Lecture 3:
From linearity in coherence spaces to Linear Logic

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A computational point of view on coherence semantics

An alternative presentation:

\[ \Delta^\emptyset, x^{\{\alpha\}} : A \vdash x^\alpha : A \]

\[ \frac{\Gamma \vdash \lambda x \ s^{(a,\beta)} : A \rightarrow B}{\Gamma, x^\alpha : A \vdash s^\beta : B} \]

\[ \frac{\Gamma_0 \vdash s^{\{\alpha_1,\ldots,\alpha_k\},\beta} : A \rightarrow B}{\Gamma \vdash \bigcup_{j=0}^k \Gamma_j \vdash (s) t^\beta : B} \]

\[ \frac{\Gamma \vdash s_1^\alpha : A_1 \quad \Gamma \vdash s_2 : A_2}{\Gamma \vdash \langle s_1, s_2 \rangle^{(1,\alpha)} : A_1 \times A_2} \]

\[ \frac{\Gamma \vdash s_1 : A_1 \quad \Gamma \vdash s_2^\alpha : A_2}{\Gamma \vdash \langle s_1, s_2 \rangle^{(2,\alpha)} : A_1 \times A_2} \]

\[ \frac{\Gamma \vdash s^{(i,\alpha)} : A_1 \times A_2}{\Gamma \vdash \pi_i s^\alpha : A_i} \]
A computational point of view on coherence semantics

An alternative presentation:

\[ \Delta^\emptyset, x^{\{\alpha\}} : A \vdash x^\alpha : A \]

\[ \Gamma, x^\alpha : A \vdash s^{\beta} : B \]

\[ \Gamma \vdash \lambda x.s^{(a, \beta)} : A \rightarrow B \]

\[ \Delta^\emptyset, x^{\{\alpha\}} : A \vdash x^\alpha : A \]

\[ \Gamma_0 \vdash s^{\{\{\alpha_1, \ldots, \alpha_k\}, \beta\}} : A \rightarrow B \]

\[ \Gamma_1 \vdash t^{\alpha_1} : A \quad \ldots \quad \Gamma_k \vdash t^{\alpha_k} : A \quad (\ast) \]

\[ \bigcup_{j=0}^k \Gamma_j \vdash (s) t^{\beta} : B \]

\[ \Gamma \vdash s_1^{\alpha} : A_1 \quad \Gamma \vdash s_2 : A_2 \]

\[ \Gamma \vdash \langle s_1, s_2 \rangle^{(1, \alpha)} : A_1 \times A_2 \]

\[ \Gamma \vdash s_1 : A_1 \quad \Gamma \vdash s_2^{\alpha} : A_2 \]

\[ \Gamma \vdash \langle s_1, s_2 \rangle^{(2, \alpha)} : A_1 \times A_2 \]

\[ \Gamma \vdash s^{(i, \alpha)} : A_1 \times A_2 \]

\[ \Gamma \vdash \pi_i s^{\alpha} : A_i \]

\[ (\ast) \text{ Coherence: } \{\alpha_1, \ldots, \alpha_k\} \text{ and the labels of } \bigcup_{j=0}^k \Gamma_j \text{ must be cliques.} \]

The stable function \([x_1 : A_1, \ldots, x_n : A_n \vdash s : B]\) has trace:

\[ \left\{ (a_1 + \ldots + a_n, \beta); \ x_1^{a_1} : A_1, \ldots, x_n^{a_n} : A_n \vdash s^{\beta} : B \right\} \]
Linearity: algebraically

Definition
A stable function \( f : \mathcal{A} \rightarrow \mathcal{B} \) is **linear** if it preserves coherent unions:

\[
a \cup a' \in \mathcal{A} \text{ implies } f(a \cup a') = f(a) \cup f(a')
\]

and

\[
f(\emptyset) = \emptyset
\]
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and

$$f(\emptyset) = \emptyset$$

Consider $\cup$ (resp. $\emptyset$) as a “qualitative” counterpart to $+$ (resp. 0).
Linearity: computationally

Definition
A stable function $f$ is linear iff all the elements of $\mathcal{T}r(f)$ are of the form $(\{\alpha\}, \beta)$.

