Differential Linear Logic and Polarization

Speaker: Michele Pagani

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Typed Lambda Calculi and Applications, 2009, Brasilia

Where is the author ?

Taking care of his newborn son:

Ulysse, born on June the 11th

Merci Michele !

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Context

Cut elimination = computation

Curry–Howard correspondence

Context

λ-calculus

Functional programming

Intuitionistic logic

Cut elimination = computation

Curry–Howard correspondence

Context

Functional programming

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Intuitionistic logic

Girard's Linear logic
$!A \multimap B$

Coherence semantics

Curry–Howard correspondence

Denotational semantics

Context

λ-calculus

Functional programming

Intuitionistic logic

Linear logic

Coherence semantics

Parigot 1992
$\lambda\mu$-calculus

Griffin 1990
Control

Classical logic

Polarized linear logic
Laurent 2002

Correlation spaces
Girard 1991

Curry–Howard correspondence

Context

Functional programming

λ-calculus

Intuitionistic logic

Linear logic

Coherence semantics

$\lambda\mu$-calculus

Control

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Correlation spaces

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Differential Linear Logic

Quantitative semantics *Girard,* predates coherence

Context

λ-calculus

Functional programming

Intuitionistic logic

Linear logic

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$\lambda\mu$-calculus

Control

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Curry–Howard correspondence

Ehrhard Laurent 2007
Concurrency

Ehrhard Regnier 2003
Differential λ-calculus

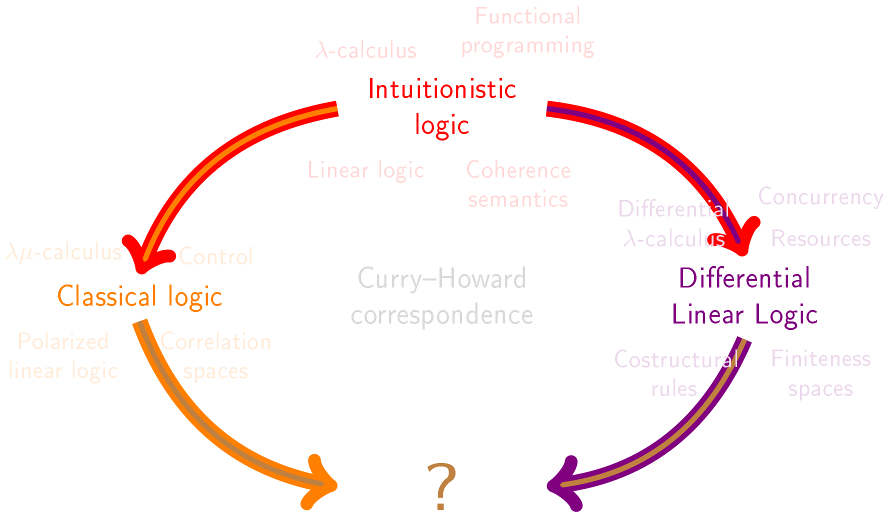
Ehrhard Regnier 2006
Resources

Differential Linear Logic

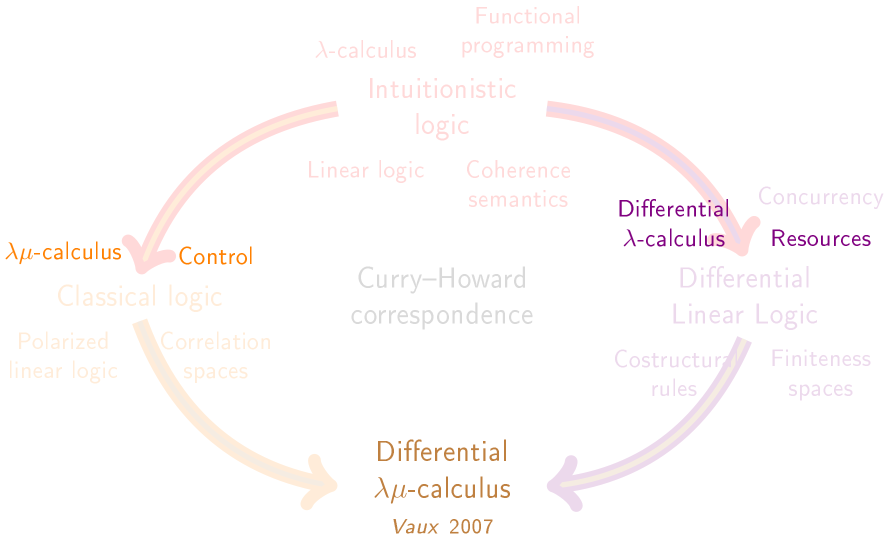
Costructural rules
Ehrhard Regnier 2006

Finiteness spaces
Ehrhard 2001

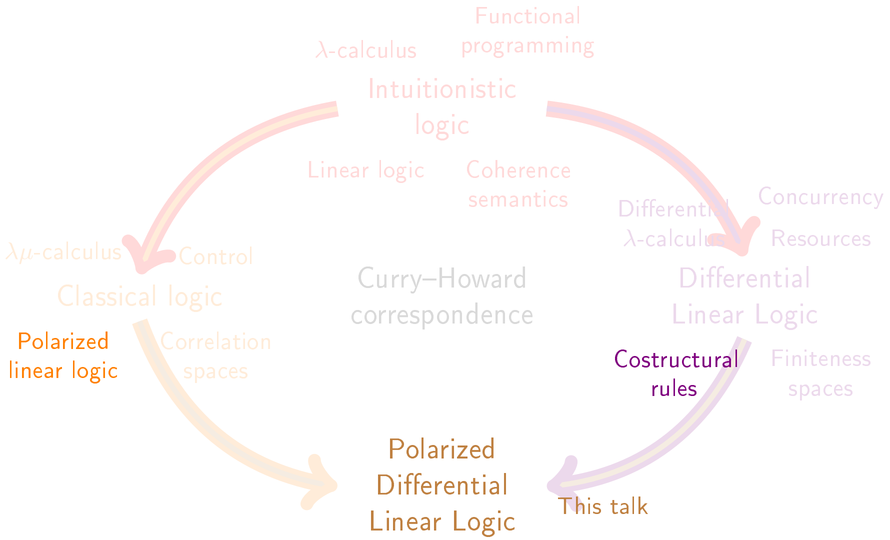
Context, Motivations



Context, Motivations



Context, Motivations, Outline



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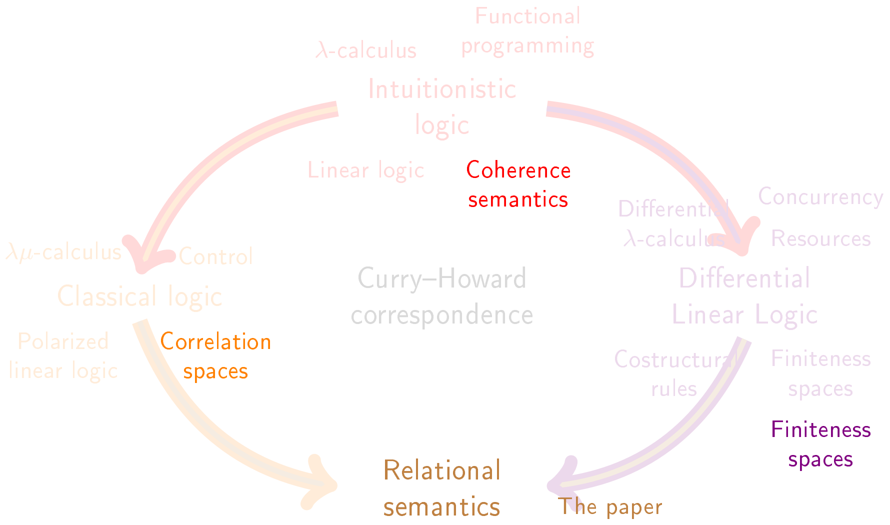

