Sur la syntaxe de la sémantique quantitative

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Soutenance d'habilitation à diriger des recherches
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Λ

A bit of λ-calculus

$$\Lambda \ni M, N, \ldots ::= x \mid \lambda x.M \mid (M)\, N$$

$$(\lambda x.M)\, N \qquad \to_\beta \qquad M[N/x]$$

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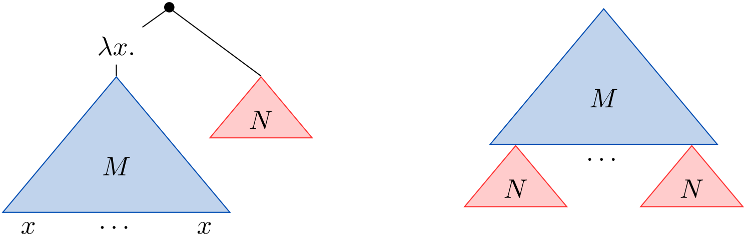

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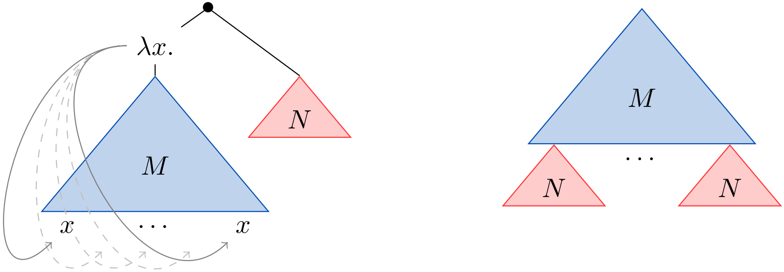

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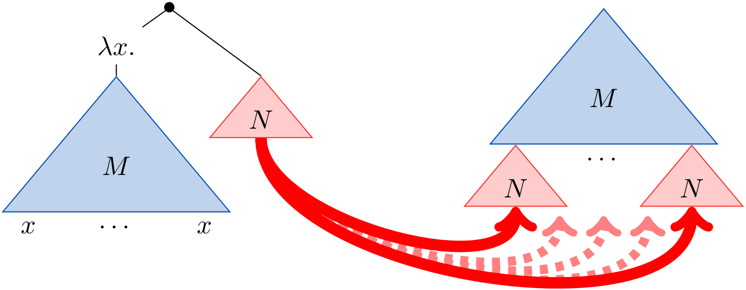

λ-terms as computable functions

Normal terms represent data:

$$\underline{n} = \lambda f.\lambda x.(f)^n\, x = \lambda f.\lambda x.(f) \cdots (f)\, x$$

Terms represent (*e.g.*, Turing-)computable functions:

$$\underline{succ} = \lambda a.\lambda f.\lambda x.(f)\,(a)\,f\,x$$

Evaluation is normalization:

$$(\underline{succ})\,\underline{n} \to_\beta \lambda f.\lambda x.(f)\,(\underline{n})\,f\,x \to_\beta^2 \lambda f.\lambda x.(f)\,(f)^n\,x = \underline{n+1}$$

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Normalization may fail (computable functions are partial functions):

- $\Omega \coloneqq (\Delta)\,\Delta \to_\beta \Omega$ with $\Delta \coloneqq \lambda x.(x)\, x$
- There is Θ s.t. $\mathsf{Fix} M \coloneqq (\Theta)\, M \to_\beta^* (M)\, \mathsf{Fix} M \to_\beta^* (M)\, \cdots (M)\, \mathsf{Fix} M$

$\sim$ *`while`-loops, recursive definitions, etc.*

λ-terms as proofs

Simple functional types: $A, B, \ldots ::= X \mid A \to B$

$$\frac{}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B} \qquad \frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash (M)\, N : B}$$

Subject reduction

If $\Gamma \vdash M : A$ and $M \to_\beta M'$ then $\Gamma \vdash M' : A$.

Strong normalization

If $\Gamma \vdash M : A$ then every β-reduction sequence from M is finite.

