Normalization by Evaluation in Linear Logic

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In programming language semantics, *normalization by evaluation* (NBE) is a technique of computing the output of a program P by appealing only to the denotational semantics of P. Denotational semantics is a methodology for giving mathematical meaning to programming languages: it interprets ("evaluates") P as a function $[\![P]\!]$ in some mathematical structure, independently of the way P effectively performs the computation. The NBE paradigm requires a mapping reify which extracts the output of P (a syntactic term) from the value of $[\![P]\!]$ (a semantic object). So, the normalization function computing the output (i.e. normal form) $\mathsf{nf}(P)$ of P is obtained by composing reify with the interpretation $[\![P]\!]$:

$$\mathsf{nf}(P) = \mathsf{reify} \circ [\![P]\!].$$

The semantic approach of NBE to computation bypasses the traditional syntactic notion of computation as a reduction relation in a term rewrite system, and is therefore sometimes referred to as *reduction-free normalization*. Of course, the trick is to define a denotational semantics by means of mathematical structures that are invariant under reduction and allow for reification.

NBE (named differently) was invented by Martin-Löf [1] as a special and constructive way of presenting an ordinary normalization proof in intuitionistic type theory: to prove that there exists a normal form of an arbitrary proof/term, means to actually be able to compute this normal form from the proof/term. Independently, NBE has been rediscovered by Berger and Schwichtenberg [2] as a tool to implement a concise and efficient normalization procedure for simply-typed λ-calculus with an extensional (“$\beta\eta$”) notion of term equality. NBE has since been extended both to weaker type systems such as the untyped λ-calculus [3] using a domain theoretic approach, and to richer type systems such as several variants of Martin-Löf type theory [4].

We aim to implement the NBE paradigm in the multiplicative-exponential fragment of linear logic without weakening ($\mathsf{MELL}_{\neg\mathsf{w}}$)[1], using the *relational semantics*[2] as denotational model. More precisely, we prove that given two cut-free, η-expanded $\mathsf{MELL}_{\neg\mathsf{w}}$ nets R and S, if they are ACC[3] then we can compute the normal form $\mathsf{nf}(\mathsf{cut}(R, S))$ of $\mathsf{cut}(R, S)$ (we denote by $\mathsf{cut}(R, S)$ the net obtained by connecting R and S by means of a cut-link) using only a *finite* subset of the interpretations $[\![R]\!]$ and $[\![S]\!]$ of R and S in the relational semantics, without performing any cut-elimination step. Moreover, we are able to bound the size of the finite subsets of $[\![R]\!]$ and $[\![S]\!]$ needed to compute $\mathsf{nf}(\mathsf{cut}(R, S))$.

In our approach the notion of *Taylor expansion* [5, 6] of a net plays a crucial role. In the multiplicative-exponential fragment of linear logic (MELL), the Taylor expansion of a MELL net R is, roughly speaking, the (possibly infinite) set of finitary approximants of R. It can be seen as an intermediate, infinite object between R and its relational semantics $[\![R]\!]$. Indeed, differential nets (which populate the Taylor expansion of a net) still retain a dynamics, even if a very simple, finitary one: the size of differential nets is strictly decreasing under cut-elimination, since duplication is not allowed. On the other hand,

[1]Remember that $\mathsf{MELL}_{\neg\mathsf{w}}$ is sufficiently expressive to encode the λI-calculus, which is Turing-complete.

[2]The relational semantics is one of the simplest denotational models of linear logic, giving an interpretation of nets in the category of sets and relations. It is a degenerate model (every formula has the same interpretation as its negation) and it provides some quantitative information (the promotion operator ! of linear logic is interpreted using multisets, which record the number of times a resource is used).

[3]A net is ACC if every correctness graph associated with it is acyclic and connected. In $\mathsf{MELL}_{\neg\mathsf{w}}$, ACC nets are all and only the ones representing a proof in the sequent calculus.

the Taylor expansion of a cut-free, η-expanded MELL net is equal (up to some irrelevant details) to its relational semantics: we will use this fact in our work, see Point 3 below.

Let us sketch how we implement the NBE paradigm in $\mathsf{MELL}_{\neg\mathsf{w}}$. Let R and S be two cut-free, η-expanded and ACC $\mathsf{MELL}_{\neg\mathsf{w}}$ nets.

1. According to a result proved in [7], from $[\![R]\!]$ and $[\![S]\!]$ we can bound the execution time of $\mathsf{cut}(R,S)$ (i.e. the number of cut-elimination steps leading from $\mathsf{cut}(R,S)$ to its normal form $\mathsf{nf}(\mathsf{cut}(R,S))$) and the size of $\mathsf{nf}(\mathsf{cut}(R,S))$, as well as the size of the element t_0 of order 2 (obtained by taking 2 copies of every box, hereditarily) in the Taylor expansion of $\mathsf{nf}(\mathsf{cut}(R,S))$.

2. By adapting to $\mathsf{MELL}_{\neg\mathsf{w}}$ nets a methodology recently developed for algebraic λ-calculus [8], we can bound the size of an element t (not necessarily unique) of the Taylor expansion of $\mathsf{cut}(R,S)$ such that $t_0 \in \mathsf{nf}(t)$; let n be this upper bound to the size of t.

3. We consider all the elements $t, t', t'', \ldots$ (there are finitely many) of the Taylor expansion of $\mathsf{cut}(R,S)$ whose size is less than n. Since the relational semantics of a MELL net is equal to the union of the relational semantics of the elements of its Taylor expansion, by necessity among $t, t', t'', \ldots$ there is a t such that its relational semantics $[\![t]\!]$ (which is finite) corresponds to the point of order 2 of $[\![\mathsf{cut}(R,S)]\!] = [\![\mathsf{nf}(\mathsf{cut}(R,S))]\!]$ and hence to the element t_0 of order 2 in the Taylor expansion of $\mathsf{nf}(\mathsf{cut}(R,S))$.

4. Since $\mathsf{cut}(R,S)$ is ACC, then $\mathsf{nf}(\mathsf{cut}(R,S))$ is ACC and in particular box-connected. So, we can apply the injectivity result proved in [9] and build $\mathsf{nf}(\mathsf{cut}(R,S))$ from t_0.

We conjecture that it is also possible to extend our NBE approach to MELL, the whole multiplicative-exponential fragment of linear logic (including weakening), both typed and untyped. Clearly, in the untyped case there are MELL nets without normal form, so we have to add the hypothesis that $\mathsf{cut}(R,S)$ is normalizable, which is equivalent of saying that the exhaustive interpretation of $\mathsf{cut}(R,S)$ (a suitable restriction of its relational semantics) is not empty, according to [7].

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