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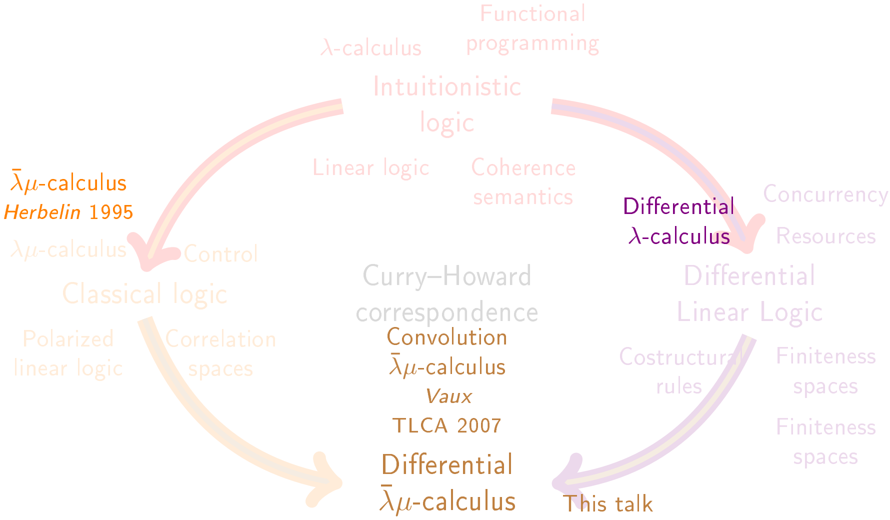

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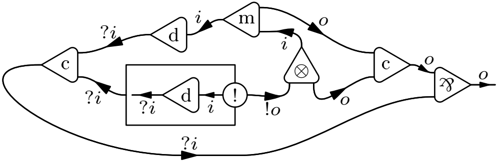
Interaction nets

An interaction net = a set of cells and wires between them (dangling wires and loops are allowed). *Lafont*, 1995



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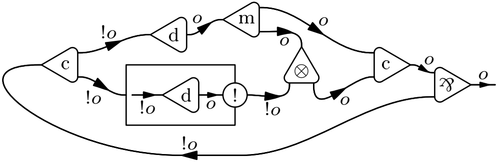

Reversing orientation = dual typing.

Interaction nets

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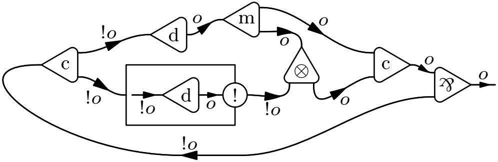
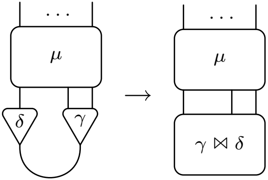


Reversing orientation = dual typing.

The core of cut elimination:



Polarization and pure types

Polarized formulas

$$\begin{array}{rcl} P, Q & ::= & X^\perp \mid P \otimes Q \mid P \oplus Q \mid \,!N \\ M, N & ::= & X \mid M \parr N \mid M \,\&\, N \mid \,?P \end{array}$$

- The polarized fragment of (ME)LL contains the pure typed λ-calculus: $M \Rightarrow N = \,!M \multimap N = \,?M^\perp \parr N$.
- It enjoys a very nice proof theory and admits structural rules on negative formulas, if not for atoms. It is the basis of Polarized Linear Logic *Laurent*, 2002.

Polarization and pure types

Polarized formulas

$$\begin{array}{rcl} P, Q & ::= & X^\perp \mid P \otimes Q \mid P \oplus Q \mid !N \\ M, N & ::= & X \mid M \parr N \mid M \,\&\, N \mid ?P \end{array}$$

Pure types (MELL for the pure untyped λ-calculus)

Let $o = ?o^\perp \parr o$ (type of all λ-terms) *Danos–Regnier*, 1990
and then $!o$ (type of arguments)
and their duals: $i = o^\perp$ (contexts) and $?i$ (free variables)

positive: i negative: o of course: $!o$ why not: $?i$

Polarization and pure types

Polarized formulas

$$P, Q \quad ::= \quad X^{\perp} \mid P \otimes Q \mid P \oplus Q \mid !N$$
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Pure types (MELL for the pure untyped λ-calculus)

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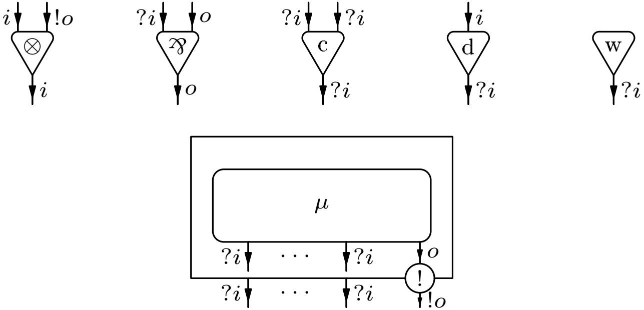
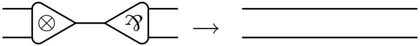
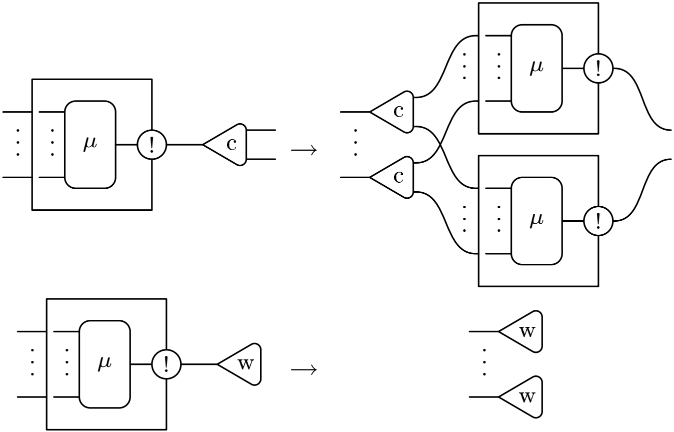
Linear logic nets



