.e.* strong confluence).

**Definition 13.** *We define an increasing sequence of algebraic relations from simple terms to terms by the following statements. $\rightarrowtriangle_0$ is algebraic equality. Assume $\rightarrowtriangle_k$ is defined. Then we set $\sigma \rightarrowtriangle_{k+1} \sigma'$ as soon as one of the following holds:*

- $\sigma \stackrel{\triangledown}{=} \lambda x\, s$ *and* $\sigma' \stackrel{\triangledown}{=} \lambda x\, S'$ *with* $s \rightarrowtriangle_k S'$*;*
- $\sigma \stackrel{\triangledown}{=} (s)\, T$ *and* $\sigma' \stackrel{\triangledown}{=} (S')\, T'$ *with* $s \rightarrowtriangle_k S'$ *and* $T \,\overline{\rightarrowtriangle_k}\, T'$*;*
- $\sigma \stackrel{\triangledown}{=} (\lambda x\, s)\, T$ *and* $\sigma' \stackrel{\triangledown}{=} S'\,[T'/x]$ *with* $s \rightarrowtriangle_k S'$ *and* $T \,\overline{\rightarrowtriangle_k}\, T'$*.*

Let $\rightarrowtriangle = \bigcup_{k\in\mathbf{N}} \rightarrowtriangle_k$. We call parallel reduction the algebraic relation $\overline{\rightarrowtriangle}$.

**Proposition 6.** $\overline{\rightarrowtriangle} = \bigcup_{k\in\mathbf{N}} \overline{\rightarrowtriangle_k}$.

**Lemma 4.** *Relation* $\overline{\rightarrowtriangle}$ *is contextual.*

*Proof.* Like in Lemma 2, this is just rephrasing the definitions of $\pi$ and $\overline{\pi}$, with the notable exception of the application case which involves Proposition 6.

**Lemma 5.** $(\lambda x\, \sigma)\, \tau \,\overline{\rightarrowtriangle}\, \sigma'\,[\tau'/x]$ *as soon as* $\sigma \,\overline{\rightarrowtriangle}\, \sigma'$ *and* $\tau \,\overline{\rightarrowtriangle}\, \tau'$*.*

*Proof.* This is a straightforward consequence of Lemma 4 and the definitions of $\overline{\rightarrowtriangle}$ and $\mathsf{can}(\lambda x\, \sigma)$.

**Lemma 6.** $\widetilde{\to} \subset \overline{\rightarrowtriangle} \subset \widetilde{\to}^*$.

*Proof.* $\widetilde{\to} \subset \overline{\rightarrowtriangle}$ should be clear. $\overline{\rightarrowtriangle} \subset \widetilde{\to}^*$ follows from contextuality of $\widetilde{\to}^*$.

*Reductions and substitution.* The main property of parallel reduction is the following one, which fails for one-step reduction.

**Lemma 7.** *Let $x$ be a variable and $\sigma, \tau, \sigma', \tau'$ be terms. If $\sigma \,\overline{\rightarrowtriangle}\, \sigma'$ and $\tau \,\overline{\rightarrowtriangle}\, \tau'$ then*

$$\sigma\,[\tau/x] \,\overline{\rightarrowtriangle}\, \sigma'\,[\tau'/x]\,.$$

*Proof.* We prove by induction on $k$ that if $\sigma \,\overline{\rightarrowtriangle_k}\, \sigma'$ and $\tau \,\overline{\rightarrowtriangle}\, \tau'$ then $\sigma\,[\tau/x] \,\overline{\rightarrowtriangle}\, \sigma'\,[\tau'/x]$. If $k = 0$ then $\sigma' \stackrel{\triangledown}{=} \sigma \stackrel{\triangledown}{=} \mathsf{can}(\sigma)$; then by Lemmas 1 and 4, and Proposition 2, we have

$$\sigma\,[\tau/x] \stackrel{\triangledown}{=} \mathsf{can}(\sigma)\,[\tau/x] \,\overline{\rightarrowtriangle}\, \mathsf{can}(\sigma)\,[\tau'/x] \stackrel{\triangledown}{=} \sigma'\,[\tau'/x]\,.$$

Suppose the result holds for some $k$, then we extend it to $k+1$ by inspecting the possible cases for reduction $\sigma \,\overline{\rightarrowtriangle_{k+1}}\, \sigma'$. We first address the case in which $\sigma$ is simple and $\sigma \rightarrowtriangle_{k+1} \sigma'$. Then one of the following statements applies:

- $\sigma \stackrel{\triangledown}{=} \lambda y\, t$ with $y \neq x$ and $y$ not free in $\tau$, and $\sigma' \stackrel{\triangledown}{=} \lambda y\, T'$ with $t \rightarrowtriangle_k T'$; hence, by induction hypothesis, $t\,[\tau/x] \,\overline{\rightarrowtriangle}\, T'\,[\tau'/x]$ and we get

  $$\sigma\,[\tau/x] \stackrel{\triangledown}{=} \lambda y\, t\,[\tau/x] \,\overline{\rightarrowtriangle}\, \lambda y\, T'\,[\tau'/x] \stackrel{\triangledown}{=} \sigma'\,[\tau'/x]$$

  by Lemma 4;
- $\sigma \stackrel{\triangledown}{=} (t)\, V$ and $\sigma' \stackrel{\triangledown}{=} (T')\, V'$ with $t \rightarrowtriangle_k T'$ and $V \,\overline{\rightarrowtriangle_k}\, V'$: by induction hypothesis, $t\,[\tau/x] \,\overline{\rightarrowtriangle}\, T'\,[\tau'/x]$ and $V\,[\tau/x] \,\overline{\rightarrowtriangle}\, V'\,[\tau'/x]$ and we get

  $$\sigma\,[\tau/x] \stackrel{\triangledown}{=} (t\,[\tau/x])\, V\,[\tau/x] \,\overline{\rightarrowtriangle}\, (T'\,[\tau'/x])\, V'\,[\tau'/x] \stackrel{\triangledown}{=} \sigma'\,[\tau'/x]$$

  by Lemma 4;

– $\sigma \stackrel{\triangledown}{=} (\lambda y\, t)\, V$ and $\sigma' \stackrel{\triangledown}{=} T'\,[V'/y]$ with $t \rightarrowtriangle_k T'$, $V \overline{\rightarrowtriangle_k}\, V'$, $x \neq y$ and $y$ not free in $\tau$: by induction hypothesis, $t\,[\tau/x] \overline{\rightarrowtriangle}\, T'\,[\tau'/x]$, $V\,[\tau/x] \overline{\rightarrowtriangle}\, V'\,[\tau'/x]$ and we get

$$\sigma\,[\tau/x] \stackrel{\triangledown}{=} (\lambda y\, t\,[\tau/x])\, V\,[\tau/x] \overline{\rightarrowtriangle}\, (T'\,[\tau'/x])\,[V'\,[\tau'/x]/y] \stackrel{\triangledown}{=} \sigma'\,[\tau'/x]\,.$$

by Lemma 5.

Now assume $\sigma \overline{\rightarrowtriangle}_{k+1}\, \sigma'$. By definition, this amounts to the following: $\sigma \stackrel{\triangledown}{=} \sum_{i=1}^{n} a_i s_i$ and $\sigma \stackrel{\triangledown}{=} \sum_{i=1}^{n} a_i S'_i$, with $s_i \rightarrowtriangle_{k+1} S'_i$ for all $i$. We have just shown that we then have $s_i\,[\tau/x] \overline{\rightarrowtriangle}\, S'_i\,[\tau'/x]$. We conclude by Lemma 4.

From previous lemma and inclusions $\widetilde{\to} \subseteq \overline{\rightarrowtriangle} \subseteq \widetilde{\to}^*$, we can derive a very similar result for $\widetilde{\to}^*$:

**Corollary 1.** *Let $x$ be a variable and $\sigma, \tau, \sigma', \tau'$ be terms. If $\sigma \widetilde{\to}^* \sigma'$ and $\tau \widetilde{\to}^* \tau'$ then*

$$\sigma\,[\tau/x] \widetilde{\to}^* \sigma'\,[\tau'/x]\,.$$

*Church-Rosser.* We finish the proof of confluence by showing that the $\rightarrowtriangle$-reducts of a fixed term $\sigma$ all $\rightarrowtriangle$-reduce to one of them (obtained by firing all redexes of $\sigma$, simultaneously).