λ-terms as proofs: Curry–Howard

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Minimal implicative logic:

$$\frac{}{\Gamma, A \vdash A} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \qquad \frac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B}$$

type $\sim$ formula term $\sim$ proof β-reduction $\sim$ cut-elimination

λ-terms as morphisms

Typed terms induce set-theoretic functions:

- $[\![\Gamma \vdash M : A]\!] \in [\![A]\!]^{[\![\Gamma]\!]}$ with $[\![B \to C]\!] = [\![C]\!]^{[\![B]\!]}$
- $[\![M]\!] = [\![M']\!]$ whenever $M \to_\beta M'$

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Denotational semantics

$$[\![\Gamma]\!] = [\![A_1]\!] \times \cdots \times [\![A_n]\!] \xrightarrow{\quad [\![\Gamma \vdash M : A]\!] \quad} [\![A]\!]$$

in any cartesian ($\times$) closed ($\to$) category.

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λ-calculus (with product types and up to $=_{\beta\eta}$) is *the* language of CCCs.

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λ-calculus (with product types and up to $=_{\beta\eta}$) is *the* language of CCCs.

- Un(i)typed calculus: use a reflexive object $D \simeq D^D$.
- Some CCCs are $\mathbf{K}$-linear for some field/ring/semiring $\mathbf{K}$: we can form linear combinations of morphisms.

Linear combinations of λ-terms

$$\Lambda_+ \ni M, N, \ldots ::= x \mid \lambda x.M \mid (M)\, N \mid M + N \mid 0 \mid aM \quad (a \in \mathbf{S}, \text{ some semiring})$$

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Retains confluence.

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Non-deterministic choice:

$$M_1 \oplus M_2 \rightarrow_+ M_i$$

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$$(M_1 \oplus M_2)\, N \to_+ (M_1)\, N + (M_2)\, N \quad (\mathbf{S} = \mathbf{B})$$

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$$(M_1 \oplus M_2)\, N \to_+ \frac{1}{2}(M_1)\, N + \frac{1}{2}(M_2)\, N \quad (\mathbf{S} = \mathbf{R}^+)$$

Quantitative non-determinism in the λ-calculus, contextually.

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- A unified framework for quantitative non-determinism?
- The internal language of **S**-linear CCCs?

A museum of horrors

$$\begin{array}{lll} \infty_M & := & \mathsf{Fix}\lambda x.(M + x) \\ & \to_\beta^* & M + \infty_M \\ & \to_\beta^* & nM + \infty_M \end{array}$$

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$$0 = M - M \to_\beta M - M'$$

$$0 = \infty_M - \infty_M =_\beta M$$

A museum of horrors

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no normal forms

up to vector space equations, β-equality is unsound!

$$0 = \infty_M - \infty_M =_\beta M$$

Positive coefficients: $a + b = 0 \Longrightarrow a = b = 0$

Subject reduction holds and typed terms are normalizable on the nose.
(V., 2007–2009)

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Weak normalizability scheme

- temporarily neutralize coefficients, replacing them with indeterminates
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- obtain a normal term by evaluating polynomials

The result depends only on the original term. (Alberti, 2014)

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Conservativity

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~~(V., 2007–2009)~~ (Kerinec–V., 2019, unpublished) algebraic λ-terms

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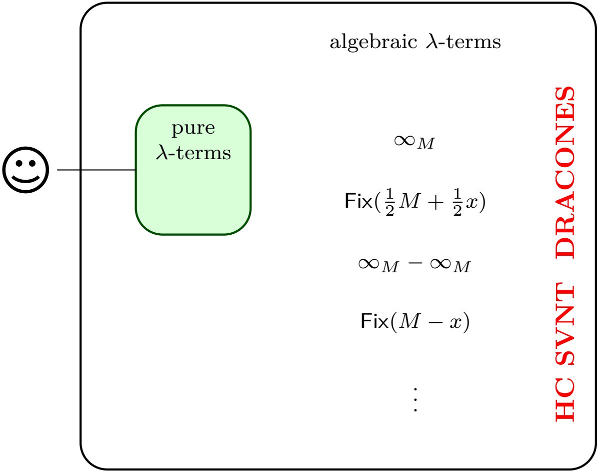
$$\infty_M - \infty_M$$