$\rightsquigarrow$ Show the equivalence with the previous definition.
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Computational linearity: $f$ uses its argument exactly once.
Linearity: computationally

Definition
A stable function $f$ is linear iff all the elements of $\mathcal{Tr}(f)$ are of the form $(\{\alpha\}, \beta)$.

$\Rightarrow$ Show the equivalence with the previous definition.

Computational linearity: $f$ uses its argument exactly once.

Example
Consider the expanded identity
$\lambda x \lambda y (x) y : (A \to B) \to A \to B$: it is linear in $x$

$\Rightarrow$ Compute the trace.
Linearity: computationally

Definition
A stable function $f$ is linear iff all the elements of $Tr(f)$ are of the form $(\{\alpha\}, \beta)$.

$\xrightarrow{\sim}$ Show the equivalence with the previous definition.

Computational linearity: $f$ “uses its argument exactly once.”

Example
Consider the expanded identity
$\lambda x \lambda y (x) y : (A \to B) \to A \to B$: it is linear in $x$ but not necessarily in $y$ (although $y$ has exactly one occurrence).

$\xrightarrow{\sim}$ Compute the trace.
Examples

The identity function is linear
No big deal.
Examples

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No big deal.

Application is linear in the function
Fix some $a \in \mathcal{A}$. The (stable) operator $f \mapsto f(a)$ from $\mathcal{A} \to \mathcal{B}$ to $\mathcal{B}$ is linear.
Examples

The identity function is linear
No big deal.

Application is linear in the function
Fix some $a \in A$. The (stable) operator $f \mapsto f(a)$ from $A \to B$ to $B$ is linear.

“Applicators”
All terms of the form $\lambda x s$ such that $x$ occurs only as the head variable of $s$, have a linear semantics.

⇝ Easy from the rules.

Projections
The projections $\mathcal{A}_1 \& \mathcal{A}_2 \to \mathcal{A}_i$ are linear.
Linear trace

Definition
Let $f$ be a linear function from $\mathcal{A}$ to $\mathcal{B}$. The linear trace of $f$ is:

$$\mathcal{T}_{rl}(f) = \{(\alpha, \beta) ; \beta \in f(\{\alpha\})\}$$

In other words:

$$\mathcal{T}_{rl}(f) = \{(\alpha, \beta) ; (\{\alpha\}, \beta) \in \mathcal{T}(f)\}.$$ 

Clearly, if $f$ is linear, then $f(a) = \mathcal{T}_{rl}(f) \cdot a$, where $\cdot$ denotes the straightforward relation composition.

Examples
$$\mathcal{T}_{rl}(\lambda x \, x) = \{(\alpha, \alpha) ; \alpha \in |\mathcal{A}|\}.$$ 

Another key example: (graphs of) rigid embeddings are linear traces.
linear implication, lollypop
Recall that... 
Traces of stable functions $\mathcal{A} \to \mathcal{B}$ form a coherence space with web $\mathcal{A}_{\text{fin}} \times |\mathcal{B}|$ and such that $(a, \beta) \preceq_{\mathcal{A} \to \mathcal{B}} (a', \beta')$ iff:

- $a \cup a' \in \mathcal{A}$ implies $\beta \preceq_{\mathcal{B}} \beta'$
- if moreover $a \neq a'$, then $\beta \prec_{\mathcal{B}} \beta'$.
Linear implication

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Linear functions actually form a coherence space by themselves:

Definition
Let $\mathcal{A} \rightarrow \circ \mathcal{B}$ be the coherence space with web $|\mathcal{A}| \times |\mathcal{B}|$, and such that: $(\alpha, \beta) \preceq_{\mathcal{A} \circ \mathcal{B}} (\alpha', \beta')$ iff:

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- $a \cup a' \in \mathcal{A}$ implies $\beta \triangleleft_{\mathcal{B}} \beta'$
- if moreover $a \neq a'$, then $\beta \bowtie_{\mathcal{B}} \beta'$.