**Definition 14.** *We define inductively on canonical term $S$ its full parallel reduct $S{\downarrow}$ by:*

$$\begin{aligned}
x{\downarrow} &\stackrel{\triangledown}{=} x \\
(\lambda x\, s)\,{\downarrow} &\stackrel{\triangledown}{=} \lambda x\, s{\downarrow} \\
((\lambda x\, s)\, T)\,{\downarrow} &\stackrel{\triangledown}{=} (s{\downarrow})\,[T{\downarrow}/x] \\
((s)\, T)\,{\downarrow} &\stackrel{\triangledown}{=} (s{\downarrow})\, T{\downarrow} \text{ if } s \text{ is a variable or an application} \\
\left(\sum_{i=1}^{n} a_i s_i\right){\downarrow} &\stackrel{\triangledown}{=} \sum_{i=1}^{n} a_i s_i{\downarrow}.
\end{aligned}$$

*For all term $\sigma$, we set $\sigma{\downarrow} \stackrel{\triangledown}{=} \mathsf{can}(\sigma){\downarrow}$.*

**Lemma 8.** *If $\sigma$ and $\sigma'$ are such that $\sigma \overline{\rightarrowtriangle}\, \sigma'$, then $\sigma' \overline{\rightarrowtriangle}\, \sigma{\downarrow}$.*

*Proof.* One simply proves by induction on $k$ that if $\sigma \overline{\rightarrowtriangle_k}\, \sigma'$ then $\sigma' \overline{\rightarrowtriangle}\, \sigma{\downarrow}$, using Lemma 7.

**Theorem 1.** *Relation $\overline{\rightarrowtriangle}$ is strongly confluent. Hence, relation $\widetilde{\to}$ enjoys the Church-Rosser property.*

*Proof.* Strong confluence of $\overline{\rightarrowtriangle}$ is a straightforward corollary of Lemma 8. It implies confluence of $\widetilde{\to}$ by Lemma 6.

*Trivia.* There is a case in which confluence is much easier to establish: if 1 admits an opposite $-1 \in \mathsf{R}$. In this case, assume $\sigma \mathrel{\widetilde{\rightarrow}^*} \sigma'$. Since $\widetilde{\rightarrow}^*$ is algebraic and contextual, $\sigma' \overset{\triangledown}{=} \sigma' + (-1)\sigma + \sigma \mathrel{\widetilde{\rightarrow}^*} \sigma' + (-1)\sigma' + \sigma \overset{\triangledown}{=} \sigma$. Hence $\widetilde{\rightarrow}^*$ is symmetric, which obviously implies Church-Rosser. But this has little meaning: in next section, we show that reduction becomes trivial as soon as $-1 \in \mathsf{R}$.

### 3.3 Conservativity

Notice that every ordinary $\lambda$-term is also a simple term of algebraic $\lambda$-calculus. Let $\Lambda$ denote the set of all $\lambda$-terms and $\rightarrow_\beta \subset \Lambda \times \Lambda$ the usual $\beta$-reduction of $\lambda$-calculus. It is clear that $\rightarrow_\beta \subset \rightarrow$.

Denote by $\leftrightarrow$ the reflexive, symmetric and transitive closure of $\widetilde{\rightarrow}$ and $\leftrightarrow_\beta$ the usual $\beta$-equivalence of $\lambda$-calculus.

**Lemma 9.** *Algebraic $\lambda$-calculus preserves the equalities of $\lambda$-calculus,* i.e. *for all $\lambda$-terms $s$ and $t$, $s \leftrightarrow_\beta t$ implies $s \leftrightarrow t$.*

*Proof.* This is a straightforward consequence of the confluence of $\rightarrow_\beta$ and the fact that $\rightarrow_\beta \subset \widetilde{\rightarrow}$.

One may wonder if the reverse also holds, *i.e.* if equivalence classes of $\lambda$-terms in algebraic $\lambda$-calculus are the same as in ordinary $\lambda$-calculus. If $\mathsf{R}$ is $\mathbf{N}$, then $\widetilde{\rightarrow}$-reductions from $\lambda$-terms are exactly $\rightarrow_\beta$-reductions ($\overset{\triangledown}{=}$ only amounts to $\alpha$-conversion on $\lambda$-terms), and the result holds by the same argument as in Lemma 9. In the general case, however, a $\lambda$-term does not necessarily reduce to another $\lambda$-term, hence the proof is not as easy.

*The positive case.* In the following, we prove that $\leftrightarrow \cap (\Lambda \times \Lambda) = \leftrightarrow_\beta$ as soon as $\mathsf{R}$ is positive.

**Definition 15.** *We define $\Lambda : \mathrm{C}_{\mathsf{R}} \longrightarrow \mathcal{P}(\Lambda)$ by the following statements:*

$$\begin{aligned}
\Lambda(x) &= \{x\} \\
\Lambda(\lambda x\, s) &= \{\lambda x\, u;\ u \in \Lambda(s)\} \\
\Lambda((s)\, T) &= \{(u)\, v;\ u \in \Lambda(s) \ \textit{and}\ v \in \Lambda(T)\} \\
\Lambda\left(\sum_{i=1}^{n} a_i s_i\right) &= \bigcup_{i=1}^{n} \Lambda(s_i)\,.
\end{aligned}$$

*For all term $\sigma$, we set $\Lambda(\sigma) = \Lambda(\mathsf{can}(\sigma))$.*

**Proposition 7.** *If $s \in \Lambda$, then $\Lambda(s) = \{s\}$.*

**Lemma 10.** *If $\mathsf{R}$ is positive and terms $\sigma \in \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$ and $\sigma' \in \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$ are such that $\sigma \mathrel{\widetilde{\rightarrow}} \sigma'$, then for all $s' \in \Lambda(\sigma')$, either $s' \in \Lambda(\sigma)$ or there exists $s \in \Lambda(\sigma)$ such that $s \rightarrow_\beta s'$.*

*Proof.* The proof is by induction on the height of the reduction $\sigma \mathrel{\widetilde{\to}} \sigma'$. All induction steps are straightforward, except for the extension from $\to_k$ to $\widetilde{\to_k}$: assume $\sigma \stackrel{\triangledown}{=} at + U$ and $\sigma' \stackrel{\triangledown}{=} aT' + U$ with $a \neq 0$ and $t \to_k T'$. By definition, $\Lambda(\sigma') = \Lambda(aT' + U) \subseteq \Lambda(T') \cup \Lambda(U)$. Moreover, since $\mathsf{R}$ is positive, the coefficient of $t$ in $at + U$ is non-zero: hence $\Lambda(\sigma) = \Lambda(at + U) = \Lambda(t) \cup \Lambda(U)$. Now assume $v' \in \Lambda(\sigma')$: either $v' \in \Lambda(U) \subset \Lambda(\sigma)$; or $v' \in \Lambda(T')$, and then, by induction hypothesis, either $v' \in \Lambda(t) \subset \Lambda(\sigma)$ or there exists $v \in \Lambda(t) \subset \Lambda(\sigma)$ such that $v \to v'$.

**Corollary 2.** *If $\mathsf{R}$ is positive and $s \in \Lambda$ and $\sigma \in \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$ are such that $s \mathrel{\widetilde{\to}^*} \sigma$, then for all $t \in \Lambda(\sigma)$, $s \to_\beta^* t$.*

**Lemma 11.** *If $\sigma$ and $\sigma' \in \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$ are such that $\sigma \mathrel{\overline{\Rightarrow}} \sigma'$ then $\sigma{\downarrow} \mathrel{\overline{\Rightarrow}} \sigma'{\downarrow}$.*

*Proof.* The proof is easy and very close to that of Lemma 8.

We define iterated full reduction by $\sigma{\downarrow}^0 \stackrel{\triangledown}{=} \sigma$ and $\sigma{\downarrow}^{n+1} \stackrel{\triangledown}{=} (\sigma{\downarrow}^n){\downarrow}$.

**Lemma 12.** *If $\sigma \mathrel{\overline{\Rightarrow}^n} \tau$ then $\tau \mathrel{\widetilde{\to}^*} \sigma{\downarrow}^n$*

*Proof.* The proof is by induction on $n$. If $n = 0$, $\sigma \stackrel{\triangledown}{=} \tau \stackrel{\triangledown}{=} \sigma{\downarrow}^0$ and this is reflexivity of $\widetilde{\to}^*$. Assume the result holds at rank $n$. If $\sigma \mathrel{\overline{\Rightarrow}^n} \tau \mathrel{\overline{\Rightarrow}} \tau'$, then, by induction hypothesis, $\tau \mathrel{\widetilde{\to}^*} \sigma{\downarrow}^n$. Since $\widetilde{\to}^*$ is also the transitive closure of $\overline{\Rightarrow}$, Lemma 11 entails $\tau{\downarrow} \mathrel{\widetilde{\to}^*} \sigma{\downarrow}^{n+1}$. By Lemma 8, we have $\tau' \mathrel{\overline{\Rightarrow}} \tau{\downarrow}$, hence $\tau' \mathrel{\widetilde{\to}^*} \sigma{\downarrow}^{n+1}$.