$$\mathsf{Fix}(M - x)$$

$$\vdots$$

HC SVNT DRACONES


algebraic λ-terms
pure λ-terms
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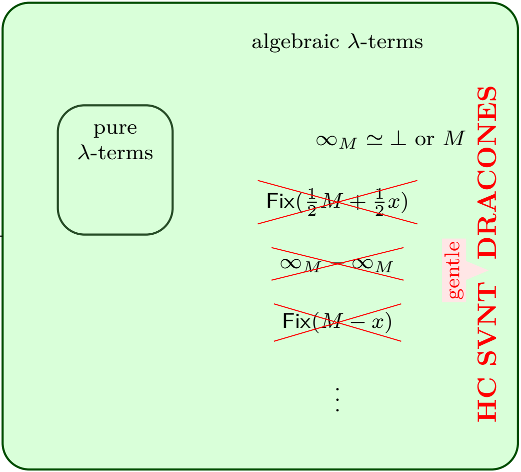
HC SVNT DRACONES

Plain n.d. choice ($\mathbf{S} = \mathbf{B}$)

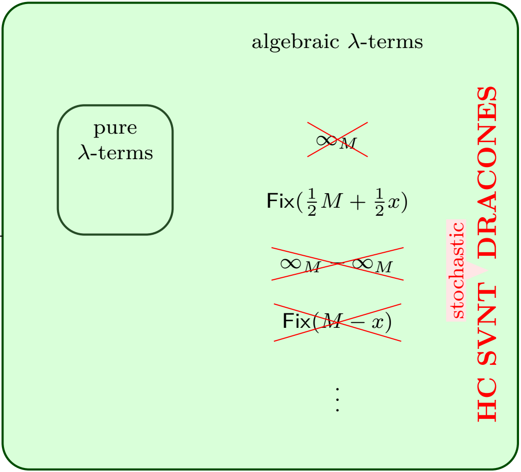
plenty of models:
e.g., De'Liguoro-Piperno's trees
or the relational model



Probabilistic (sub)distributions

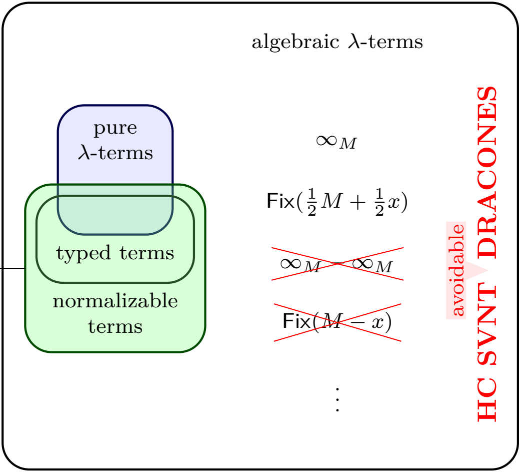
probabilistic Böhm trees
(Leventis, 2016)

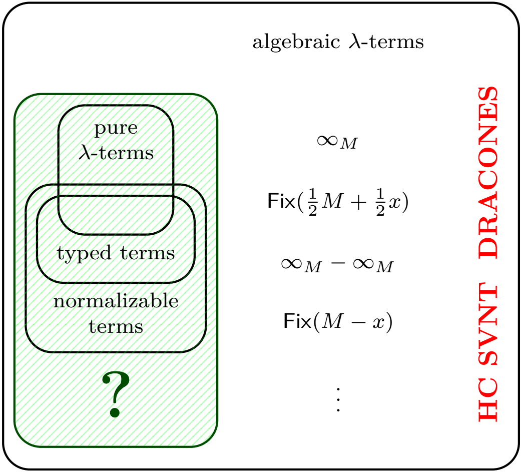
many models from
various communities
(game semantics,
domain theory,
linear logic, *etc.*)



Positive coefficients ($a + b = 0 \Rightarrow a = b = 0$)

typed terms normalize

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The syntax of quantitative semantics

Quantitative semantics

Normal functors (Girard, '80s, before LL)