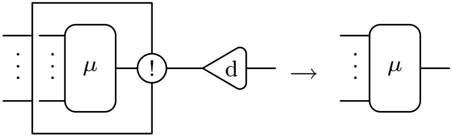
Multiplicative reduction $\langle \parr , \otimes \rangle$



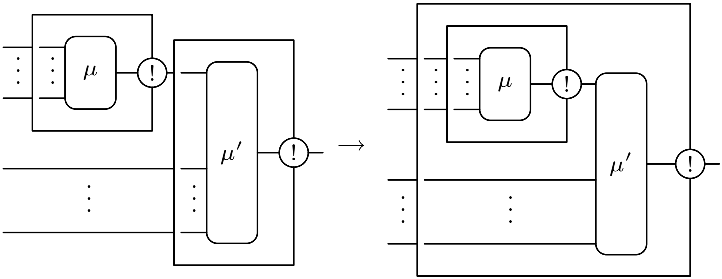
Duplication $\langle ! , \mathrm{c} \rangle$ and erasing $\langle ! , \mathrm{w} \rangle$



Box opening $\langle ! , \mathrm{d} \rangle$



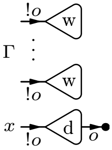
Box commutation $\langle ! , ! \rangle$



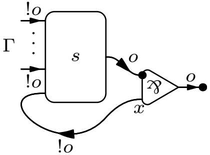
λ-calculus

Danos–Regnier, 1990

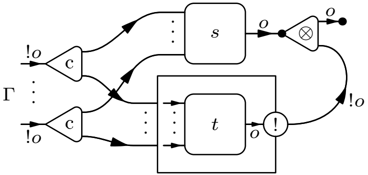
$$\frac{}{\Gamma, x \vdash x} \quad \rightsquigarrow$$



$$\frac{\Gamma, x \vdash s}{\Gamma \vdash \lambda x\, s} \quad \rightsquigarrow$$



$$\frac{\Gamma \vdash s \qquad \Gamma \vdash t}{\Gamma \vdash (s)\, t} \quad \rightsquigarrow$$



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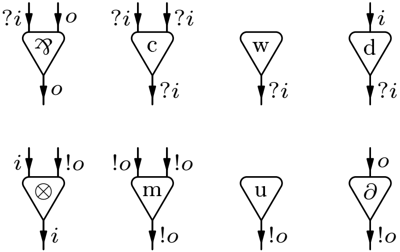
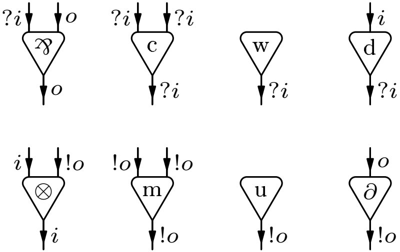
Differential nets

Ehrhard–Regnier, 2006



Differential nets

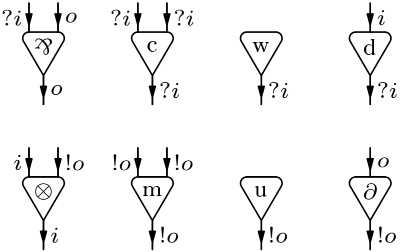
Ehrhard–Regnier, 2006



∂	:	A	$\multimap$	$!A$	derivative at 0
m	:	$!A \otimes !A$	$\multimap$	$!A$	sum of arguments of type A
u	:	$\mathbf{1}$	$\multimap$	$!A$	zero argument of type A

Differential nets

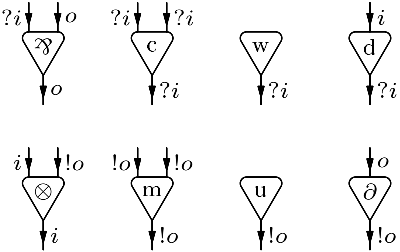
Ehrhard–Regnier, 2006



∂	:	o	$\multimap$	$!o$	make singleton packet
m	:	$!o \otimes !o$	$\multimap$	$!o$	merge
u	:	$\mathbf{1}$	$\multimap$	$!o$	empty packet

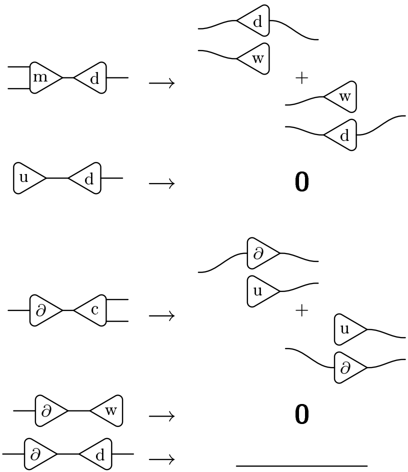
Differential nets

Ehrhard–Regnier, 2006



d	:	$!o$	$\multimap$	o	open singleton packet
c	:	$!o$	$\multimap$	$!o \otimes !o$	split
w	:	$!o$	$\multimap$	$\mathbf{1}$	erase

