A sheaf model of sequentiality

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Sequential algorithms and friends — Marseille 2022

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Introduction

- The O'Hearn-Riecke construction (OHR): full completeness for PCF.
  [O'Hearn & Riecke '95]
  - *Fully complete model*: every morphism is definable.

The O'Hearn-Riecke construction is an application of concrete sheaves.

- Concrete sheaves:
  - General framework for adding higher-order types (and recursion).
  - E.g. quasi-Borel spaces, diffeological spaces, ...

See [Matache, Moss, Staton: FSCD '21, LICS '22].

Introduction

The concrete sheaf model of sequentiality:

- Not more explicit (cf. [Loader '01]).
- Similar character:
  - extensional, not a quotient, syntax-free, logical relations.
- Originally a CCC; now a bi-CCC with strong monad.
  - We use CBV and have sum types modelled by categorical sums.
- Following [Riecke & Sandholm '97], [Marz '00], [Streicher '06]:
  Formulate the OHR logical relations in terms of 'SSP': a category of simple 'sequential data types'.

(Finitary) $\mathrm{PCF_v}$: A call-by-value language

$$\text{Types:} \quad \tau ::= 0 \mid 1 \mid \sout{\mathsf{nat}} \mid \tau + \tau \mid \tau \times \tau \mid \tau \to \tau$$

$$\text{Values:} \quad v, w ::= \ldots \mid \lambda x.\, t \mid \sout{\mathsf{rec}\, f\, x.\, t} \mid \mathsf{diverge}$$

$$\text{Computations:} \quad t ::= \ldots \mid v\ w \mid \mathsf{let}\ x = t\ \mathsf{in}\ t'$$

Typing judgements: $\Gamma \vdash^{\mathbf{v}} v : \tau$ and $\Gamma \vdash^{\mathbf{c}} t : \tau$.

Semantics in a bi-CCC with strong monad L with a point $\bot : 1 \Rightarrow L$.

$$[\![0]\!] = 0 \qquad [\![1]\!] = 1 \qquad [\![\sigma + \tau]\!] = [\![\sigma]\!] + [\![\tau]\!] \qquad [\![\sigma \times \tau]\!] = [\![\sigma]\!] \times [\![\tau]\!]$$

$$[\![\sigma \to \tau]\!] = [\![\sigma]\!] \Rightarrow L[\![\tau]\!]$$

$$[\![\Gamma \vdash^{\mathbf{v}} v : \tau]\!] : [\![\Gamma]\!] \to [\![\tau]\!] \qquad [\![\Gamma \vdash^{\mathbf{c}} t : \tau]\!] : [\![\Gamma]\!] \to L[\![\tau]\!]$$

Outline

Outline

1. Background on the OHR construction
2. Concrete sheaves
3. SSP - a category of sequential data types
4. Building a sequential sheaf model
5. Conclusion

The ‘domains’ model of $\mathsf{PCF_v}$

The category of posets Pos, with lifting monad $P \mapsto P_\perp$. Not fully complete:

$$\mathrm{por} : [\![(1 \to 2) \times (1 \to 2)]\!] \cong 2_\perp \times 2_\perp \to 2_\perp \cong [\![2]\!]_\perp$$

- Definability problem: characterize the definable morphisms?
- Full completeness: give a model where all morphisms are definable?

These problems are linked to the *full abstraction problem* [Milner ’77].

Acceptability criteria

Definability: characterize the definable elements in a given model.
Full completeness: give a model where all elements are definable.

For both problems:

- Without reference to syntax contrast with [Milner '77].
- ! Effective? impossible by [Loader '01].

For full completeness:

- In a well-pointed category — every $[\![\sigma]\!]$ has an underlying set $|[\![\sigma]\!]|$,
  - Additionally, $|LX| \cong |X| + 1$.
- Types denote the corresponding categorical object.
  - E.g. sum types really denote categorical sums,...

Sketch of the logical relations approach to definability

...,[Plotkin '80], [Sieber '92], [Jung & Tiuryn '93],...