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- $\alpha \triangleleft_{\mathcal{A}} \alpha'$ implies $\beta \triangleleft_{\mathcal{B}} \beta'$
- $\alpha \bowtie_{\mathcal{A}} \alpha'$ implies $\beta \bowtie_{\mathcal{B}} \beta'$.
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- $\alpha \preceq_{\mathcal{A}} \alpha'$ implies $\beta \preceq_{\mathcal{B}} \beta'$
- $\beta \sim_{\mathcal{B}} \beta'$ implies $\alpha \sim_{\mathcal{B}} \alpha'$. 
“of course”, bang
Linearization of stable maps

The similarity between the conditions for $\circ$:

- $\alpha \circ_\mathcal{A} \alpha' \implies \beta \circ_\mathcal{B} \beta'$
- $\alpha \circ_\mathcal{A} \alpha' \implies \beta \circ_\mathcal{B} \beta'$.

and those for $\rightarrow$:

- $a \cup a' \in \mathcal{A} \implies \beta \circ_\mathcal{B} \beta'$
- if moreover $a \neq a'$, then $\beta \circ_\mathcal{B} \beta'$.

suggests the introduction of a space of finite cliques $\mathcal{A}$:

**Definition**
The space $\mathcal{A}$ has web $\mathcal{A}_{\text{fin}}$ and coherence:

$$a \circ_\mathcal{A} \mathcal{A} a' \iff a \cup a' \in \mathcal{A}.$$
Linearization of stable maps

The similarity between the conditions for $\circ$:

- $\alpha \subseteq_\mathcal{A} \alpha' \implies \beta \subseteq_\mathcal{B} \beta'$
- $\alpha \cap_\mathcal{A} \alpha' \implies \beta \cap_\mathcal{B} \beta'$.

and those for $\rightarrow$:

- $a \cup a' \in \mathcal{A} \implies \beta \subseteq_\mathcal{B} \beta'$
- if moreover $a \neq a'$, then $\beta \cap_\mathcal{B} \beta'$.

suggests the introduction of a space of finite cliques $!\mathcal{A}$:

**Definition**

The space $!\mathcal{A}$ has web $\mathcal{A}_{\text{fin}}$ and coherence:

$$a \subseteq !_\mathcal{A} a' \iff a \cup a' \in \mathcal{A}.$$ 

Then $\mathcal{A} \rightarrow \mathcal{B} = !_\mathcal{A} \rightarrow \mathcal{B}$, and stable functions from $\mathcal{A} \rightarrow \mathcal{B}$ “are” linear functions from $!\mathcal{A}$ to $\mathcal{B}$.

$\sim$ Describe the bijection extensionally.
linear negation, dual, polar, orthogonal
Linear negation

The symmetry of conditions:

▶ \( \alpha \succeq_A \alpha' \) implies \( \beta \succeq_B \beta' \)

▶ \( \beta \succeq_B \beta' \) implies \( \alpha \succeq_B \alpha' \)

suggests the introduction of linear negation:

**Definition**

The dual space of \( A \), denoted \( A^\perp \), has web \(|A|\) and coherence:

\[
\alpha \succeq_{A^\perp} \alpha' \iff \alpha \succeq_A \alpha'
\]

**Transposition**

Clearly \( A^{\perp\perp} = A \) and there is a linear involution from \( A \rightarrow \mathcal{B} \) to \( B^\perp \rightarrow A^\perp \).

(we could even write \( A \rightarrow B = A^\perp \leftarrow B^\perp \))

\( \rightsquigarrow \) Find the (linear) trace of this involution...
Linear negation

The symmetry of conditions:

- $\alpha \preceq_\mathcal{A} \alpha' \implies \beta \preceq_\mathcal{B} \beta'$
- $\beta \succeq_\mathcal{B} \beta' \implies \alpha \succeq_\mathcal{B} \alpha'$

suggests the introduction of linear negation:

**Definition**

The dual space of $\mathcal{A}$, denoted $\mathcal{A}^{\perp}$, has web $|\mathcal{A}|$ and coherence:

$$\alpha \preceq_{\mathcal{A}^{\perp}} \alpha' \iff \alpha \succeq_{\mathcal{A}} \alpha'$$

**Transposition**

Clearly $\mathcal{A}^{\perp\perp} = \mathcal{A}$ and there is a linear involution from $\mathcal{A} \rightarrow \mathcal{B}$ to $\mathcal{B}^{\perp} \rightarrow \mathcal{A}^{\perp}$.