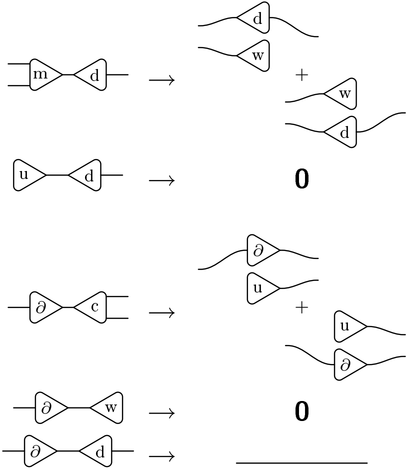
Routing and communication



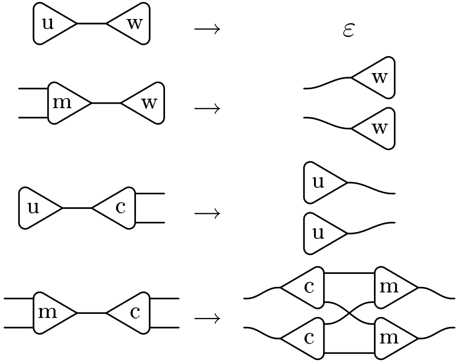
Routing and communication

Sum $\simeq$ non-deterministic choice

0 $\simeq$ failure

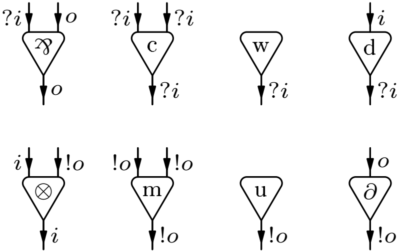


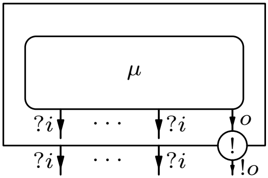
Bialgebras à la Hopf



Differential nets

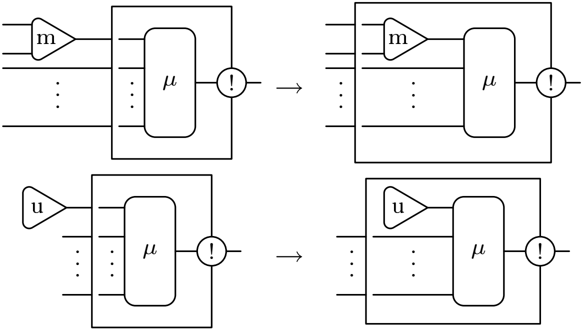
Ehrhard–Regnier, 2006



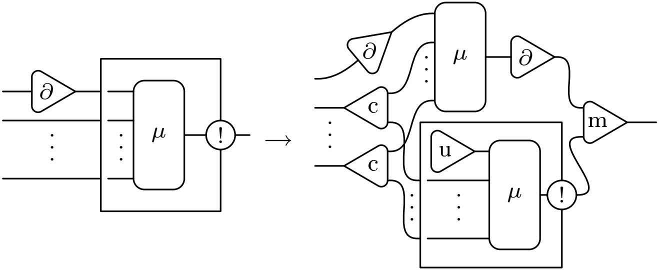


Vaux, PhD, 2007
Tranquilli, PhD, 2009
Pagani, TLCA 2009

Commutations



Chain law



$$(f[u])' = f'[u] \times u' = (\mathrm{D}f \cdot u')[u]$$

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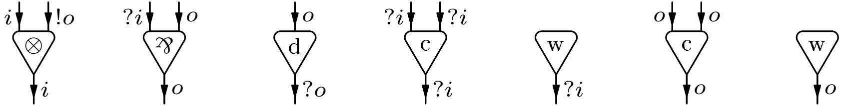
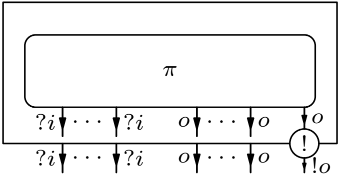
Polarized linear logic

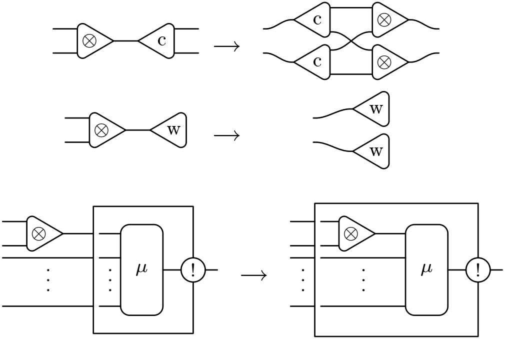
Laurent, PhD, 2002

Allow structural rules on negative types.

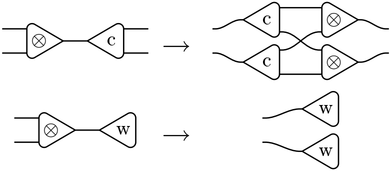




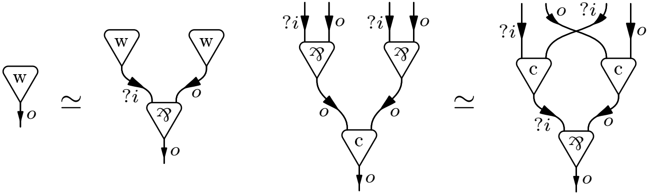
Polarized linear logic



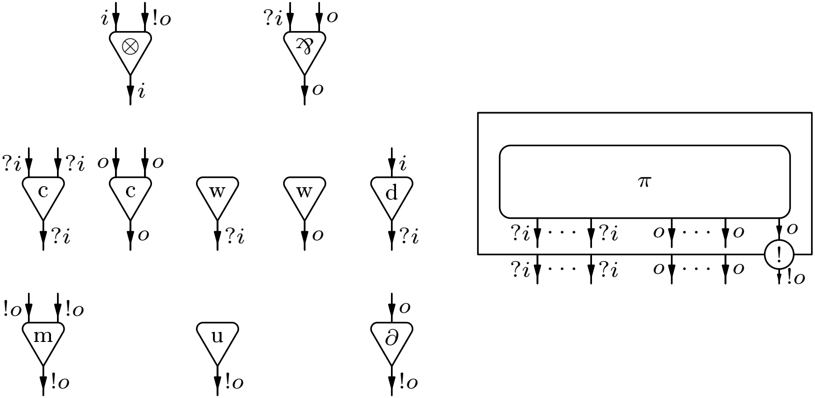
Polarized linear logic



Structural equivalence

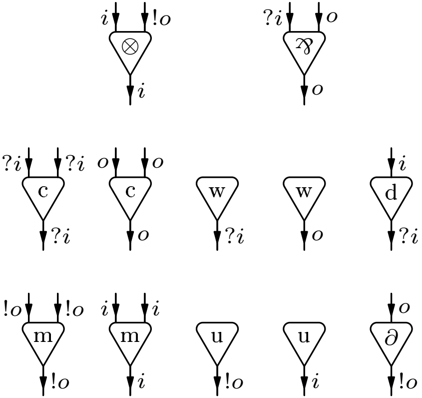


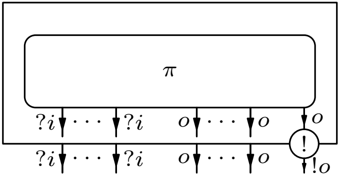
Differential nets ∪ Polarized nets



- Corresponds with the differential $\lambda\mu$-calculus.
- No new logical feature or computational effect.