Write $\mathsf{Def}(\Gamma;\sigma) := \{|\llbracket M \rrbracket| \mid \Gamma \vdash^{\mathbf{c}} M : \sigma\} \subseteq \mathsf{Set}(|\llbracket \Gamma \rrbracket|, |\llbracket \sigma \rrbracket| + 1)$.

1) For $f : |\llbracket \sigma \rrbracket| \to |\llbracket \tau \rrbracket| + 1$, postcomposition with f

$$\mathsf{Set}(|\llbracket \Gamma \rrbracket|, |\llbracket \sigma \rrbracket| + 1) \xrightarrow{g \mapsto f \circ g} \mathsf{Set}(|\llbracket \Gamma \rrbracket|, |\llbracket \tau \rrbracket| + 1)$$

maps $\mathsf{Def}(\Gamma;\sigma)$ into $\mathsf{Def}(\Gamma;\tau)$ for all $\Gamma \iff f$ is PCF_v-definable.

2) Generalize from $(\Gamma \mapsto |\llbracket \Gamma \rrbracket|) : \mathsf{Ctxt} \to \mathsf{Set}$ to more general functors $F : \mathcal{C} \to \mathsf{Set}$ and consider $(\mathrm{ob}\,\mathcal{C} \times \mathsf{Typ})$-indexed families of predicates

$$A(c;\sigma) \subseteq \mathsf{Set}(F(c), |\llbracket \sigma \rrbracket| + 1)$$

preserved by $f \circ (-)$ when f is PCF_v-definable.

We 'lose' Def in a larger class, characterized without the syntax.

The O'Hearn-Riecke construction: 'predictive' logical relations

- Collect suitable 'guesses'/'predictions' for
  - the sets $|[\![\Gamma]\!]|$,
  - the subsets $\mathsf{Def}(\Gamma; 1 + \ldots + 1) \subseteq \mathsf{Set}(|[\![\Gamma]\!]|, |[\![1 + \ldots + 1]\!]| + 1)$.
- Force preservation by every morphism in the category.
- Folklore: logical relations ≈ presheaves.
- Refinement: *reflexive* logical relations ≈ concrete presheaves.
  - ...that respect sum types ≈ concrete *sheaves*

Outline

Well-pointed categories and concrete sites

A category $\mathbb{C}$ is *well-pointed* if

- it has a terminal object $\star$
- $\mathbb{C}(\star, -) : \mathbb{C} \to \mathbf{Set}$ is faithful
  i.e. maps $h : d \to c$ are distinguished *functions* $|h| : |d| \to |c|$

where $|c| = \mathbb{C}(\star, c)$. So $\mathbb{C}$ is a category of sets and certain functions.

Concrete site $(\mathbb{C}, J)$

- A small well-pointed category $\mathbb{C}$.
- For every $c \in \mathbb{C}$ a set $J(c)$ of **covering families** $\{f_i : c_i \to c\}_{i \in I}$ of c s.t.
  (C) pullback stability
  ($\star$) If $\{f_i : c_i \to c\}_{i \in I}$ covers c, then $\bigcup_{i \in I} \mathrm{Im}(|f_i|) = |c|$

Concrete sheaf on a concrete site $(\mathbb{C}, J)$

[*Concrete quasitopoi*, Dubuc '77]
[*Convenient categories of smooth spaces*, Baez & Hoffnung '11]

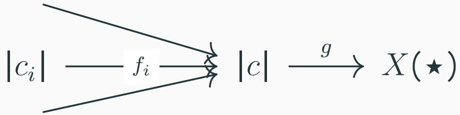
Well-pointed category $\mathbb{C}$
- has a terminal $\star$
- a map $h : d \to c$ is a function between sets $|d| = \mathbb{C}(\star, d)$ etc.

Concrete site $(\mathbb{C}, J)$
- small well-pointed $\mathbb{C}$
- For every $c \in \mathbb{C}$ a set $J(c)$ of **covering families** $\{f_i : c_i \to c\}_{i \in I}$ of c, with axioms (C) and $\star$.