**Theorem 2.** *If $\mathsf{R}$ is positive and $s, t \in \Lambda$ are such that $s \leftrightarrow t$ then $s \leftrightarrow_\beta t$.*

*Proof.* Assume $s, t \in \Lambda$ and $s \leftrightarrow t$. By the Church-Rosser property of $\widetilde{\to}$ (Theorem 1), there exists $\sigma \in \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$ such that $s \mathrel{\widetilde{\to}^*} \sigma$ and $t \mathrel{\widetilde{\to}^*} \sigma$. By Lemma 12, there exists some $n \in \mathbf{N}$ such that $\sigma \mathrel{\widetilde{\to}^*} v = s{\downarrow}^n$. Notice that if $w \in \Lambda$, then $w{\downarrow} \in \Lambda$, hence $v \in \Lambda$. We have $s \mathrel{\widetilde{\to}^*} v$ and $t \mathrel{\widetilde{\to}^*} v$, hence by positivity of $\mathsf{R}$ and Corollary 2, for all $v' \in \Lambda(v)$ there are $s' \in \Lambda(s)$ and $t' \in \Lambda(t)$ such that $s' \to_\beta^* v'$ and $t' \to_\beta^* v'$. By proposition 7, $\Lambda(s) = \{s\}$, $\Lambda(t) = \{t\}$ and $\Lambda(v) = \{v\}$, hence the conclusion.

*Collapse.* If $\mathsf{R}$ is not positive, we show that reductional equality collapses: $\leftrightarrow$ identifies terms which bear absolutely no relationship with each other.

**Lemma 13.** *Assume, there are $a, b \in \mathsf{R}^\bullet$ such that $a + b = 0$, then for all term $\sigma$, $\mathbf{0} \mathrel{\widetilde{\to}^*} a\sigma \mathrel{\widetilde{\to}^*} \mathbf{0}$.*

*Proof.* Take $Y$ a fixed point combinator of $\lambda$-calculus, such that $(Y)\,s \to_\beta^* (s)\,(Y)\,s$ for all $\lambda$-term $s$. Write $\Upsilon_\sigma$ for $(Y)\,\lambda x\,(\sigma + x)$; then $\Upsilon_\sigma \mathrel{\widetilde{\to}^*} \sigma + \Upsilon_\sigma$. We get:

$$\mathbf{0} \stackrel{\triangledown}{=} a\Upsilon_\sigma + b\Upsilon_\sigma \mathrel{\widetilde{\to}^*} a\sigma + a\Upsilon_\sigma + b\Upsilon_\sigma \stackrel{\triangledown}{=} a\sigma$$

and

$$a\sigma \stackrel{\triangledown}{=} a\sigma + a\Upsilon_\sigma + b\Upsilon_\sigma \mathrel{\widetilde{\to}^*} a\sigma + a\Upsilon_\sigma + b\sigma + b\Upsilon_\sigma \stackrel{\triangledown}{=} \mathbf{0}.$$

**Corollary 3.** *If $\mathsf{R}$ is such that $1$ has an opposite,* i.e. $-1 \in \mathsf{R}$ *with* $1 + (-1) = 0$, *then for all terms $\sigma$ and $\tau$, $\sigma \mathrel{\widetilde{\to}^*} \tau$.*

## 4 On normalization

Unsurprisingly, if $\mathsf{R}$ is not positive, there is no normal term: assume there are $a, b \in \mathsf{R}$ such that $a + b = 0$ and $a \neq 0$ and let $s \in \mathrm{A}_\mathsf{R}$ and $S' \in \mathrm{C}_\mathsf{R}$ be such that $s \to S'$; then for all $\sigma \in \mathsf{R}\langle\Delta_\mathsf{R}\rangle$, $\sigma \stackrel{\triangledown}{=} as + bs + \sigma$ and then $\sigma \mathrel{\widetilde{\rightarrow}} aS' + bs + \sigma$. Hence every term $\sigma$ reduces.

Moreover, positivity is not a sufficient condition for strong normalization to hold: assume $\mathsf{R}$ is the set $\mathbf{Q}^+$ of non-negative rational numbers and let $s \in \mathrm{A}_\mathsf{R}$ and $S' \in \mathrm{C}_\mathsf{R}$ be such that $s \to S'$; then there is an infinite sequence of reductions from $s$:

$$s \stackrel{\triangledown}{=} \frac{1}{2}s + \frac{1}{2}s \mathrel{\widetilde{\rightarrow}} \frac{1}{2}s + \frac{1}{2}S' \mathrel{\widetilde{\rightarrow}} \frac{1}{4}s + \frac{3}{4}S' \mathrel{\widetilde{\rightarrow}} \dots \mathrel{\widetilde{\rightarrow}} \frac{1}{2^n}s + \frac{2^n - 1}{2^n}S' \mathrel{\widetilde{\rightarrow}} \dots$$

whether $s$ is typable or not.

In [ER03], it is proved that if $\mathsf{R}$ is the set $\mathbf{N}$ of all natural numbers, then simply typed terms are strongly normalizing. The associated type system is defined on canonical terms, by adding to usual typing rules for variable, abstraction and application, the following rules for linear combinations:

$$\frac{}{\Gamma \vdash \mathbf{0} : A} \qquad \frac{\Gamma \vdash \sigma : A}{\Gamma \vdash a\sigma : A} \qquad \frac{\Gamma \vdash \sigma : A \quad \Gamma \vdash \tau : A}{\Gamma \vdash \sigma + \tau : A} .$$

Then one extends typing to all terms: for all $\sigma \in \mathsf{R}\langle\Delta_\mathsf{R}\rangle$, write $\Gamma \vdash \sigma : A$ iff $\Gamma \vdash \mathsf{can}(\sigma) : A$. The strong normalization proof is by an variation of Tait's reducibility method, using the following key lemma:

**Lemma 14.** *The set of all strongly normalizing terms is the* $\mathsf{R}$*-submodule of* $\mathsf{R}\langle\Delta_\mathsf{R}\rangle$ *generated by simple strongly normalizing terms:* i.e. $\sigma$ *is strongly normalizing iff, for all* $s \in \mathrm{Supp}(\sigma)$*,* $s$ *is strongly normalizing.*

which is easily established in the case $\mathsf{R} = \mathbf{N}$.

In [Vau06], the author showed that Lemma 14 can be generalized to any rig $\mathsf{R}$ such that:

(i) $\mathsf{R}$ is finitely splitting in the sense that, for all $a \in \mathsf{R}$, the following set is finite
$$\{(a_1, \dots, a_n) \in (\mathsf{R}^\bullet)^n \,;\ n \in \mathbf{N} \text{ and } a = a_1 + \dots + a_n\} ;$$

(ii) the $\mathsf{width}$ function $\mathrm{w} : \mathsf{R} \longrightarrow \mathbf{N}$ defined by
$$\mathrm{w}(a) = \max\{n \in \mathbf{N};\ \exists (a_1, \dots, a_n) \in (\mathsf{R}^\bullet)^n \text{ s.t. } a = a_1 + \dots + a_n\}$$
is a morphism of rigs: $\mathrm{w}(a + b) = \mathrm{w}(a) + \mathrm{w}(b)$ and $\mathrm{w}(ab) = \mathrm{w}(a)\,\mathrm{w}(b)$.

Clearly, these conditions also imply $\mathsf{R}$ is positive.

*Example 1.* Setting $\mathsf{R} = \mathbf{N}$ satisfies these conditions, with $\mathrm{w}(n) = n$ for all $n \in \mathbf{N}$. One more interesting instance is the rig of all polynomials over indeterminates $\xi_1, \dots, \xi_n$ with non-negative integer coefficients $\mathbf{N}[\xi_1, \dots, \xi_n]$: the width of polynomial $P$ is its value at point $(1, \dots, 1)$, *i.e.* the sum of its coefficients.

The proof of strong normalization from [ER03] generalizes to this setting and one obtains the following theorem:

**Theorem 3.** *If* $\mathsf{R}$ *is finitely splitting and* w *is a morphism of rigs, then all typable terms are strongly normalizing.*

As a consequence of this theorem, one also derives the following corollary:

**Corollary 4.** *If* $\mathsf{R}$ *is positive, then every typable term admits a normal form.*

The algorithm behind Corollary 4 can be sketched as follows:

- replace scalars with formal indeterminates in the canonical form of a typable term $\sigma$;
- the object $\tau$ thus obtained can be considered as a term with coefficients in the free rig generated by indeterminates;
- this rig of polynomials enjoys (i) and (ii), hence Theorem 3 applies, since $\tau$ is also typable;
- replace indeterminates by their values in the normal form of $\tau$: this is the normal form of $\sigma$.