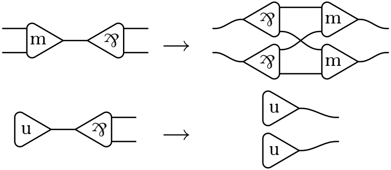
Polarized differential nets





- Polarized costructural rules.

Cut elimination



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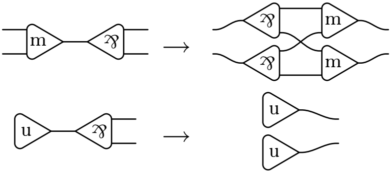

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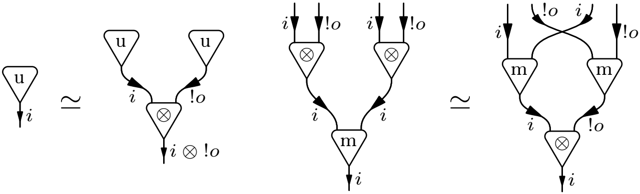

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$\bar{\lambda}\mu$-calculus

Herbelin, PhD, 1995

$$
\begin{array}{lrcl}
\text{Terms:} & s, t & ::= & x \mid \lambda x\, s \mid \mu\alpha\, c \\
\text{Stacks:} & e & ::= & \alpha \mid s \cdot e \\
\text{Commands:} & c & ::= & \langle s \,,\, e \rangle
\end{array}
$$

$\bar{\lambda}\mu$-calculus: a term calculus for sequents *Herbelin*, PhD, 1995

Terms:	s, t	$::=$	$x \mid \lambda x\, s \mid \mu\alpha\, c$	*(conclusions)*
Stacks:	e	$::=$	$\alpha \mid s \cdot e$	*(hypotheses)*
Commands:	c	$::=$	$\langle s \,,\, e \rangle$	*(cuts)*

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Stacks:	e	$::=$	$\alpha \mid s \cdot e$	*(hypotheses)*
Commands:	c	$::=$	$\langle s\, ,\, e \rangle$	*(cuts)*

$$\begin{array}{rcl} \langle \lambda x\, s\, ,\, t \cdot e \rangle & \rightarrow & \langle s\,[t/x]\, ,\, e \rangle \\ \langle \mu\alpha\, c\, ,\, e \rangle & \rightarrow & c\,[e/\alpha] \end{array}$$

$\bar{\lambda}\mu$-calculus: a term calculus for sequents *Herbelin*, PhD, 1995

$$
\begin{array}{llcll}
\text{Terms:} & s, t & ::= & x \mid \lambda x\, s \mid \mu\alpha\, c & \textit{(conclusions)} \\
\text{Stacks:} & e & ::= & \alpha \mid s \cdot e & \textit{(hypotheses)} \\
\text{Commands:} & c & ::= & \langle s \,,\, e \rangle & \textit{(cuts)}
\end{array}
$$

$$
\begin{array}{rcl}
\langle \lambda x\, s \,,\, t \cdot e \rangle & \to & \langle s\,[t/x] \,,\, e \rangle \\
\langle \mu\alpha\, c \,,\, e \rangle & \to & c\,[e/\alpha]
\end{array}
$$

Encoding λ-calculus application:

$$(s)\, t = \mu\alpha\, \langle s \,,\, t \cdot \alpha \rangle$$

$\bar{\lambda}\mu$-calculus: a term calculus for sequents *Herbelin*, PhD, 1995

$$\begin{array}{llcll} \text{Terms:} & s, t & ::= & x \mid \lambda x\, s \mid \mu\alpha\, c & \textit{(conclusions)} \\ \text{Stacks:} & e & ::= & \alpha \mid s \cdot e & \textit{(hypotheses)} \\ \text{Commands:} & c & ::= & \langle s\, ,\, e \rangle & \textit{(cuts)} \end{array}$$

$$\begin{array}{rcl} \langle \lambda x\, s\, ,\, t \cdot e \rangle & \rightarrow & \langle s\, [t/x]\, ,\, e \rangle \\ \langle \mu\alpha\, c\, ,\, e \rangle & \rightarrow & c\, [e/\alpha] \end{array}$$

Encoding λ-calculus application:

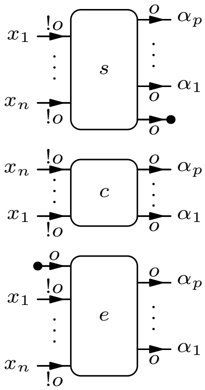
$$(s)\, t = \mu\alpha\, \langle s\, ,\, t \cdot \alpha \rangle$$

η-expansion:

$$s \leftarrow_\eta \lambda x\, \mu\alpha\, \langle s\, ,\, x \cdot \alpha \rangle$$

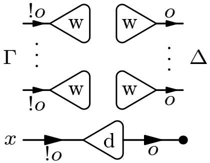
Translation into in polarized nets

Simplified version of Laurent's translation of Curien–Herbelin's $\bar{\lambda}\mu$-calculus. *Laurent*, PhD, 2002

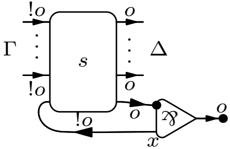


Translation: terms

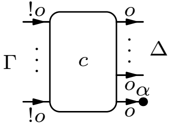
$\Gamma \vdash x \mid \Delta \quad \rightsquigarrow$



$\Gamma \vdash \lambda x\, s \mid \Delta \quad \rightsquigarrow$

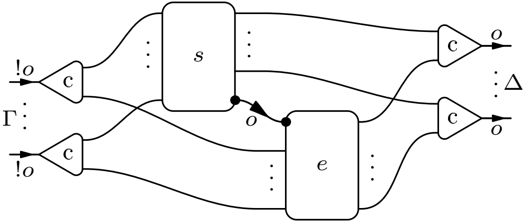


$\Gamma \vdash \mu\alpha\, c \mid \Delta \quad \rightsquigarrow$



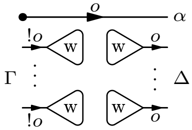
Translation: commands

$\langle s \, , e \rangle : (\Gamma \vdash \Delta) \quad \rightsquigarrow$

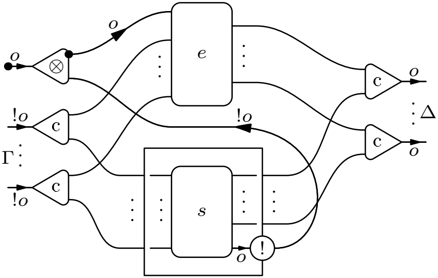


Translation: stacks

$\Gamma \mid \alpha \vdash \Delta \quad \rightsquigarrow$



$\Gamma \mid s \cdot e \vdash \Delta \quad \rightsquigarrow$



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Convolution $\bar{\lambda}\mu$-calculus

Vaux, TLCA 2007

Computational interpretation of polarized costructural rules

Positive formulas (i.e. i) appear as the type of stacks.