A concrete presheaf $X : \mathbb{C}^{\mathsf{op}} \to \mathsf{Set}$ is:
- a set $X(\star)$
- $X(c) \subseteq [|c| \to X(\star)]$

with $\forall \phi \in X(c).\ \phi \circ |h| \in X(d)$.

Sheaf condition: for $g : |c| \to X(\star)$ and $\{f_i : c_i \to c\}_{i \in I} \in J(c)$, if each $g \circ |f_i| \in X(c_i)$, then $g \in X(c)$.



A morphism $\alpha : X \to Y$ is a function $\alpha : X(\star) \to Y(\star)$ with $\forall \phi \in X(c).\ \alpha \circ \phi \in Y(c)$.

Representables: for $c \in \mathbb{C}$, $y(c)(\star) = |c|$ and $\mathbb{C}(d, c) \subseteq y(c)(d)$, but might need to close under the sheaf condition!

Bi-CCC structure

Interpretation in *concrete* (pre)sheaves = *reflexive* logical relation.

Products: $(X \times Y)(\star) = X(\star) \times Y(\star)$

$$\phi \in (X \times Y)(c) \iff \pi_1 \circ \phi \in X(c) \wedge \pi_2 \circ \phi \in Y(c).$$

Exponentials $(X \Rightarrow Y)(\star) = \mathsf{ConcSh}(\mathbb{C}, J)(X, Y)$

$$\phi \in (X \Rightarrow Y)(c) \iff$$

$$\forall (h : d \to c) \in \mathbb{C}, \psi \in X(d).\ (\lambda x \in |d|.\ \phi(|h|(x))(\psi(x)) \in Y(d).$$

Sums : $(X + Y)(\star) = X(\star) + Y(\star)$

$$\phi \in X(c) \implies \mathrm{inl} \circ \phi \in (X + Y)(c), \quad \psi \in Y(c) \implies \mathrm{inr} \circ \psi \in (X + Y)(c)$$

Close under the sheaf condition.

Partiality

Let $(\mathbb{C}, J)$ be a concrete site. Let $\mathcal{M}$ be a class of monomorphisms in $\mathbb{C}$ satisfying conditions given in [MMS, '22].

Conditions: pullback-stability, closure under composition, 'concreteness', 'sheaf condition',...

Theorem

There is a strong monad $L_{\mathcal{M}} = L$ on $\mathsf{ConcSh}(\mathbb{C}, J)$ given by

$$(LX)(\star) = X(\star) + \{\bot\}$$

$$\phi \in (LX)(c) \iff \exists (m : d \to c) \in \mathcal{M}.\ \operatorname{dom}\phi = |d| \wedge \phi|_{|d|} \in X(d).$$

Equivalently,

$$(LX)(c) = \sum_{(m:d \to c) \in \mathcal{M}} X(d).$$

First attempt with a sequential *presheaf* model

Semidecidable subset of a type τ = program $x : \tau \vdash^{\mathbf{c}} s : 1$.

Category Syn (modulo a suitable equivalence relation to make it well-pointed):

- Objects: (τ, s) type + semidecidable subset
- Morphisms: $f : (\tau, s) \to (\tau', s')$ is a program $x : \tau \vdash^{\mathbf{c}} f : \tau'$ with domain s and image in s'.

Monos: $(x : \tau \vdash^{\mathbf{c}} x : \tau) : (\tau, s') \to (\tau, s)$ where $s' \downarrow \implies s \downarrow$. In Conc(Syn),

$$(LX)((\tau, s)) = \sum_{s'\downarrow \implies s\downarrow} X((\tau, s')).$$

Check: $y(\sigma \to \tau, \text{return} \star) \cong y(\sigma, \text{return} \star) \Rightarrow Ly(\tau, \text{return} \star)$.

Second attempt with a sequential *sheaf* model

Yoneda lemma $\implies$ $\mathsf{Syn} \to \mathsf{Conc}(\mathsf{Syn}) \subseteq [\mathsf{Syn}^{\mathsf{op}}, \mathsf{Set}]$ full and faithful, so ≈ fully complete interpretation of $\mathsf{PCF_v}$ with $\sigma \mapsto y(\sigma, \mathsf{return} \star)$.