A more thorough development on normalization, including a full proof of Theorem 3 is provided in Appendix A.

## 5 Other approaches and future work

It is noteworthy that the collapse we described in section 3.3 involves a term of the form $\Upsilon_\sigma - \Upsilon_\sigma$ where $\Upsilon_\sigma$ does not normalize. More strikingly, $\Upsilon_\sigma \mathrel{\widetilde{\rightarrow}}^* n\sigma + \Upsilon_\sigma$, for all $n \in \mathbf{N}$: reduction of $\Upsilon_\sigma$ generates a potentially infinite amount of $\sigma$. This is not a surprise, since untyped algebraic $\lambda$-calculus involves both linear algebra and arbitrary fixed points. The term $\Upsilon_\sigma - \Upsilon_\sigma$ is then analoguous to the well know indeterminate form $\infty - \infty$ of the affinely extended real line.

The collapse of equality by reduction in presence of negative scalars follows from the fact that we consider $\mathbf{0} \stackrel{\triangledown}{=} \Upsilon_\sigma + (-1)\Upsilon_\sigma$. Also, we have seen that the way we defined reduction is problematic w.r.t. normalization properties: even if $\mathsf{R}$ is positive, typable terms needn't be strongly normalizing. Here we briefly review some possible fixes, each addressing one of these problems, or both.

*Restricting reduction.* One seemingly natural variant on reduction is the following one. Rather than (4), extend reduction of simple terms to all terms by:

$$\sigma \mathrel{\widehat{\rightarrow}} \sigma' \text{ iff } \sigma \stackrel{\triangledown}{=} as + T \text{ and } \sigma' \stackrel{\triangledown}{=} aS' + T, \text{ with } a \neq 0,\ T_{(s)} = 0 \text{ and } s \rightarrow S'. \quad (6)$$

This amounts to restrict the contextuality of reduction to the canonical forms of terms. This reduction, however, is not confluent as soon as $\mathsf{R}$ is not positive: if $a + b = 0$ with $a \neq 0$ and $y \in \mathsf{V}$, then $a\Upsilon_y + b\,(\lambda x\, x)\,\Upsilon_y$ $\widehat{\rightarrow}^*$-reduces both to $ay$ and $\mathbf{0}$, and those two terms are $\widehat{\rightarrow}$-normal forms.

Also, even assuming $\mathsf{R}$ is positive, we did not manage to prove confluence of $\widehat{\rightarrow}$: as we hinted in the introduction, the diamond property fails for the corresponding notion of parallel reduction. Nonetheless, $\widehat{\rightarrow}$-should be well-behaved as far as normalization is concerned: the trick involving rational coefficients is no longer possible.

*Typing.* The restriction we have just suggested diminishes the role of scalar operations during reduction. Another possible fix to the collapse might involve typing, in order to ward arbitrary fixed points off.

The main problem with that idea is that, unless the set of scalars is positive (and we have seen that, in this case, reduction is conservative over usual $\beta$-reduction), typing is not preserved by reduction. Hence, it should be better to study the denotational semantics of ordinary typed $\lambda$-calculus in finiteness spaces [Ehr05] more thoroughly, before investigating further in that direction.

*Restricting equality.* Last, we mention a completely different point of view on linear combinations of terms, that ought to lead to interesting results. In [AD06], Arrighi and Dowek introduce linear algebraic $\lambda$-calculus. The background setting is quite unrelated: their work provides a framework for quantum computation; in particular, terms represent linear operators, hence application is bilinear rather than linear in the function only. Notwithstanding this distinction, their approach to $\lambda$-calculus with linear combinations of terms contrasts with ours: consider terms up to $\equiv$ rather than some variant of $\stackrel{\triangledown}{=}$, and handle the identities between linear combinations, together with analogues of (1) and (2), as reduction rules.

Confronted to problems similar to those we exposed above in presence of negative coefficients, they opted for a completely different solution, far more natural in their setting: restrict those reduction rules involving rewriting of linear combinations to closed terms in normal form. This allows to tame some of the intrinsic potential infinity of pure $\lambda$-calculus, avoiding to consider indeterminate forms. Up to these restrictions, they prove confluence for the whole system.

Although this defines a reduction strategy, in contrast with usual $\beta$-reduction, and they did not consider typing nor normalization properties, it should prove quite enlightening to check that a similar system can be defined for algebraic $\lambda$-calculus. In particular it will be interesting to find out whether the same restrictions on reduction will ensure confluence.

# A Typing and normalization in algebraic $\lambda$-calculus

We explicit the simple type system for algebraic $\lambda$-calculus, then give full proofs of Lemma 14 and Theorem 3. Also, we present a weak normalization scheme in case $\mathsf{R}$ is positive but not finitely splitting.

## A.1 Simple type system

Algebraic $\lambda$-terms may be given implicative propositional types in a natural way. Assume we have a denumerable set of basic types $\phi, \psi, \ldots$; we build types from basic types using intuitionistic arrow: if $A$ and $B$ are types, then so is $A \Rightarrow B$.

Typing rules are given in figure 1.

**Proposition 8.** *Typing in algebraic $\lambda$-calculus enjoys the following properties:*

*(i) If $\Gamma \vdash \sigma : A$ then free variables of $\sigma$ are declared in $\Gamma$.*

$$\frac{}{\Gamma, x : A \vdash x : A}$$

$$\frac{\Gamma, x : A \vdash \sigma : B}{\Gamma \vdash \lambda x\, \sigma : A \Rightarrow B} \qquad \frac{\Gamma \vdash \sigma : A \Rightarrow B \quad \Gamma \vdash \tau : A}{\Gamma \vdash (\sigma)\, \tau : B}$$

$$\frac{}{\Gamma \vdash \mathbf{0} : A} \qquad \frac{\Gamma \vdash \sigma : A}{\Gamma \vdash a\sigma : A} \qquad \frac{\Gamma \vdash \sigma : A \quad \Gamma \vdash \tau : A}{\Gamma \vdash \sigma + \tau : A}$$

**Fig. 1.** Typing rules for algebraic $\lambda$-calculus.

*(ii)* *If $\Gamma \vdash \sigma : A$ then, for all $\Gamma'$, $\Gamma, \Gamma' \vdash \sigma : A$.*
*(iii)* *If $\sigma \equiv \sigma'$ then $\Gamma \vdash \sigma : A$ iff $\Gamma \vdash \sigma' : A$.*
*(iv)* *For all canonical term $S$, $\Gamma \vdash S : A$ if and only if, for all $u \in \Lambda(S)$, $\Gamma \vdash u : A$.*

Notice that typing is not preserved by $\overset{\triangledown}{=}$: for all term $\sigma$, if $\Gamma \vdash \sigma : A$ then $\Gamma \vdash \mathsf{can}(\sigma) : A$, but the converse does not necessarily hold. Hence we weaken typing judgements as follows:

**Definition 16.** *We say term $\sigma$ is weakly typable of type $A$ in context $\Gamma$ if $\Gamma \vdash \mathsf{can}(\sigma) : A$ is derivable. We write $\Gamma \Vdash \sigma : A$ for $\Gamma \vdash \mathsf{can}(\sigma) : A$.*

**Proposition 9.** *Weak typing enjoys the following properties:*

*(i)* *If $\sigma \overset{\triangledown}{=} \sigma'$ then $\Gamma \Vdash \sigma : A$ iff $\Gamma \Vdash \sigma' : A$.*
*(ii)* *For all term $\sigma$, $\Gamma \Vdash \sigma : A$ if and only if, for all $u \in \Lambda(\sigma)$, $\Gamma \vdash u : A$.*

Now we show that subject reduction holds, as soon as $\mathsf{R}$ is positive.

**Lemma 15.** *Let $\sigma, \tau \in \Lambda_{\mathsf{R}}$. If $\Gamma, x : A \vdash \sigma : B$ and $\Gamma \vdash \tau : A$ then $\Gamma \vdash \sigma[\tau/x] : B$.*

*Proof.* The proof is straightforward by induction on the typing derivation of $\Gamma, x : A \vdash \sigma : B$.