λ-terms $\rightsquigarrow$ set-valued power series (cf. Joyal's analytic functors)

$$[\![M + N]\!]_a = [\![M]\!]_a + [\![N]\!]_a \qquad \text{(disjoint sum of sets)}$$

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Finiteness spaces (Ehrhard, early 2000's)

- types $\rightsquigarrow$ particular topological vector spaces (or semimodules):
  $[\![A]\!] \subseteq \mathbf{S}^{|A|}$ + some additional structure (with $\mathbf{S}$ an arbitrary semifield)
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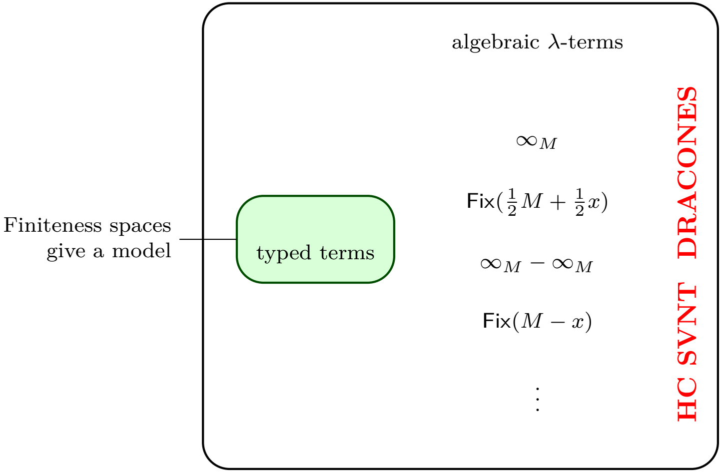
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Differentiation of λ-terms (Ehrhard-Regnier, 2003-2004)

- differential λ-calculus (requires sums of terms)

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Differentiation of λ-terms (Ehrhard-Regnier, 2003-2004)

- differential λ-calculus (requires sums of terms)
- a finitary fragment: resource λ-calculus = the target of Taylor expansion

Resource λ-calculus

$$
\begin{array}{lclcll}
\Delta & \ni & s, t, \ldots & ::= & x \mid \lambda x.s \mid \langle s \rangle \, \bar{t} & \text{(terms)} \\
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Resource reduction

$$\langle \lambda x.s \rangle\, \overline{t} \to_\partial \partial_x s \cdot \overline{t} \quad \text{(anywhere)}$$

Semantically: *(at least in a typed setting)*

$$\partial_x s \cdot [t_1, \ldots, t_n] = \left(\frac{\partial^n s}{\partial x^n}\right)_{x=0} \cdot (t_1, \ldots, t_n)$$

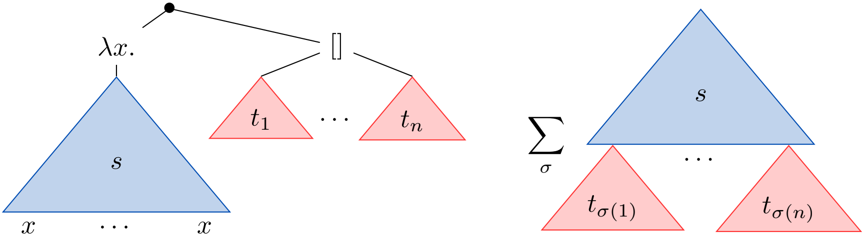
Syntactically:

$$\partial_x s \cdot \overline{t} = \begin{cases} \sum_{\sigma \in \mathfrak{S}_n} s[t_{\sigma(1)}, \ldots, t_{\sigma(n)}/x_1, \ldots, x_n] & \text{if } n_x(s) = \#\overline{t} = n \\ 0 & \text{otherwise} \end{cases}$$

- Linearity: $\lambda x.0 = 0$, $\langle s \rangle\, [t_1 + t_2, u] = \langle s \rangle\, [t_1, u] + \langle s \rangle\, [t_2, u]$, $\ldots$

Resource reduction

$$\langle \lambda x.s \rangle [t_1, \ldots, t_n] \quad \to_\partial \quad \partial_x s \cdot [t_1, \ldots, t_n]$$