Problems:

1. $y(\sigma, \mathsf{return} \star) + y(\tau, \mathsf{return} \star) \to y(\sigma + \tau, \mathsf{return} \star)$ not an isomorphism.
2. We'd like a non-syntactic model.

For 1: add covering families $J((\tau, s))$ where, for each $x : \tau \vdash^{\mathbf{c}} t : 1_1 + \ldots + 1_n$, with $s \downarrow \iff t \downarrow$,

$$\{(x : \tau \vdash^{\mathbf{c}} x : \tau) : (\tau, \mathsf{let}\, y = t\, \mathsf{in}\, \nu_i) \to (\tau, s)\}_{i=1\ldots,n}$$

where $y : 1_1 + \ldots + 1_n \vdash^{\mathbf{c}} \nu_i : 1$ terminates on the ith summand only.

Outline

SSP: A category of sequential data types [Marz '00], [Streicher '06]

Objects: $X = (|X|, \mathcal{A}^X)$ where $|X|$ is a finite set and $\mathcal{A}^X$ is a set of partial functions $|X| \rightharpoonup \mathbb{N}$ such that:

- $\mathcal{A}^X$ contains all constant functions: $\lambda x.n, \lambda x.\bot \in \mathcal{A}^X$;
- $\mathcal{A}^X$ is closed under postcomposition:
  $f \in \mathcal{A}^X, \phi : \mathbb{N} \rightharpoonup \mathbb{N} \implies \phi \circ f \in \mathcal{A}^X$;
- $\mathcal{A}^X$ is closed under 'sequencing': $f, g_n \in \mathcal{A}^X \implies \lambda x.g_{f(x)}(x) \in \mathcal{A}^X$.

Morphisms $X \to Y$ are functions $f : |X| \to |Y|$ such that

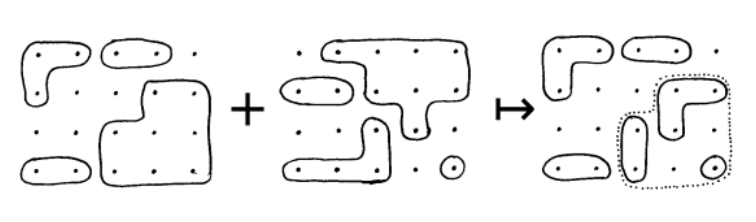
$$g \in \mathcal{A}^Y \implies g \circ f \in \mathcal{A}^X.$$

SSP objects as 'Structural Systems of Partitions'

For $X \in \mathsf{SSP}$, let $S^X = \{\{f^{-1}(\{n\}) \mid n \in \mathbb{N}\} \setminus \{\varnothing\} \mid f \in \mathcal{A}^X\}$.

partial equivalence relations on X, or 'partial partitions' of X.

We can equivalently axiomatize SSP in terms of S^X, e.g. sequencing:



- If $\{u\} \in S^X$, then $u \subseteq |X|$ is a 'semidecidable subset'.
- $\{u_1, \ldots, u_n\} \in S^X$ is a 'coherent' collection of semidecidable subsets.

Categorical structure in SSP

- Sums: $X + Y = (|X| + |Y|, \mathcal{A}^{X+Y})$:
  - $f \in \mathcal{A}^{X+Y} \iff f|_{|X|} \in \mathcal{A}^X \wedge f|_{|Y|} \in \mathcal{A}^Y$.
- Products: $X \times Y = (|X| \times |Y|, \mathcal{A}^{X \times Y})$:
  - Have $f \circ \pi_X, g \circ \pi_Y \in \mathcal{A}^{X \times Y}$ for $f \in \mathcal{A}^X$, $g \in \mathcal{A}^Y$.
  - Then close under sequencing!
- Lifting monad: $LX = (|X| + \{\bot\}, S^{LX})$:
  - $S^{LX} = S^X \cup \{\{|X| + \{\bot\}\}\}$.