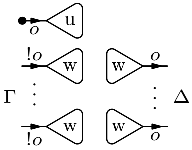
Convolution $\bar{\lambda}\mu$-calculus

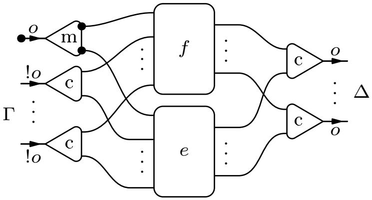
Vaux, TLCA 2007

Computational interpretation of polarized costructural rules

Positive formulas (i.e. i) appear as the type of stacks.

$$\begin{array}{rcl} s,t & ::= & x \mid \lambda x\, s \mid \mu\alpha\, c \\ e,f & ::= & \alpha \mid s\cdot e \mid \mathbf{1} \mid e * f \\ c & ::= & \langle s\, ,\, e\rangle \end{array}$$

$\mathbf{1} \rightsquigarrow$ 

$e * f \rightsquigarrow$ 

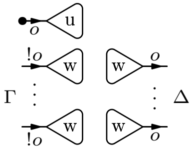
Convolution $\bar{\lambda}\mu$-calculus

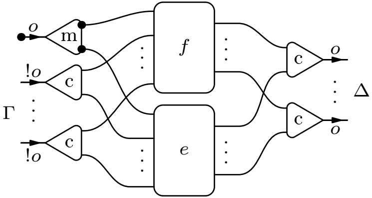
Vaux, TLCA 2007

Computational interpretation of polarized costructural rules

Positive formulas (i.e. i) appear as the type of stacks.

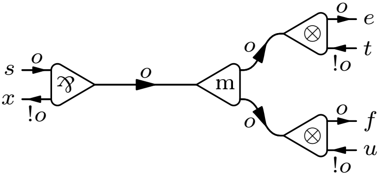
$$
\begin{aligned}
s,t &::= x \mid \lambda x\, s \mid \mu\alpha\, c \mid \mathbf{0} \mid s+t\\
e,f &::= \alpha \mid s\cdot e \mid \mathbf{1} \mid e*f \mid \mathbf{0} \mid e+f\\
c &::= \langle s\,,\,e\rangle \mid \mathbf{0} \mid c+c'
\end{aligned}
$$

$\mathbf{1} \rightsquigarrow$ 

$e*f \rightsquigarrow$ 

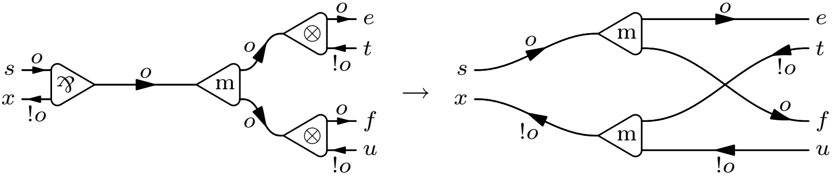
Routing $\rightsquigarrow$ sums.

Reduction



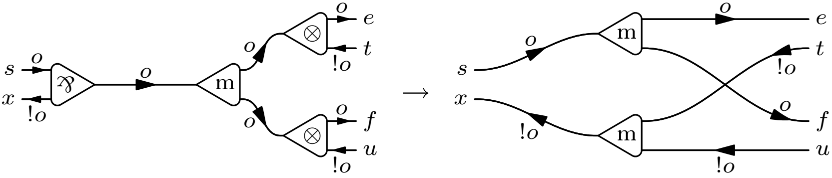
$$\langle \lambda x\, s\,,\, (t \cdot e) * (u \cdot f) \rangle$$

Reduction



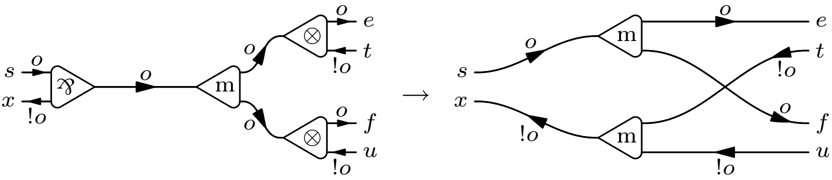
$$\langle \lambda x\, s\,,\, (t \cdot e) * (u \cdot f) \rangle$$

Reduction



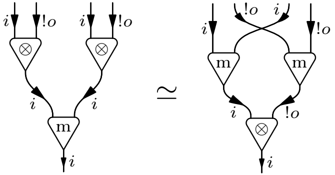
$$\langle \lambda x\, s\, ,\, (t \cdot e) * (u \cdot f) \rangle \rightarrow \langle s\,[t + u/x]\, ,\, e * f \rangle$$

Reduction



$$\langle \lambda x\, s \,,\, (t \cdot e) * (u \cdot f) \rangle \;\rightarrow\; \langle s\,[t + u/x] \,,\, e * f \rangle$$

$(u{\cdot}e){*}(v{\cdot}f) \simeq (t{+}u){\cdot}(e{*}f)$ cf.



Reduction

$$
\begin{aligned}
\langle \mu\alpha\, c \,,\, e \rangle &\rightarrow c\,[e/\alpha] \\
\langle \lambda x\, s \,,\, (t_1 \cdot e_1) * \cdots * (t_k \cdot e_k) \rangle &\rightarrow \langle s\,[t_1 + \cdots + t_k / x] \,,\, e_1 * \cdots * e_k \rangle
\end{aligned}
$$

Reduction

$$
\begin{aligned}
\langle \mu\alpha\, c \,,\, e\rangle &\rightarrow c\,[e/\alpha] \\
\langle \lambda x\, s \,,\, (t \cdot e) * f\rangle &\rightarrow \langle \lambda y\, \mu\beta\, \langle s\,[y + t/x] \,,\, e * \beta\rangle \,,\, f\rangle \\
\langle \lambda x\, s \,,\, \mathbf{1}\rangle &\rightarrow \langle s\,[\mathbf{0}/x] \,,\, \mathbf{1}\rangle
\end{aligned}
$$

Reduction

$$\begin{array}{rcl} \langle \mu\alpha\, c\,,\, e\rangle & \rightarrow & c\,[e/\alpha] \\ \langle \lambda x\, s\,,\, (t \cdot e) * f\rangle & \rightarrow & \langle \lambda y\, \mu\beta\, \langle s\,[y + t/x]\,,\, e * \beta\rangle\,,\, f\rangle \\ \langle \lambda x\, s\,,\, \mathbf{1}\rangle & \rightarrow & \langle s\,[\mathbf{0}/x]\,,\, \mathbf{1}\rangle \end{array}$$