(we could even write $\mathcal{A} \rightarrow \mathcal{B} = \mathcal{A}^{\perp} \leftarrow \mathcal{B}^{\perp}$)

$\leadsto$ Find the (linear) trace of this involution...

(hint in the paragraph title)
Linear logic?

The story so far

- logic: $\lambda$-calculus, system $F$, ...
Linear logic?

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- logic: $\lambda$-calculus, system $F$, ...
- semantics: stable functions between coherence spaces
Linear logic?

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- logic: $\lambda$-calculus, system $F$, ...
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- decomposition of implication semantically
Linear logic?

The story so far

- logic: $\lambda$-calculus, system $F$, ...
- semantics: stable functions between coherence spaces
- decomposition of implication semantically

Is there a logical system behind that refinement, based on linear implication?
Linear category of coherence spaces

Linear functions compose
More precisely, coherence spaces and linear functions between them form a category $\text{Coh}_l$.

Product
The direct product $\&$ is still a categorical product (pairing).
Linear category of coherence spaces

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More precisely, coherence spaces and linear functions between them form a category $\text{Coh}_l$.

Product
The direct product $\&$ is still a categorical product (pairing) but not properly related with $\circ$:

$A \& B \circ C$ is not the type of bilinear functions.
Linear category of coherence spaces

Linear functions compose
More precisely, coherence spaces and linear functions between them form a category $\text{Coh}_l$.

Product
The direct product $\&$ is still a categorical product (pairing) but not properly related with $\rightarrow_{\circ}$:
$A \& B \rightarrow_{\circ} C$ is not the type of bilinear functions.

$\rightsquigarrow$ Do you see why?
tensor
Tensor product

Definition
The tensor product of $\mathcal{A}$ and $\mathcal{B}$, denoted $\mathcal{A} \otimes \mathcal{B}$ has web $|\mathcal{A}| \times |\mathcal{B}|$, and coherence:

$$(\alpha, \beta) \circ_{\mathcal{A} \otimes \mathcal{B}} (\alpha', \beta') \iff (\alpha \circ_{\mathcal{A}} \alpha' \wedge \beta \circ_{\mathcal{B}} \beta')$$

Then $\mathcal{A} \Rightarrow (\mathcal{B} \Rightarrow \mathcal{C})$ is isomorphic to $\mathcal{A} \otimes \mathcal{B} \Rightarrow \mathcal{C}$.

\[ \Rightarrow \]

Write down the isomorphism.

This product is associative and symmetric ($\mathcal{A} \otimes \mathcal{B} \cong \mathcal{B} \otimes \mathcal{A}$), and has a unit:

Definition
Let $1$ be the only coherence space with web $\{\emptyset\}$.
Tensor product

Definition
The tensor product of \( A \) and \( B \), denoted \( A \otimes B \) has web \(|A| \times |B|\), and coherence:

\[(\alpha, \beta) \in A \otimes B \quad (\alpha', \beta') \iff (\alpha \in A \alpha' \land \beta \in B \beta')\]

Then \( A \rightarrow (B \rightarrow C) \) is isomorphic to \((A \otimes B) \rightarrow C\).

\( \Rightarrow \) Write down the isomorphism.
Tensor product

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The tensor product of $\mathcal{A}$ and $\mathcal{B}$, denoted $\mathcal{A} \otimes \mathcal{B}$ has web $|\mathcal{A}| \times |\mathcal{B}|$, and coherence:

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Then $\mathcal{A} \circ (\mathcal{B} \circ \mathcal{C})$ is isomorphic to $(\mathcal{A} \otimes \mathcal{B}) \circ \mathcal{C}$.

$\Rightarrow$ Write down the isomorphism.