Full completeness of SSP at first order + thunking

Consider a simple CBV language with types $\tau ::= \mathbf{0} \mid \mathbf{1} \mid \tau + \tau \mid \tau \times \tau \mid T\tau$

Values:

$$v ::= \ldots \mid \mathsf{thunk}\ t$$

Computations:

$$t ::= \ldots \mid \mathsf{diverge} \mid \mathsf{force}\ v$$

$$\frac{\Gamma \vdash^{\mathbf{v}} v : T\tau}{\Gamma \vdash^{\mathbf{c}} \mathsf{force}\ v : \tau} \qquad \frac{\Gamma \vdash^{\mathbf{c}} T : \tau}{\Gamma \vdash^{\mathbf{v}} \mathsf{thunk}\ t : T\tau}$$

(Equivalently, restrict $\mathsf{PCF_v}$ function types to $\mathbf{1} \to (-)$).

Theorem

The interpretation in SSP is fully complete, i.e. every Kleisli morphism $[\![\Gamma]\!] \to L[\![\tau]\!]$ is the interpretation of some term $\Gamma \vdash^{\mathbf{c}} t : \tau$.

SSP in the OHR construction

In logical relations, $F : \mathcal{C} \to \mathsf{Set}$ generalizes $\Gamma \mapsto |[\![\Gamma]\!]| : \mathsf{Ctxt} \to \mathsf{Set}$.

- In [Marz '00] & [Streicher '06], the construction ranges over (a sufficiently large set of) faithful functors $F : \mathcal{C} \to \mathsf{SSP}$.
- For CBV, we will instead range over faithful functors $F : \mathcal{C} \to \mathsf{SSP}_L$.
  - Construct a category like Syn using $(\mathcal{C}, F)$ instead of the syntax.

Objects of Syn are (τ, s) where $x : \tau \vdash^{\mathbf{c}} s : 1$.

Outline

Defining sites via systems of partitions

$X = (|X|, S^X) \in \mathsf{SSP}$: $|X|$ = finite set, $S^X \subseteq$ {partial partitions of $|X|$} +axioms
SSP_L has Kleisli maps $X \to LY$

For each faithful functor $F : \mathcal{C} \to \mathsf{SSP}_L$ define a category $\mathcal{I}_{\mathcal{C},F}$ (approximating **Syn**):

- Objects: a terminal object $\star$ and also (c, U), for each $c \in \mathcal{C}$ and $\{U\} \in S^{F(c)}$.
- Morphisms: $f : (c, U) \to (d, W)$ is a function $f : U \to W$
  - either constant
  - or s.t. there is $(\phi : c \to d) \in \mathcal{C}$ with $U \subseteq \operatorname{dom} F(\phi)$ and $f = F(\phi)|_U$.

Defining sites via systems of partitions

Fix $F : \mathcal{C} \to \mathsf{SSP}_L$. $\mathcal{I}_{\mathcal{C},F}$ has objects (c, U) for $c \in \operatorname{ob}\mathcal{C}$, $\{U\} \in S^{F(c)}$.

Define a coverage $J_{\mathcal{C},F}$ on $\mathcal{I}_{\mathcal{C},F}$ and class of monos $\mathcal{M}_{\mathcal{C},F}$:

- For $c \in \operatorname{ob}\mathcal{C}$, $\{U_1, \ldots, U_n\} \in S^{F(c)}$, the object $(c, \bigcup_i U_i)$ is covered by the set of inclusions
$$(c, U_k) \hookrightarrow (c, \textstyle\bigcup_i U_i).$$
- $\mathcal{M}_{\mathcal{C},F}$ is generated by inclusions
$$(c, U) \hookrightarrow (c, V)$$
for $\{U\}, \{V\} \in S^{F(c)}$ with $U \subseteq V$.