**Theorem 4.** *Assume $\mathsf{R}$ is positive. Subject reduction holds on canonical terms: if $S \widetilde{\to} S'$ and $\Gamma \vdash S : A$ then $\Gamma \vdash S' : A$.*

*Proof.* One proves that property by induction on the typing derivation $\Gamma \vdash S : A$. Each induction step is proved by inspecting all possible cases for the reduction $S \widetilde{\to} S'$ using previous lemma and Proposition 4 in the case of a redex. The positivity condition is used to handle the case in which $S \overset{\triangledown}{=} at + U$ and $S' \overset{\triangledown}{=} aT' + U$ with $t \to T'$: by positivity, we necessarily have $\Gamma \vdash t : A$ and $\Gamma \vdash U : A$.

**Corollary 5.** *Assume $\mathsf{R}$ is positive. Subject reduction for weak typing holds: if $\sigma \widetilde{\to} \sigma'$ and $\Gamma \Vdash \sigma : A$ then $\Gamma \Vdash \sigma' : A$.*

### A.2 Scalars and normalization

Recall that, if $\mathsf{R}$ is not positive, there is no normal term, and that positivity is not even a sufficient condition for strong normalization to hold. In sections A.2 to A.5, we assume that $\mathsf{R}$ is finitely splitting in the following sense: for all $a \in \mathsf{R}$, the set

$$\{(a_1, \ldots, a_n) \in (\mathsf{R}^\bullet)^n\,;\ n \in \mathbf{N} \text{ and } a = a_1 + \cdots + a_n\}$$

is finite. Moreover, we assume that the $\mathsf{width}$ function $\mathrm{w} : \mathsf{R} \longrightarrow \mathbf{N}$ defined by

$$\mathrm{w}\,(a) = \max\{n \in \mathbf{N};\ \exists (a_1, \ldots, a_n) \in (\mathsf{R}^\bullet)^n \ \text{ s.t. } a = a_1 + \cdots + a_n\}$$

is a morphism of rigs: $\mathrm{w}\,(a + b) = \mathrm{w}\,(a) + \mathrm{w}\,(b)$ and $\mathrm{w}\,(ab) = \mathrm{w}\,(a)\,\mathrm{w}\,(b)$ (which entails $\mathrm{w}\,(0) = 0$ and $\mathrm{w}\,(1) = 1$).

**Proposition 10.** *For all $a \in \mathsf{R}$, $\mathrm{w}\,(a) = 0$ iff $a = 0$. Hence $\mathsf{R}$ is positive and has no zero divisor.*

Recall that the rig of all polynomials over indeterminates $\xi_1, \ldots, \xi_n$ with non-negative integer coefficients, denoted by $\mathsf{P}_n = \mathbf{N}[\xi_1, \ldots, \xi_n]$, satisfies these conditions, with $\mathrm{w}\,(P) = P(1, \ldots, 1)$, for all $P \in \mathsf{P}_n$. It is the archetypal rig the structure of which inspired the proof. Conversely, notice that any rig $\mathsf{R}$ with a width morphism resembles a rig of polynomials with coefficients in $\mathbf{N}$: call "unitary monomials" those $a \in \mathsf{R}$ such that $\mathrm{w}\,(a) = 1$.

Such a rig of polynomials is also involved in the weak normalization scheme we develop in section A.6.

*The $\mathsf{R}$-module of strongly normalizing terms*

**Lemma 16.** *If $\sigma \overset{\triangledown}{=} a\tau + \upsilon$ with $a \neq 0$ then $\mathrm{Supp}(\tau) \subseteq \mathrm{Supp}(\sigma)$.*

*Proof.* This is just positivity of $\mathsf{R}$ together with the fact that $\mathsf{R}$ has no zero divisor.

**Lemma 17.** *If $\sigma \in \Delta_{\mathsf{R}}$ is a simple term, then for all $\sigma' \in \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$, $\sigma \mathrel{\widetilde{\rightarrow}} \sigma'$ iff $\sigma \rightarrow \sigma'$.*

*Proof.* This is a straightforward consequence of the previous lemma together with the fact that $\mathrm{w}\,(1) = 1$.

**Definition 17.** *We define the height of a term as follows. First, on atomic and canonical terms:*

- $\mathrm{h}(x) = 1$;
- $\mathrm{h}(\lambda x\, s) = 1 + \mathrm{h}(s)$;
- $\mathrm{h}((s)\,T) = 1 + \max(\mathrm{h}(s), \mathrm{h}(T))$;
- $\mathrm{h}(\sum_{i=1}^{n} a_i s_i) = \max\{\mathrm{h}(s_i);\ i \in [1;n]\}$.

*Then we generalize this definition to all terms: for all $\sigma \in \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$, $\mathrm{h}(\sigma) = \mathrm{h}(\mathsf{can}(\sigma))$.*

**Lemma 18.** *Let $\sigma \in \mathsf{R}\langle \Delta_{\mathsf{R}} \rangle$. There are only finitely many terms $\sigma'$ (up to $\overset{\triangledown}{=}$) such that $\sigma \mathrel{\widetilde{\rightarrow}} \sigma'$.*

*Proof.* The proof is by induction on $\mathrm{h}(\sigma)$. If $\mathrm{h}(\sigma) = 0$ then $\sigma \overset{\triangledown}{=} 0$ and the property holds trivially by Lemma 16. Assume that property holds for all $\sigma$ such that $\mathrm{h}(\sigma) < k$. Let $\sigma \in \mathsf{R}\langle \Delta_{\mathsf{R}} \rangle$ be such that $\mathrm{h}(\sigma) = k$. For each term $\sigma' \in \mathsf{R}\langle \Delta_{\mathsf{R}} \rangle$ such that $\sigma \mathrel{\widetilde{\rightarrow}} \sigma'$, there are $t \in \mathrm{A}_{\mathsf{R}}$, $T', U \in \mathrm{C}_{\mathsf{R}}$ and $a \in \mathsf{R}^{\bullet}$ such that $\sigma \overset{\triangledown}{=} at + U$, $\sigma' \overset{\triangledown}{=} aT' + U$ and $t \to T'$. By Lemma 16, $t \in \mathrm{Supp}(\sigma)$: there are finitely many such atomic terms. Moreover, due to the finite splitting condition on $\mathsf{R}$, for each such $t$ there exist finitely many $a \in \mathsf{R}^{\bullet}$ and $U \in \mathrm{C}_{\mathsf{R}}$ such that $\sigma \overset{\triangledown}{=} at + U$. A simple inspection of the definition of $\to$ shows that, by inductive hypothesis applied to strict subterms of $t$ (all of height strictly less than $\mathrm{h}(t) \leq k$), $t$ $\to$-reduces to finitely many canonical terms, which are all the possible choices for $T'$.

**Definition 18.** *We denote by $\mathcal{N}$ the set of strongly normalizing simple terms, with equality $\overset{\triangledown}{=}$.*

Then $\mathsf{R}\langle \mathcal{N} \rangle$ is the set of linear combinations of strongly normalizing simple terms:

$$\mathsf{R}\langle \mathcal{N} \rangle = \{\sigma \in \mathsf{R}\langle \Delta_{\mathsf{R}} \rangle;\ \mathrm{Supp}(\sigma) \subseteq \mathcal{N}\}\,.$$

**Definition 19.** *If $\tau \in \Delta_{\mathsf{R}}$ is a simple term, and $\sigma \in \mathsf{R}\langle \Delta_{\mathsf{R}} \rangle$ is any term, we write $\mathrm{w}_\tau(\sigma)$ for the width of the coefficient of $\tau$ in $\sigma$: $\mathrm{w}_\tau(\sigma) = \mathrm{w}\left(\sigma_{(\tau)}\right)$.*

By Lemma 18, König's lemma allows for the following definition:

**Definition 20.** *If $\sigma$ is a strongly normalizing term, we denote by $|\sigma|$ the length of the longest sequence of $\widetilde{\rightarrow}$-reductions from $\sigma$ to its normal form. If $\sigma \in \mathsf{R}\langle \mathcal{N} \rangle$, we define $\|\sigma\| = \sum_{\tau \in \mathrm{Supp}(\sigma)} \mathrm{w}_\tau(\sigma)\,|\tau|$.*

Recall that we consider the elements of $\mathrm{Supp}(\sigma)$ up to $\overset{\triangledown}{=}$: for instance, if $\sigma$ is a strongly normalizing simple term, $\|\sigma\| = |\sigma|$. By definition, If $\sigma \overset{\triangledown}{=} \sigma'$ then $\|\sigma\| = \|\sigma'\|$.