This product is associative and symmetric ($\mathcal{A} \otimes \mathcal{B} \cong \mathcal{B} \otimes \mathcal{A}$), and has a unit:

Definition
Let $\mathbf{1}$ be the only coherence space with web $\{\emptyset\}$.
par
par
Multiplicatives

By de Morgan, we obtain the connective $\&$ as dual to $\otimes$:

$$A \& B = (A^\perp \otimes B^\perp)^\perp$$

and then

$$A \to B = A^\perp \& B$$

$\rightsquigarrow$ Write down an explicit definition of coherence for $\&$
Multiplicatives

By de Morgan, we obtain the connective $\otimes$ as dual to $\otimes$:

$$A \otimes B = (A^\perp \otimes B^\perp)^\perp$$

and then

$$A \multimap B = A^\perp \otimes B$$

$\leadsto$ Write down an explicit definition of coherence for $\otimes$

We thus have three connectives based on set theoretical product, aka. **multiplicatives**:

- linear implication: $\multimap$;
- the associated conjunction: $\otimes$;
- the associated disjunction: $\otimes$. 
Multiplicatives

By de Morgan, we obtain the connective \( \otimes \) as dual to \( \otimes \):

\[
A \otimes B = (A \perp \otimes B \perp) \perp
\]

and then

\[
A \implies B = A \perp \otimes B
\]

\( \rightsquigarrow \) Write down an explicit definition of coherence for \( \otimes \)

We thus have three connectives based on set theoretical product, aka. multiplicatives:

- linear implication: \( \implies \);
- the associated conjunction: \( \otimes \);
- the associated disjunction: \( \otimes \).

The unit of \( \otimes \) is \( \perp = 1 \perp \).

A (minor) degeneracy in this case: \( \perp = 1 \)
Sequents

Linear negation is classical (because it is involutive).
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The framework of choice is thus that of sequents: a proof of

\[ \mathcal{A}_1, \ldots, \mathcal{A}_n \vdash \mathcal{B}_1, \ldots, \mathcal{B}_p \]

denotes a linear map

\[ \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n \to \mathcal{B}_1 \simeq \ldots \simeq \mathcal{B}_p \]

and negation swaps sides

\[ (\mathcal{A} \vdash \mathcal{B}) \cong (\mathcal{B}^\perp \vdash \mathcal{A}^\perp) \]

so we can freely move things around.
Recall that... The product $\&$ is not adjoint to $\rightarrow$. Hence, in the process of proving, we can’t:

- duplicate hypotheses (there is no diagonal for $\otimes$)
- discard hypotheses (the terminal object is not $1$)

In other words, the structural rules of contraction and weakening do not hold.
Multiplicative linear logic

identity (axiom) \[ \frac{}{A \vdash A} \]

composition (cut) \[ \frac{\Gamma \vdash A \quad A \vdash \Delta}{\Gamma \vdash \Delta} \]

tensor \[ \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \]

unit \[ \frac{}{\vdash 1} \]

par \[ \frac{\Gamma \vdash A, B}{\Gamma \vdash A \& B} \]

bottom \[ \frac{}{\Gamma \vdash \bot} \]
direct product, with
Recall that... The direct product $A_1 \& A_2$ has web $|A_1| + |A_2|$ and coherence:

$$(i, \alpha) \circ_{A_1 \& A_2} (j, \alpha') \iff (i = j \Rightarrow \alpha \circ_{A_i} \alpha')$$

- It is the type of pairs: a clique in $A_1 \& A_2$ can be uniquely written as $a_1 + a_2$.
- Its unit is $\top$: the only space with empty web.
Recall that...  
The direct product $A_1 \& A_2$ has web $|A_1| + |A_2|$ and coherence:

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- It is the type of pairs:  
  a clique in $A_1 \& A_2$ can be uniquely written as $a_1 + a_2$.  
- Its unit is $\top$: the only space with empty web.

pairing (with) $\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B}$

terminal object (top) $\frac{}{\Gamma \vdash \top}$
direct sum, plus
The category \textbf{Coh}_l also has coproducts (or sums), dual to \&:

**Definition**

The direct sum \( A_1 \oplus A_2 \) has web \(|A_1| + |A_2|\) and coherence:

\[
(i, \alpha) \preceq_{A_1 \oplus A_2} (j, \alpha') \iff (i = j \land \alpha \preceq_{A_i} \alpha')
\]

The unit is \( 0 = \top^\perp \).