The same, modulo:

$$\langle s\,,\, e * f\rangle \leftarrow_{\eta'} \langle \lambda x\, \mu\alpha\, \langle s\,,\, e * (x \cdot \alpha)\rangle\,,\, f\rangle\,.$$

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De-synchronisation

Recall that: $(u \cdot e) * (v \cdot f) \simeq (t + u) \cdot (e * f)$

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$$\begin{aligned} s^! &= s \cdot \mathbf{1} \\ \uparrow e &= \mathbf{0} \cdot e \end{aligned}$$

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Then: $s \cdot e \simeq s^! * \uparrow e$

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Then: $s \cdot e \simeq s^! * \uparrow e$

$$\begin{aligned} \left\langle \lambda x\, s \,,\, t^! * f \right\rangle &\to \langle \lambda x\, s\,[x + t/x] \,,\, f \rangle \\ \langle \lambda x\, s \,,\, \uparrow e * f \rangle &\to \langle \lambda x\, \mu\alpha\, \langle s \,,\, e * \alpha \rangle \,,\, f \rangle\,. \end{aligned}$$

De-synchronisation

Recall that: $(u \cdot e) * (v \cdot f) \simeq (t + u) \cdot (e * f)$

Let:

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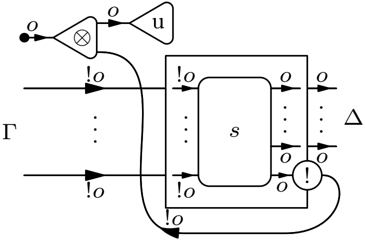
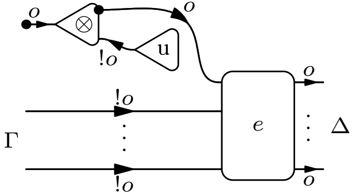
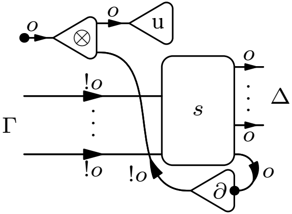
Then: $s \cdot e \simeq s^{!} * \uparrow e$





Differential $\bar{\lambda}\mu$-calculus

Linear argument for derivative: $[s]$



Reduction

$$
\begin{array}{rcl}
\langle \lambda x\, s\,,\, \mathbf{1} \rangle & \rightarrow & \langle s\,[\mathbf{0}/x]\,,\, \mathbf{1} \rangle \\
\left\langle \lambda x\, s\,,\, t^{!} * f \right\rangle & \rightarrow & \langle \lambda x\, s\,[t + x/x]\,,\, f \rangle \\
\langle \lambda x\, s\,,\, \uparrow e * f \rangle & \rightarrow & \langle \lambda x\, \mu\alpha\, \langle s\,,\, e * \alpha \rangle\,,\, f \rangle \\
\langle \lambda x\, s\,,\, [t] * f \rangle & \rightarrow & \left\langle \lambda x \left(\dfrac{\partial\, s}{\partial\, x} \cdot t \right),\, f \right\rangle \\
\langle \mu\alpha\, c\,,\, e \rangle & \rightarrow & \langle c\,,\, e \rangle_{\alpha}
\end{array}
$$

Linear substitution *aka.* partial derivative

Cf. differential λ-calculus *Ehrhard–Regnier*, 2003

$$\frac{\partial\, y}{\partial\, x}\cdot t = \delta_{x,y} t \qquad \frac{\partial\, \alpha}{\partial\, x}\cdot t = \mathbf{0}$$

$$\frac{\partial\, \lambda y\, s}{\partial\, x}\cdot t = \lambda y\left(\frac{\partial\, s}{\partial\, x}\cdot t\right) \qquad \frac{\partial\,[s]}{\partial\, x}\cdot t = \left[\frac{\partial\, s}{\partial\, x}\cdot t\right]$$

$$\frac{\partial\, \mu\alpha\, c}{\partial\, x}\cdot t = \mu\alpha\left(\frac{\partial\, c}{\partial\, x}\cdot t\right) \qquad \frac{\partial\, s^!}{\partial\, x}\cdot t = s^! * \left[\frac{\partial\, S}{\partial\, x}\cdot t\right]$$

$$\frac{\partial\, \langle s\,,\, e\rangle}{\partial\, x}\cdot t = \left\langle \frac{\partial\, s}{\partial\, x}\cdot t\,,\, e\right\rangle + \left\langle s\,,\, \frac{\partial\, e}{\partial\, x}\cdot t\right\rangle \qquad \frac{\partial\, \uparrow e}{\partial\, x}\cdot t = \uparrow\left(\frac{\partial\, e}{\partial\, x}\cdot t\right)$$

$$\frac{\partial\, \mathbf{0}}{\partial\, x}\cdot t = \mathbf{0} \qquad \frac{\partial\, \mathbf{1}}{\partial\, x}\cdot t = \mathbf{0}$$

$$\frac{\partial\, \theta+\theta'}{\partial\, x}\cdot t = \frac{\partial\, \theta}{\partial\, x}\cdot t + \frac{\partial\, \theta'}{\partial\, x}\cdot t \qquad \frac{\partial(e * f)}{\partial\, x}\cdot t = \left(\frac{\partial\, e}{\partial\, x}\cdot t\right) * f + e * \left(\frac{\partial\, f}{\partial\, x}\cdot t\right)$$

Linear substitution *aka.* partial derivative

Cf. differential λ-calculus *Ehrhard–Regnier*, 2003

$$\frac{\partial\, y}{\partial\, x}\cdot t = \delta_{x,y}\, t \qquad \frac{\partial\, \alpha}{\partial\, x}\cdot t = \mathbf{0}$$

$$\frac{\partial\, \lambda y\, s}{\partial\, x}\cdot t = \lambda y\left(\frac{\partial\, s}{\partial\, x}\cdot t\right) \qquad \frac{\partial\, [s]}{\partial\, x}\cdot t = \left[\frac{\partial\, s}{\partial\, x}\cdot t\right]$$

$$\frac{\partial\, \mu\alpha\, c}{\partial\, x}\cdot t = \mu\alpha\left(\frac{\partial\, c}{\partial\, x}\cdot t\right) \qquad \frac{\partial\, s^!}{\partial\, x}\cdot t = s^! * \left[\frac{\partial\, S}{\partial\, x}\cdot t\right]$$

$$\frac{\partial\, \langle s\,,\, e\rangle}{\partial\, x}\cdot t = \left\langle \frac{\partial\, s}{\partial\, x}\cdot t\,,\, e\right\rangle + \left\langle s\,,\, \frac{\partial\, e}{\partial\, x}\cdot t\right\rangle \qquad \frac{\partial\, {\uparrow} e}{\partial\, x}\cdot t = {\uparrow}\left(\frac{\partial\, e}{\partial\, x}\cdot t\right)$$

$$\frac{\partial\, \mathbf{0}}{\partial\, x}\cdot t = \mathbf{0} \qquad \frac{\partial\, \mathbf{1}}{\partial\, x}\cdot t = \mathbf{0}$$