For all $\sigma, \sigma'$ such that $\sigma \mathrel{\widetilde{\rightarrow}} \sigma'$, $a\sigma + \tau \mathrel{\widetilde{\rightarrow}} a\sigma' + \tau$ also holds as soon as $a \neq 0$: this is a straightforward consequence of Proposition 10. Hence the following proposition:

**Proposition 11.** *The support of every strongly normalizing term $\sigma$ is a finite subset of $\mathcal{N}$,* i.e. *$\sigma \in \mathsf{R}\langle \mathcal{N} \rangle$.*

**Lemma 19.** *Let $\sigma \in \mathcal{N}$ and let $\sigma'$ be such that $\sigma \to \sigma'$. Then $\|\sigma'\| < |\sigma|$.*

*Proof.* We write

$$\mathsf{can}(\sigma') \equiv \sum_{i=1}^{n} \sigma'_{(t_i)} t_i.$$

By Proposition 11, each $t_i \in \mathcal{N}$. We expand this sum as follows:

$$\sigma' \stackrel{\triangledown}{=} \sum_{i=1}^{n} \sum_{j=1}^{\mathrm{w}_{t_i}(\sigma')} a_j^{(i)} t_i$$

where, for all $i \in [1; n]$,

$$\sum_{j=1}^{\mathrm{w}_{t_i}(\sigma')} a_j^{(i)} = \sigma'_{(t_i)}$$

and, for all $j$, $a_j^{(i)} \neq 0$. For each $i \in [1; n]$, one can find a reduction of length $|t_i|$ from $t_i$ to its normal form and concatenating these reductions, we get a reduction from $\sigma'$ of length $\|\sigma'\|$. Hence $\|\sigma'\| + 1 \leq |\sigma|$.

The following proposition follows from w being a morphism of rigs.

**Proposition 12.** *For all terms $\sigma$ and $\tau$ and all scalar $a$, one has $\|\sigma + \tau\| = \|\sigma\| + \|\tau\|$ and $\|a\sigma\| = \mathrm{w}(a)\, \|\sigma\|$.*

We are now able to prove the following reformulation of Lemma 14

**Lemma 20.** *The set of all strongly normalizing terms is $\mathsf{R}\langle\mathcal{N}\rangle$.*

*Proof.* One inclusion is Proposition 11. It remains to prove that if $\sigma \in \mathsf{R}\langle\mathcal{N}\rangle$ then $\sigma$ is strongly normalizing. This is proved by induction on $\|\sigma\|$.

- If $\|\sigma\| = 0$, then positivity of $\mathsf{R}$ implies that for all simple term $\tau \in \mathrm{Supp}(\sigma)$, $\mathrm{w}_\tau(\sigma)\, |\tau| = 0$: since $\mathsf{R}$ has no zero divisor, we have $|\tau| = 0$. Hence, as soon as $\sigma$ can be written $as + T$ with $a \neq 0$, since Lemma 16 implies $s \in \mathrm{Supp}(\sigma)$, $s$ is normal and doesn't give rise to a reduction from $\sigma$.
- Suppose the result holds for all $\tau \in \mathsf{R}\langle\mathcal{N}\rangle$ such that $\|\tau\| < \|\sigma\|$. It is sufficient to prove that, for all $\sigma'$ such that $\sigma \mathrel{\widetilde{\to}} \sigma'$, $\sigma'$ is strongly normalizing. Such a $\sigma'$ is given by $a \in \mathsf{R}^\bullet$, $u \in \mathrm{A}_{\mathsf{R}}$ and $T, U' \in \mathrm{C}_{\mathsf{R}}$ such that $\sigma \stackrel{\triangledown}{=} au + T$, $u \to U'$ and $\sigma' \stackrel{\triangledown}{=} aU' + T$. By Lemma 16 and since $\mathrm{Supp}(\sigma) \subset \mathcal{N}$, $u \in \mathcal{N}$ (so $U'$ has to be strongly normalizing, which by Proposition 11 implies $U' \in \mathsf{R}\langle\mathcal{N}\rangle$) and $T \in \mathsf{R}\langle\mathcal{N}\rangle$; hence $\sigma' \in \mathsf{R}\langle\mathcal{N}\rangle$. By Proposition 12, we have $\|\sigma'\| = \|aU' + T\| = \mathrm{w}(a)\, \|U'\| + \|T\|$ and $\|\sigma\| = \|au + T\| = \mathrm{w}(a)\, |u| + \|T\|$. By Lemma 19, $\|U'\| < |u|$. Hence $\|\sigma'\| < \|\sigma\|$ and induction hypothesis applies.

### A.3 Saturated sets

**Definition 21.** *A* stack *is a sequence $(\sigma_1, \ldots, \sigma_n)$ of terms, which we denote by $\sigma_1 \ldots \sigma_n$. If $\pi = \sigma_1 \ldots \sigma_n$ is a stack and $\sigma$ is a term, we write $\sigma :: \pi$ for stack $\sigma\sigma_1 \ldots \sigma_n$. If $\theta = \tau_1 \ldots \tau_m$ is another stack, we write $\pi\theta$ for the concatenation $\sigma_1 \ldots \sigma_n \tau_1 \ldots \tau_m$.*

*Let $\mathcal{X} \subseteq \Delta_{\mathsf{R}}$ be a set of simple terms. A stack $\pi = \sigma_1 \ldots \sigma_n$ is an $\mathcal{X}$-stack if, for all $i \in [1; n]$, $\sigma_i \in \mathsf{R}\langle\mathcal{X}\rangle$.*

Notice that any term $\sigma$ with $\sigma \in \mathsf{R}\langle\mathcal{X}\rangle$ can be considered as an $\mathcal{X}$-stack of length 1 so that definitions and results about stacks generally apply to terms.

**Definition 22.** *We extend algebraic equality on stacks as follows:* $\sigma_1 \dots \sigma_n \stackrel{\triangledown}{=} \tau_1 \dots \tau_p$ *iff* $n = p$ *and, for all* $i \in [1; n]$, $\sigma_i \stackrel{\triangledown}{=} \tau_i$.

We generalize application with stacks in argument position: if $\pi = \sigma_1 \dots \sigma_n$ is a stack and $\sigma \in \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$, then we set $(\sigma)\,\pi = ((\sigma)\,\sigma_1 \dots)\,\sigma_n$ so that, if $\sigma$ is a simple term and $\pi$ is a stack, then $(\sigma)\,\pi$ is also a simple term.

**Definition 23.** *Let $\mathcal{X}$ be a set of simple terms. An $\mathcal{X}$-redex is a simple term of the following shape:*

$$\sigma \stackrel{\triangledown}{=} (\lambda x\, s)\, T$$

*where $s \in \mathcal{X}$ and $T \in \mathsf{R}\langle\mathcal{X}\rangle$. We write $\mathsf{Red}\,(\sigma)$ for the term obtained by firing this redex: $\mathsf{Red}\,(\sigma) \stackrel{\triangledown}{=} s\,[T/x]$.*

*We say set $\mathcal{X}$ is* saturated *if, for all $\mathcal{N}$-redex $\sigma$ and all $\mathcal{N}$-stack $\pi$, $(\mathsf{Red}\,(\sigma))\,\pi \in \mathsf{R}\langle\mathcal{X}\rangle$ implies $(\sigma)\,\pi \in \mathcal{X}$.*

**Lemma 21.** *$\mathcal{N}$ is saturated.*

*Proof.* We have to prove that, for all $\mathcal{N}$-redex $\sigma$ and all $\mathcal{N}$-stack $\pi$, $(\mathsf{Red}\,(\sigma))\,\pi \in \mathsf{R}\langle\mathcal{N}\rangle$ implies $(\sigma)\,\pi \in \mathcal{N}$. We write $\sigma \stackrel{\triangledown}{=} (\lambda x\, s)\, T_0$ and $\pi \stackrel{\triangledown}{=} T_1 \dots T_n$, where $s \in \mathcal{N}$ and, for all $i \in [0; n]$, $T_i \in \mathsf{R}\langle\mathcal{N}\rangle$. With these notations, we are led to prove that, for all $s \in \mathcal{N}$ and all $\mathcal{N}$-stack $T_0 \dots T_n$, if

$$(s\,[T_0/x])\, T_1 \dots T_n \in \mathsf{R}\langle\mathcal{N}\rangle, \tag{7}$$

then

$$\tau \stackrel{\triangledown}{=} (\lambda x\, s)\, T_0 \dots T_n \in \mathcal{N}.$$

By Lemma 14, each $T_i$ is strongly normalizing. We prove the result by induction on $|s| + \sum_{i=0}^{n} |T_i|$. By Lemma 17, it is sufficient to show that for all $\tau'$ such that $\tau \to \tau'$, $\tau'$ is strongly normalizing. The reduction $\tau \to \tau'$ can occur at the following positions:

- at the root of the $\mathcal{N}$-redex;
- inside $s$;
- inside one of the $T_i$'s.