Again: \( 0 = \top \)
Sums

The category $\textbf{Coh}_l$ also has coproducts (or sums), dual to $\&$:

**Definition**
The direct sum $A_1 \oplus A_2$ has web $|A_1| + |A_2|$ and coherence:

$$(i, \alpha) \triangleleft_{A_1 \oplus A_2} (j, \alpha') \iff (i = j \land \alpha \triangleleft_{A_i} \alpha')$$

The unit is $0 = \top \bot$.

Again: $0 = \top$

injections (plus) \[
\frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B} \quad \frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}
\]
Sums

The category $\textbf{Coh}_I$ also has coproducts (or sums), dual to $\&$:

**Definition**

The direct sum $A_1 \oplus A_2$ has web $|A_1| + |A_2|$ and coherence:

$$(i, \alpha) \simeq_{A_1 \oplus A_2} (j, \alpha') \iff (i = j \land \alpha \simeq_{A_i} \alpha')$$

The unit is $0 = \top \bot$.

Again: $0 = \top$

injections (plus)

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B} \quad \frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}$$

By contrast, there was no sum type in the stable semantics!
Together \& and \oplus are the additive connectives of linear logic. Moreover:

\[ A \otimes (B \oplus C) \cong (A \otimes B) \oplus (A \otimes C) \]

(and dually for \wedge and \&).
“of course” and “why not”: the exponentials
Recall that... The space $\mathcal{A}$ has web $\mathcal{A}_{\text{fin}}$ and coherence:

$$a \subseteq_{\mathcal{A}} a' \iff a \cup a' \in \mathcal{A}.$$ 

Its de Morgan dual is denoted $?A = (\!A)^{\perp\!\perp}$. 

Structural rules hold on exponentials:

- **contraction**
  $$\frac{\!A, \!A \vdash \Delta}{\!A \vdash \Delta}$$

- **weakening**
  $$\frac{\vdash \Delta}{\!A \vdash \Delta}$$
Recall that... The space $!A$ has web $A_{\text{fin}}$ and coherence:

$$a \triangleleft !A \ a' \iff a \cup a' \in A.$$ 

Its de Morgan dual is denoted $?A = (!A^\perp)^\perp$. 

Structural rules hold on exponentials

**contraction**

$$a : !A, a' : !A \vdash \Delta$$

$$a \cup a' : !A \vdash \Delta$$

**weakening**

$$\vdash \Delta$$

$$\emptyset : !A \vdash \Delta$$
Introduction rules for exponentials: dereliction

Linearization of the (stable) identity $\lambda x . x : A \rightarrow A$ gives

$$\{\{\alpha\}, \alpha\}; \alpha \in |A| \} \in !A \rightarrow A$$

Hence the rule

$$ \frac{A \vdash \Delta}{!A \vdash \Delta} \quad \text{or equivalently} \quad \frac{\vdash A, \Delta}{\vdash ?A, \Delta}$$
Exponentials are functors

If \( f : A \to B \), define \( !f : !A \to !B \) by its trace

\[
\{(\{\alpha_1, \ldots, \alpha_n\}, \{\beta_1, \ldots, \beta_n\}) \in A_{\text{fin}} \times B_{\text{fin}}; \forall i, (\alpha_i, \beta_i) \in f\}
\]

So we could have a rule

\[
\frac{\Gamma \vdash A}{!\Gamma \vdash !A}
\]
Introduction rules for exponentials: promotion

Exponentials are (co-)monads
We only miss the “multiplication” \( !A \rightarrow !!A \), given by

\[
\left\{ \left( \bigcup_{j=1}^{n} a_j, \{a_1, \ldots, a_n\} \right) \in A_{\text{fin}} \times !A_{\text{fin}} \right\}
\]

which would give a rule

\[
\frac{!!A \vdash \Delta}{!A \vdash \Delta}
\]

called digging.
Introduction rules for exponentials: promotion

The previous two are subsumed by the promotion rule:

\[
\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}
\]
The previous two are subsumed by the promotion rule:

\[
\left( !\Gamma_j \vdash \alpha_j : A \right)_{j=1}^k \\
\bigcup_{j=1}^k !\Gamma_j \vdash \{\alpha_1, \ldots, \alpha_k\} : !A
\]
Cut elimination

- Next talk!
- Reflects properties of the things we used (properties of functors, products, sums, ...).
- Makes the system logically relevant.
Cut elimination

- Next talk!
- Reflects properties of the things we used (properties of functors, products, sums, ...).
- Makes the system logically relevant.
- Refines $\beta$-reduction...
Lambda-calculus in LL

Principle

The decomposition $A \rightarrow B = !A \leftarrow B$ leads to translate a typed term $\Gamma \vdash s : A$ as a proof of $!\Gamma \vdash A$.