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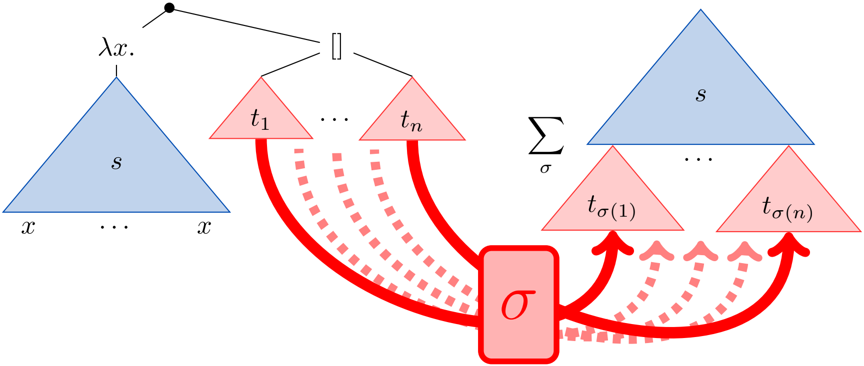

Resource λ-calculus

$$\begin{array}{llcll} \Delta & \ni & s, t, \ldots & ::= & x \mid \lambda x.s \mid \langle s \rangle\, \bar{t} & \text{(terms)} \\ \Delta^! & \ni & \bar{s}, \bar{t}, \ldots & ::= & [s_1, \ldots, s_n] & \text{(monomials)} \end{array}$$

Resource reduction

$$\langle \lambda x.s \rangle\, \bar{t} \to_\partial \partial_x s \cdot \bar{t} \quad \text{(anywhere)}$$

Semantically: *(at least in a typed setting)*

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Syntactically:

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- Linearity: $\lambda x.0 = 0$, $\langle s \rangle\, [t_1 + t_2, u] = \langle s \rangle\, [t_1, u] + \langle s \rangle\, [t_2, u]$, $\ldots$
- Resource reduction preserves free variables, is size-decreasing, strongly confluent and strongly normalizing.

Taylor expansion of λ-terms

Taylor expansion: $M^* \in \mathbf{Q}^{\Delta}$

$$((M)\,N)^* = \sum_{n \in \mathbf{N}} \frac{1}{n!} \langle M^* \rangle\, N^{*n}$$

$$x^* = x \qquad (\lambda x.M)^* = \lambda x.M^*$$

Many models related with LL validate $[\![M]\!] = [\![M^*]\!]$.

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Normalizing Taylor expansions

We want to set

$$\mathcal{N}\Big(\sum_{i \in I} a_i s_i\Big) = \sum_{i \in I} a_i \mathcal{N}(s_i)$$

Normalizing vectors fails in general!

$\mathcal{N}(\infty_x^*) = ?$ $\qquad$ ∞_x^* contains infinitely many terms s_i such that $\mathcal{N}(s_i) = x$.

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For all $M \in \Lambda$ and $t \in \Delta$, there is at most one $s \in M^*$ such that $\mathcal{N}(s)_t \neq 0$.

Proof. λ-terms are uniform: their approximants all have the same structure. □

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Theorem (Ehrhard-Regnier, CiE 2006)

$\mathcal{N}(M^*) \simeq \mathcal{B}(M)$ *(in particular $\mathcal{N}(\Omega^*) = 0 \simeq \bot$)*

$\mathcal{N}(M^*) = \mathcal{N}(M)^*$

We want:

- If $M \to_\beta N$ then $M^* \widetilde{\rightrightarrows}_\partial N^*$.
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... or if we do not consider coefficients:
this simplifies the proof in the uniform case (Olimpieri–V., 2018, 2021)

Key technique: bounding the size of antireducts

The resource λ-calculus is *extremely* regular:

Lemma

If $s \to_\partial S' \ni s'$, *then* $\mathsf{size}(s') < \mathsf{size}(s)$

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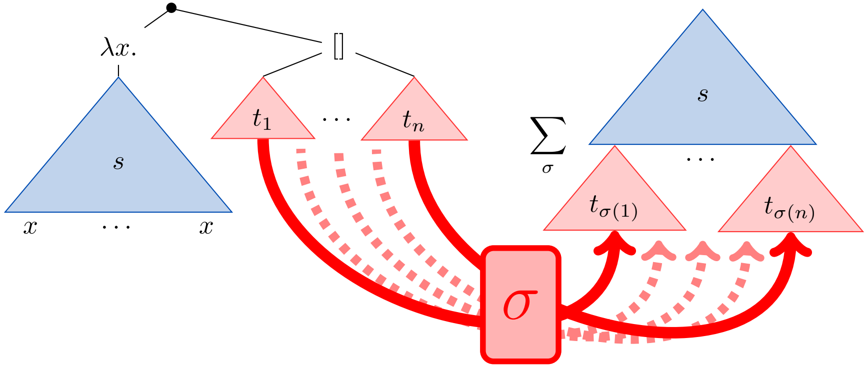

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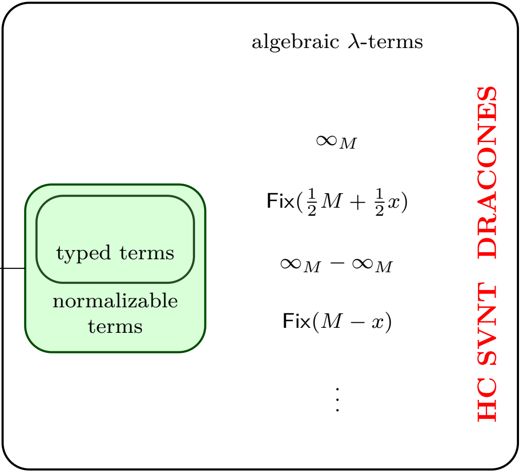
Lemma

If $s \rightrightarrows_\partial S' \ni s'$, *then* $\mathsf{height}(s') \leq \psi(\mathsf{height}(s))$.

Hence we can iterate.

M^* is normalizable and
$\mathcal{N}(M^*) = \mathcal{N}(M)^*$
(V., 2017–2019)



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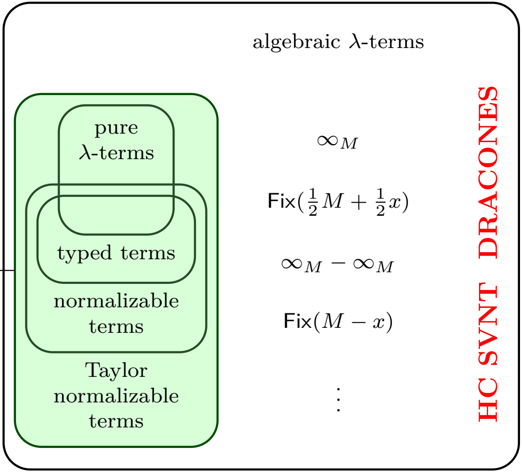

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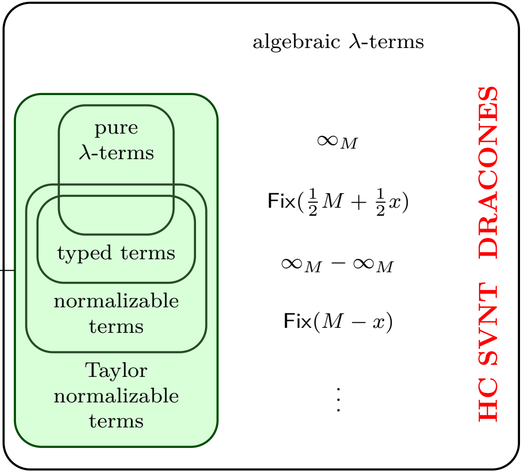
Proposal

$\mathcal{B}(M) := \mathcal{N}(M^*)$



Taylor expansion in MELL

Taylor expansion in MELL proof nets

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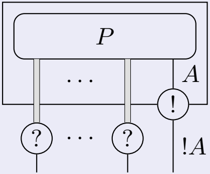
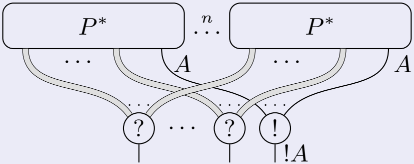
Taylor expansion of promotion