A sequential sheaf model of $\mathsf{PCF_v}$

Partiality monad on $\mathcal{G} = \mathsf{ConcSh}(\sum_{F:\mathcal{C}\to\mathsf{SSP}_L} \mathcal{I}_{\mathcal{C},F}, \sum_{F:\mathcal{C}\to\mathsf{SSP}_L} J_{\mathcal{C},F})$:

$$(L_{\mathcal{G}}X)(\star) = X(\star) + \{\bot\}$$
$$(L_{\mathcal{G}}X)(c,U) = \sum_{W\subseteq U, \{W\}\in S^{F(c)}} X(c,W)$$

Theorem

$\mathcal{G}$ is bicartesian closed with a strong pointed monad $L_{\mathcal{G}}$. The canonical interpretation of $\mathsf{PCF_v}$ is adequate and fully complete.

Notes on proof of full completeness

- Pick $\mathcal{C}_0$ to be the category whose objects are $\mathsf{PCF_v}$ types with $\mathcal{C}_0(\sigma, \tau) \subseteq \mathsf{Set}(|[\![\sigma]\!]|, |[\![\tau]\!]|)$ given by the definable functions.
- Let $F_0 : \mathcal{C}_0 \to \mathsf{SSP}_L$ send σ to the SSP-structure induced by the definable functions $|[\![\sigma]\!]| \to |L_{\mathcal{G}}(1 + \ldots + 1)|$.
- Write $y_0 : \mathcal{I}_{\mathcal{C}_0, F_0} \to \mathcal{G}$ for 'sheafified Yoneda'.
- There is an evident bijection $|y_0(\sigma, |[\![\sigma]\!]|)| \to |[\![\sigma]\!]|$, but it doesn't obviously lift to a natural transformation $y_0(\sigma, |[\![\sigma]\!]|) \to [\![\sigma]\!]$.
- By induction on σ, show it becomes a natural isomorphism after applying $\mathsf{res}_0 : \mathcal{G} \to \mathsf{ConcSh}(\mathcal{I}_{\mathcal{C}_0, F_0}, J_{\mathcal{C}_0, F_0})$ (faithful and preserves points).

Remarks on adding recursion and infinite types

Could take ωCPO-valued (pre)sheaves. Instead, use logical relations/sheaves:

- Let $\mathrm{V} = \{0 < 1 < \ldots < \infty\} \in \omega\mathsf{CPO}$, let $\mathbb{V}_0 = \{\varnothing, 1, \mathrm{V}\} \subseteq \omega\mathsf{CPO}$.
- Take $\mathcal{M}_{\mathbb{V}} = \{\text{Scott open inclusions}\}$.
- Sum $\mathbb{V}_0$ with $\sum_{F:\mathcal{C}\to\mathsf{SSP}_\perp} \mathcal{I}_{\mathcal{C},F}$.
- Interpret $[\![\mathsf{nat}]\!]$ as $\sum_0^\infty 1$.

$\implies$ 'synthetic domain theory' gives relevant fixed point operators

- Full completeness fails (e.g. for cardinality reasons).
- Following [Milner '77]: full abstraction follows from 'full completeness' for the truncated types for $n \in \mathbb{N}$:

$$[\![\mathsf{nat}]\!]_n = \sum_0^n 1 \qquad [\![\sigma \to \tau]\!]_n = [\![\sigma]\!]_n \Rightarrow L_{\mathcal{G}}[\![\tau]\!]_n \qquad \ldots$$

Outline

Summary

- The OHR technique fits into the general framework of concrete sheaves.
  - Therefore connected to useful techniques for differentiable programming, measurable programming, ...
- We give a fully complete model of finitary $\mathrm{PCF_v}$, and fully abstract model of $\mathrm{PCF_v} + \mathsf{nat}, \mathsf{rec}$.
  - As the 'canonical' interpretation of types in a model of intuitionistic set theory.
- Principled interpretation of sums, as well as function spaces.

Outlook

- [Kammar, Katsumata, Saville, ’22] Full completeness for effects with well-pointed monadic models (without recursion).
- Other effects? With recursion and not necessarily well-pointed models?
- [Colson, Ehrhard ’94]: Hypercoherences + strongly stable functions embed in presheaves on $\mathbb{N}_{\perp}^{\omega}$.
- [van Oosten ’99], [Longley ’02]: A realizability topos of strongly stable functionals.