Variable

$$\Delta^\emptyset, x^\{\alpha\} : A \vdash x^\alpha : A$$

becomes

$$\frac{\alpha : A \vdash \alpha : A}{\{\alpha\} : !A \vdash \alpha : A} \quad \text{(der)}$$

$$\frac{\alpha : A \vdash \alpha : A}{\emptyset : !\Delta, \{\alpha\} : !A \vdash \alpha : A} \quad \text{(weak)}^*$$
Lambda-calculus in LL

Abstraction

\[ \Gamma, x^a : A \vdash s^\beta : B \]
\[ \Gamma \vdash \lambda x \ s^{(a,\beta)} : A \rightarrow B \]

becomes

\[ !\Gamma, a : !A \vdash \beta : B \]
\[ !\Gamma \vdash (a, \beta) : !A \multimap B \] (→)}
Lambda-calculus in LL

Application

\[ \Gamma_0 \vdash s(\{\alpha_1, \ldots, \alpha_k\}, \beta) : A \rightarrow B \quad (\Gamma_j \vdash t^{\alpha_j} : A)_{j=1}^k \]

\[ \bigcup_{j=0}^k \Gamma_j \vdash (s) t^\beta : B \]

becomes

\[ !\Gamma_0 \vdash (\{\alpha_1, \ldots, \alpha_k\}, \beta) : !A \rightarrow B \]

\[ \vdash (\bigcup_{j=1}^k !\Gamma_j \vdash \alpha_j : A)_{j=1}^k \quad \text{(prom)} \]

\[ \bigcup_{j=1}^k !\Gamma_j \vdash \{\alpha_1, \ldots, \alpha_k\} : !A \quad \text{(ax)} \]

\[ \beta : B \vdash \beta : B \quad \text{(cut)} \]

\[ \Gamma_0, \bigcup_{j=1}^k !\Gamma_j \vdash \beta : B \quad \text{cont}^* \]

\[ \bigcup_{j=0}^k \Gamma_j \vdash \beta : B \quad \text{(cont)}^* \]
Digression 1: Orthogonality

Coherence spaces: a third definition

- If \( a, a' \subseteq A \), write \( a \perp a' \) iff \( a \cap a' \) is at most a singleton.
- If \( A \subseteq \mathcal{P}(A) \), write \( A^\perp = \{ a' \subseteq A; \forall a \in A, \ a \perp a' \} \).
- Coherence spaces of web \( A \) are those \( A \) such that \( A = A^\perp \perp \).

Then a relation \( f \subseteq |A| \times |B| \) is a linear trace iff

\[
\forall a \in A, \ \forall b' \in B^\perp, \ f \cdot a \perp b' \land a \perp f^\perp \cdot b'
\]

Such orthogonality constructions are very common in the design of LL models.
Digression 2: Quantitative semantics

Why do we use such linear algebraic vocabulary, notations and concepts? Historically:
coh. spaces ⇐ qualitative domains ⇐ quantitative semantics
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coh. spaces $\iff$ qualitative domains $\iff$ quantitative semantics

Rough intuition
Interpret terms as a linear combinations: $s = \sum_{\alpha \in s} s_\alpha \alpha$
so that application is given by a power series:

$((s \ t))_{\beta} = \sum_{(a, \beta) \in s} s_{(a, \beta)} t^a$

where $t^{[\alpha_1, \ldots, \alpha_k]} = t_{\alpha_1} \cdots t_{\alpha_k}$. 
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Taken litterally, it is only meaningful if we can ensure a form of convergence.
The end

Thanks.
The end

Thanks.
Questions?
The end

Thanks.
Questions?

Next talk, Emmanuel Beffara:
sequent calculus, polarities, focalization, phase semantics