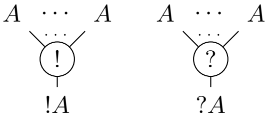


expands to $\sum_{n \in \mathbf{N}} \frac{1}{n!}$

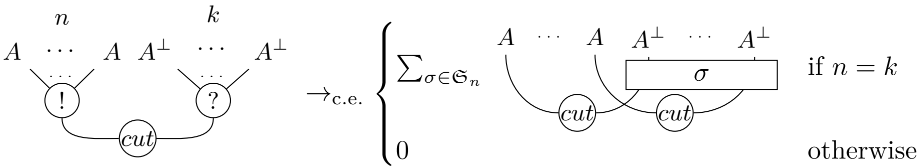


Resource nets

MLL nets + ? and ! links of any arity:



with cut elimination:



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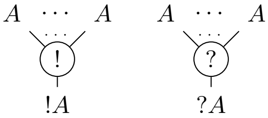

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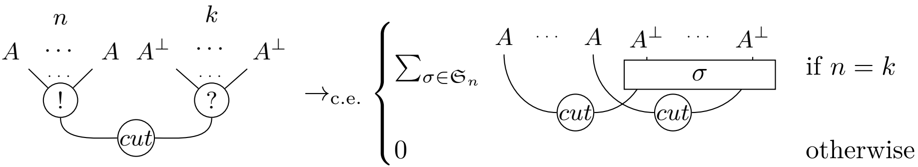

Geometrically, this is essentially multiplicative:

- ! works like an n-ary $\otimes$
- ? works like an n-ary $\parr$

(at least if we forget about sums, typing and the order of premisses)

Cut elimination and Taylor expansion

Fact

If $P \to_{\text{c.e.}} P'$ *then* $P^* \widetilde{\rightrightarrows}_{\text{c.e.}} P'^*$.

$$\sum_{i \in i} a_i s_i \; \widetilde{\rightrightarrows}_{\text{c.e.}} \sum_{i \in I} a_i S'_i$$

whenever $s_i \rightrightarrows_{\text{c.e.}} S'_i$ for all $i \in I$, where $\rightrightarrows_{\text{c.e.}}$ is parallel cut elimination.

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Same issue

Given $S = \sum_{i \in i} a_i s_i$ and a family of reductions $(s_i \rightrightarrows_{\text{c.e.}} S'_i)_{i \in I}$
$\sum_{i \in I} a_i S'_i$ might not converge.

Same solution, replacing $\mathsf{height}(s)$ with the length of switching paths in s
(+ jump-in-degree for treating weakening/coweakening)

(Chouquet–V., 2018–2021)

À suivre…

Three ongoing research directions

Taylor expansion for the infinitary λ-calculus

- Infinite λ-terms generalize both pure λ-terms and Böhm trees
- Extending Taylor expansion to these is easy (at least for Λ_∞^{001})

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We must also consider infinite β-reduction sequences:
in general, these are unwieldy.

Claim

$\widetilde{\rightrightarrows_\partial^*}$ *simulates strongly convergent sequences*

*w/ **Rémy Cerda** (PhD started 2020)*

Should give a new proof of standardization.

Revisiting operational properties of proof nets

- Parallel cut elimination is very well-behaved in MLL and resource nets
- Can we extend it to MELL?

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The difficult part is the definition…

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In some versions of g.s., strategies look like sets of normal resource terms.

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Claim

This can be made formal, and then $[\![M]\!]_{games} \cong \mathcal{N}(M^*)$.

w/ ***Lison Blondeau-Patissier*** *(PhD started 2021, w/ P. C.)*
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A trick: enforce infinite extensionality in the syntax:

$$s ::= \lambda\vec{y}.\langle s\rangle\,\pi \mid \lambda\vec{y}.\langle x\rangle\,\pi \qquad (\vec{y} = (y_i)_{i\in\mathbf{N}})$$

$$\pi ::= \epsilon \mid [s_1, \ldots, s_n]\cdot\pi \qquad (\epsilon = []\cdot\epsilon = ([])_{i\in\mathbf{N}})$$

Merci !

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