*Head reduction.* In the first case, which is the only possible one if $|s| + \sum_{i=0}^{n} |T_i| = 0$, $\tau' \stackrel{\triangledown}{=} (\mathsf{Red}\,(\sigma))\,\pi$ so hypothesis (7) applies directly.

*Reduction in the function.* Consider the case in which reduction occurs inside $s$. So $\tau' \stackrel{\triangledown}{=} (\lambda x\, S')\, T_0 :: \pi$ with $s \to S'$. Write $S' \equiv \sum_{l=1}^{q} a_l s'_l$ and, for all $l \in [1; q]$, define $\tau'_l \stackrel{\triangledown}{=} (\lambda x\, s'_l)\, T_0 :: \pi$ so that $\tau' \stackrel{\triangledown}{=} \sum_{l=1}^{q} a_l \tau'_l$. It is then sufficient to prove that, for all $l \in [1; q]$, $\tau'_l \in \mathcal{N}$. For all $l$, $|s'_l| < |s|$ and induction hypothesis applies to the data $s'_l, T_0, \dots, T_m$. Hence it is sufficient to show that $(s'_l\,[T_0/x])\,\pi \in \mathsf{R}\langle\mathcal{N}\rangle$. By hypothesis (7), $(\mathsf{Red}\,(\sigma))\,\pi \in \mathsf{R}\langle\mathcal{N}\rangle$. But, since $s \to S'$, Corollary 1 and Lemma 3 imply $(\mathsf{Red}\,(\sigma))\,\pi \mathrel{\widetilde{\to}}^{*} \sum_{l=1}^{q} a_l\,(s'_l\,[T_0/x])\,\pi$. Hence each $(s'_l\,[T_0/x])\,\pi \in \mathsf{R}\langle\mathcal{N}\rangle$ (Lemma 16).

*Reduction in an argument.* Consider the case in which reduction occurs inside one $T_i$. So $\tau' \overset{\triangledown}{=} (\lambda x\, s)\, T_0 \ldots T_i' \ldots T_m$ with $T_i \,\widetilde{\to}\, T_i'$. Since $|T_i'| < |T_i|$, induction hypothesis applies to the data $s, T_0, \ldots, T_i', \ldots, T_m$. It is sufficient to show that $(s\,[T_0/x])\, T_1 \ldots T_i' \ldots T_m \in \mathsf{R}\langle\mathcal{N}\rangle$ — or something similar if $i = 0$. The end of the proof is the same as before.

### A.4 Reducibility

**Definition 24.** *If $\mathcal{X}$ and $\mathcal{Y}$ are sets of simple terms, one defines $\mathcal{X} \Rightarrow \mathcal{Y} \subseteq \Delta_{\mathsf{R}}$ by:*

$$\mathcal{X} \Rightarrow \mathcal{Y} = \{\sigma \in \Delta_{\mathsf{R}};\ \text{ for all } \tau \in \mathsf{R}\langle\mathcal{X}\rangle, (\sigma)\,\tau \in \mathcal{Y}\}\,.$$

*More generally, if $\mathcal{P}$ is a set of stacks, one defines $\mathcal{P} \Rightarrow \mathcal{Y} \subseteq \Delta_{\mathsf{R}}$ by:*

$$\mathcal{P} \Rightarrow \mathcal{Y} = \{\sigma \in \Delta_{\mathsf{R}};\ \forall \pi \in \mathcal{P},\ (\sigma)\,\pi \in \mathcal{Y}\}\,.$$

**Proposition 13.** *If $\mathcal{P} \subseteq \mathcal{P}'$ are sets of stacks and $\mathcal{Y}' \subseteq \mathcal{Y} \subseteq \Delta_{\mathsf{R}}$ then $\mathcal{P}' \Rightarrow \mathcal{Y}' \subseteq \mathcal{P} \Rightarrow \mathcal{Y}$.*

**Lemma 22.** *If $\mathcal{S}$ is a saturated set and $\mathcal{P}$ is a set of $\mathcal{N}$-stacks, then $\mathcal{P} \Rightarrow \mathcal{S}$ is saturated.*

*Proof.* We have to show the following: for all $\mathcal{N}$-redex $\tau$ and all $\mathcal{N}$-stack $\pi$, if $(\mathsf{Red}\,(\tau))\,\pi \in \mathsf{R}\langle\mathcal{P} \Rightarrow \mathcal{S}\rangle$, then $(\tau)\,\pi \in \mathcal{P} \Rightarrow \mathcal{S}$. By definition of $\mathcal{P} \Rightarrow \mathcal{S}$, it amounts to prove that for all $\theta \in \mathcal{P}$, $(\tau)\,\pi\theta \in \mathcal{S}$. But since $\pi$ and $\theta$ are $\mathcal{N}$-stacks, $\pi\theta$ is an $\mathcal{N}$-stack too; thus by saturation of $\mathcal{S}$, it is sufficient to prove that $(\mathsf{Red}\,(\tau))\,\pi\theta \in \mathsf{R}\langle\mathcal{S}\rangle$. By hypothesis, $(\mathsf{Red}\,(\tau))\,\pi \in \mathsf{R}\langle\mathcal{P} \Rightarrow \mathcal{S}\rangle$, which ends the proof, using the definition of $\mathcal{P} \Rightarrow \mathcal{S}$ and the fact that $\theta \in \mathcal{P}$.

**Definition 25.** *We define the interpretation $A^*$ of type $A$ by induction on $A$:*

- $\phi^* = \mathcal{N}$ *if $\phi$ is a basic type;*
- $(A \Rightarrow B)^* = A^* \Rightarrow B^*$.

**Definition 26.** *Let $\mathcal{N}_0$ be the set of all simple terms $\sigma$ of shape $\sigma \overset{\triangledown}{=} (x)\,\pi$, where $\pi$ is an $\mathcal{N}$-stack.*

**Lemma 23.** *The following inclusions hold:*

$$\mathcal{N}_0 \subseteq (\mathcal{N} \Rightarrow \mathcal{N}_0) \subseteq (\mathcal{N}_0 \Rightarrow \mathcal{N}) \subseteq \mathcal{N}.$$

*Proof.* Of course, $\mathcal{N}_0 \subseteq \mathcal{N}$, hence the central inclusion, by Proposition 13. The first inclusion holds by definition of $\mathcal{N}_0$. If $\tau \in \mathcal{N}_0 \Rightarrow \mathcal{N}$, let $x$ be any variable, $x \in \mathcal{N}_0$ and we have $(\tau)\,x \in \mathcal{N}$, which clearly implies $\tau \in \mathcal{N}$ by contextuality of $\widetilde{\to}$; hence the last inclusion.

**Corollary 6.** *For all type $A$, $\mathcal{N}_0 \subseteq A^* \subseteq \mathcal{N}$.*

## A.5 Adequation

**Theorem 5.** *Let $\sigma$ be a term and assume*

$$x_1 : A_1, \ldots, x_m : A_m \Vdash \sigma : A$$

*is derivable. Let $\sigma_1 \in \mathsf{R}\langle A_1^* \rangle$, ..., $\sigma_m \in \mathsf{R}\langle A_m^* \rangle$. Then*

$$\tau \stackrel{\triangledown}{=} \sigma\,[\sigma_1, \ldots, \sigma_m / x_1, \ldots, x_m] \in \mathsf{R}\langle A^* \rangle.$$

*Proof.* We prove the theorem by induction on type derivation

$$x_1 : A_1, \ldots, x_m : A_m \vdash \mathsf{can}(\sigma) : A.$$

*Variable.* $\sigma \stackrel{\triangledown}{=} x_i$ for some $i$ and $A = A_i$. Then $\tau \stackrel{\triangledown}{=} \sigma_i \in \mathsf{R}\langle A_i^* \rangle$ by hypothesis.

*Application.* $\sigma \stackrel{\triangledown}{=} (s)\,T$ with $x_1 : A_1, \ldots, x_m : A_m \vdash s : B \Rightarrow A$ and $x_1 : A_1, \ldots, x_m : A_m \vdash T : B$. By inductive hypothesis,

$$s\,[\sigma_1, \ldots, \sigma_m / x_1, \ldots, x_m] \in \mathsf{R}\langle (B \Rightarrow A)^* \rangle$$

and

$$T\,[\sigma_1, \ldots, \sigma_m / x_1, \ldots, x_m] \in \mathsf{R}\langle B^* \rangle.$$

Hence $\tau \in \mathsf{R}\langle A^* \rangle$ by definition of $B^* \Rightarrow A^*$.