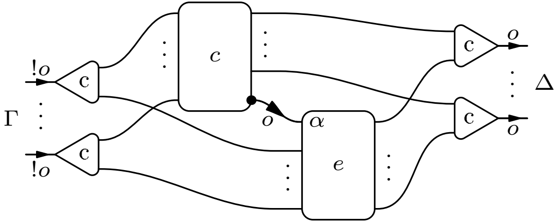
$$\frac{\partial\, \theta + \theta'}{\partial\, x}\cdot t = \frac{\partial\, \theta}{\partial\, x}\cdot t + \frac{\partial\, \theta'}{\partial\, x}\cdot t \qquad \frac{\partial (e * f)}{\partial\, x}\cdot t = \left(\frac{\partial\, e}{\partial\, x}\cdot t\right) * f + e * \left(\frac{\partial\, f}{\partial\, x}\cdot t\right)$$

Reduction

$$
\begin{array}{rcl}
\langle \lambda x\, s\,,\, \mathbf{1} \rangle & \rightarrow & \langle s\, [\mathbf{0}/x]\,,\, \mathbf{1} \rangle \\
\left\langle \lambda x\, s\,,\, t^{!} * f \right\rangle & \rightarrow & \langle \lambda x\, s\, [t + x/x]\,,\, f \rangle \\
\langle \lambda x\, s\,,\, \uparrow e * f \rangle & \rightarrow & \langle \lambda x\, \mu\alpha\, \langle s\,,\, e * \alpha \rangle\,,\, f \rangle \\
\langle \lambda x\, s\,,\, [t] * f \rangle & \rightarrow & \left\langle \lambda x \left(\frac{\partial\, s}{\partial\, x} \cdot t \right),\, f \right\rangle \\
\langle \mu\alpha\, c\,,\, e \rangle & \rightarrow & \langle c\,,\, e \rangle_{\alpha}
\end{array}
$$

Named cut

$\langle c \,,\, e \rangle_\alpha \rightsquigarrow$



Reduction

$$
\begin{array}{rcl}
\langle \lambda x\, s\, ,\, \mathbf{1} \rangle & \to & \langle s\, [\mathbf{0}/x]\, ,\, \mathbf{1} \rangle \\
\left\langle \lambda x\, s\, ,\, t^{!} * f \right\rangle & \to & \langle \lambda x\, s\, [t + x/x]\, ,\, f \rangle \\
\langle \lambda x\, s\, ,\, \uparrow e * f \rangle & \to & \langle \lambda x\, \mu\alpha\, \langle s\, ,\, e * \alpha \rangle\, ,\, f \rangle \\
\langle \lambda x\, s\, ,\, [t] * f \rangle & \to & \left\langle \lambda x \left(\dfrac{\partial\, s}{\partial\, x} \cdot t \right) ,\, f \right\rangle \\
\langle \mu\alpha\, c\, ,\, \beta \rangle & \to & c\, [\beta/\alpha] \\
\left\langle \mu\alpha\, c\, ,\, s^{!} \right\rangle & \to & c\, \left[s^{!}/\alpha \right] \\
\langle \mu\alpha\, c\, ,\, \mathbf{1} \rangle & \to & c\, [\mathbf{1}/\alpha] \\
\langle \mu\alpha\, c\, ,\, \uparrow e \rangle & \to & \langle \mu\alpha\, c\, [\uparrow\alpha\, /\alpha]\, ,\, e \rangle \\
\langle \mu\alpha\, c\, ,\, (e * f) \rangle & \to & \langle \mu\alpha''\, \langle \mu\alpha'\, c\, [\alpha' * \alpha''/\alpha]\, ,\, e \rangle\, ,\, f \rangle \\
\langle \mu\alpha\, c\, ,\, [s] \rangle & \to & \left(\dfrac{\partial\, c}{\partial\, \alpha} \cdot [s] \right) [\mathbf{1}/\alpha]
\end{array}
$$

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Outcome

Polarized differential logic

- Proper extension of both differential and polarized linear logics
- Preserves the symmetries of differential linear logic
- Breaks the intuitionnistic flavour of polarization (cf. paper)
- Non uniform semantics in the relational model (cf. paper)

Differential $\bar{\lambda}\mu$-calculus

- Polarized costructural rules = [Vaux, TLCA 2007]
- Stacks as “bags of resources”: breaks sequentiality between arguments
- *A good playground to explore concurrency in a functional, logically founded setting.*

Towards concurrency

Resource λ-calculus

Boudol, CONCUR 1993

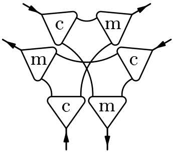
$$\begin{array}{rcl} s & ::= & x \mid \lambda x\, s \mid (s)\, S \\ S & ::= & \mathbf{1} \mid S * S \mid [s] \mid s^! \end{array}$$

As discriminative as Milner's π-calculus w.r.t. λ-calculus
Boudol–Laneve, 1995

Translation: $(s)\, S \rightsquigarrow \mu\alpha\, \langle s\,,\, S \cdot \alpha \rangle$

DiLL as a model of concurrency

Ehrhard–Laurent, CONCUR 2007



- Costructural rules are crucial.
- Also makes use of *non sequential* nets. How much is that essential ?
- What is the concurrent expressivity of the diff. $\bar{\lambda}\mu$-calculus (fragment of π-calculus) ?
- What about concurrent semantics (event structures, cf. recent work by Mazza) ?

A word on the Taylor expansion

The finitary fragment of the resource λ-calculus encodes the pure λ-calculus *Ehrhard–Regnier*, 2006

Similarly

$$s^{!} \simeq \sum_{n=0}^{\infty} \frac{1}{n!} [s]^n$$

To be related with Carvalho's study of LL promotion in models of DiLL... *de Carvalho*, PhD, 2007

...and recent work by Tasson and Pagani demonstrating an inverse Taylor expansion in DiLL *Tasson–Pagani*, LICS, 2009

Can this bring new light on the properties of replication in process algebras?

Thank you

for your attention

e grazie a Michele

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