*Abstraction.* $A = B \Rightarrow C$ and $\sigma \stackrel{\triangledown}{=} \lambda x\, s$ with

$$x_1 : A_1, \ldots, x_m : A_m, x : B \vdash s : C.$$

We assume $x$ is distinct from every $x_i$ and does not occur free in any $\sigma_i$. Then $\tau \stackrel{\triangledown}{=} \lambda x\, S'$ with

$$S' \stackrel{\triangledown}{=} s\,[\sigma_1, \ldots, \sigma_m / x_1, \ldots, x_m]\,.$$

We show that $\tau \in \mathsf{R}\langle (B \Rightarrow C)^* \rangle$ using the definition of $B^* \Rightarrow C^*$: let $T \in \mathsf{R}\langle B^* \rangle$, we have to prove $(\lambda x\, S')\,T \in \mathsf{R}\langle C^* \rangle$. Since $C^*$ is saturated, it is sufficient to show that $S'\,[T/x] \in \mathsf{R}\langle C^* \rangle$. By Proposition 1,

$$S'\,[T/x] \stackrel{\triangledown}{=} s\,[T, T_1, \ldots, T_m / x, x_1, \ldots, x_m]$$

and we conclude by induction hypothesis applied to $s$.

*Linear combinations.* $\sigma \stackrel{\triangledown}{=} \sum_{i=1}^{n} a_i s_i$ and $\Gamma \vdash s_i : A$ for all $i \in [1; n]$. Then, by induction hypothesis, each $s_i\,[\sigma_1, \ldots, \sigma_m / x_1, \ldots, x_m] \in \mathsf{R}\langle A^* \rangle$ and we conclude directly.

We get the following theorem as a corollary of theorem 5.

**Theorem 6.** *All weakly typable term are strongly normalizing.*

*Proof.* Let $\sigma \in \mathsf{R}\langle \Delta_{\mathsf{R}} \rangle$ be such that $x_1 : A_1, \ldots, x_m : A_m \Vdash \sigma : A$ is derivable. For all $i \in [1; n]$, since $\mathcal{N}_0 \subset A_i^*$, $x_i \in \mathsf{R}\langle A_i^* \rangle$. Hence $\sigma \stackrel{\triangledown}{=} \sigma\,[x_1, \ldots, x_m / x_1, \ldots, x_m] \in \mathsf{R}\langle A^* \rangle$ by Theorem 5 and we conclude by lemma 6.

### A.6 Weak normalization scheme

Remember that we forced strong conditions on $\mathsf{R}$ in the beginning of this section: we assumed that the width function $\mathrm{w} : \mathsf{R} \longrightarrow \mathbf{N}$ could be defined and was a homomorphism of rigs (which in particular entails positivity of $\mathsf{R}$). One can however get rid of this problem by slightly changing the notion of normal form and still obtain a weak normalization result.

**Definition 27.** *We define passive terms by the following statements:*

- $\sigma \in \Delta_{\mathsf{R}}$ *is a neutral term if* $\sigma \overset{\triangledown}{=} x \in \mathsf{V}$, *or* $\sigma \overset{\triangledown}{=} (s)\,T$, *where* $s$ *is a neutral term and* $T$ *is a passive term;*
- $\sigma \in \Delta_{\mathsf{R}}$ *is a simple passive term if* $\sigma$ *is neutral, or* $\sigma \overset{\triangledown}{=} \lambda x\, s$ *where* $s$ *is a simple passive term;*
- $\sigma$ *is a passive term if, for all* $s \in \mathrm{Supp}(\sigma)$, $s$ *is a simple passive term.*

Intuitively, passive terms are those terms $\sigma$ such that $\mathsf{can}(\sigma)$ contains no redex.

**Proposition 14.** *Any normal term is passive. Moreover, if* $\mathsf{R}$ *is positive then passive terms are exactly normal terms.*

*A rig of polynomials.* Let $\mathsf{R}$ be any rig and $\Xi$ be a set of indeterminates in bijection with $\mathsf{R}$: to every $a \in \mathsf{R}$ we associate $\xi_a \in \Xi$ such that $\xi_a = \xi_b$ iff $a = b$, and $\Xi = \{\xi_a;\, a \in \mathsf{R}\}$.

**Definition 28.** *Let* $\mathsf{P} = \mathbf{N}\,[\Xi]$ *be the rig of polynomials with non-negative integer coefficients over indeterminates in* $\Xi$. *If* $P \in \mathsf{P}$, *and* $f : \mathsf{R} \longrightarrow \mathsf{R}'$ *where* $\mathsf{R}'$ *is any rig, we denote by*

$$P\{a \mapsto f(a)\}$$

*the valuation of* $P$ *at* $f$, i.e. *the scalar (in* $\mathsf{R}'$*) obtained by replacing each* $\xi_a$ *in* $P$ *by* $f(a)$, *for all* $a \in \mathsf{R}$.

**Definition 29.** *If* $P \in \mathsf{P}$, *we denote by* $[\![P]\!]$ *the value of* $P$ *in* $\mathsf{R}$*:*

$$[\![P]\!] = P\{a \mapsto a\} \in \mathsf{R}.$$

**Lemma 24.** *The width function is well-defined and is a morphism of rigs from* $\mathsf{P}$ *to* $\mathbf{N}$.

*Proof.* The width function is exactly the sum of all coefficients:

$$\mathrm{w}\,(P) = P\{a \mapsto 1\} \in \mathbf{N}.$$

**Corollary 7.** *All weakly typable terms in* $\mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$ *are strongly normalizing.*

We extend valuation to terms as follows.

**Definition 30.** *Let* $\sigma \in \Lambda_{\mathsf{P}}$*:* $[\![\sigma]\!] \in \Lambda_{\mathsf{R}}$ *is the term obtained by replacing every coefficient* $P$ *in* $\sigma$ *by its value* $[\![P]\!]$.

If $\sigma, \tau \in \Lambda_{\mathsf{P}}$, then $\llbracket\sigma\rrbracket \equiv \llbracket\tau\rrbracket$ iff $\sigma \equiv \tau$, and $\llbracket\sigma\rrbracket \stackrel{\triangledown}{=} \llbracket\tau\rrbracket$ as soon as $\sigma \stackrel{\triangledown}{=} \tau$. In general, however, $\llbracket\sigma\rrbracket \stackrel{\triangledown}{=} \llbracket\tau\rrbracket$ does not imply $\sigma \stackrel{\triangledown}{=} \tau$.

**Proposition 15.** *For all $\sigma \in \mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$, if $\sigma$ is a passive term, then $\llbracket\sigma\rrbracket \in \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$ is a passive term.*

**Lemma 25.** *For all $\sigma, \sigma' \in \mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$, if $\sigma \mathrel{\widetilde{\to}} \sigma'$, then $\llbracket\sigma\rrbracket \mathrel{\widetilde{\to}}^{*} \llbracket\sigma\rrbracket'$.*

*Proof.* The proof is easy by induction on reduction $\sigma \mathrel{\widetilde{\to}} \sigma'$.

**Definition 31.** *Let $\sigma \in \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$. We define $\check{\sigma} \in \mathsf{P}\langle\Delta_{\mathsf{P}}\rangle$ as the term obtained from $\mathsf{can}(\sigma)$ by replacing every coefficient $a$ by the monomial $\chi_a$, so that $\sigma \stackrel{\triangledown}{=} \mathsf{can}(\sigma) \equiv \llbracket\check{\sigma}\rrbracket$.*

**Lemma 26.** *Let $\sigma \in \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$. If $\Gamma \Vdash \sigma : A$ then $\Gamma \vdash \check{\sigma} : A$.*

*Proof.* One easily proves by induction on canonical term $S$ that that if $\Gamma \vdash S : A$ then $\Gamma \vdash \check{S} : A$.

**Theorem 7.** *Let $\sigma \in \mathsf{R}\langle\Delta_{\mathsf{R}}\rangle$ be a weakly typable term. Then $\sigma$ is weakly normalizing in the sense that it reduces to a passive form.*

*Proof.* If $\sigma$ is weakly typable then, by lemma 26, $\check{\sigma}$ is typable. By Theorem 3, $\check{\sigma}$ is strongly normalizing, hence $\check{\sigma} \mathrel{\widetilde{\to}}^{*} \tau$ where $\tau$ is normal. By Proposition 14, $\tau$ is passive, and so is $\llbracket\tau\rrbracket$ by Proposition 15. By Lemma 25, $\sigma \mathrel{\widetilde{\to}}^{*} \llbracket\tau\rrbracket$, hence the conclusion.

Recall that if $\mathsf{R}$ is positive, then every passive form is a normal form; in this case Theorem 7 states a genuine weak normalization property.

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