A Non-Uniform Finitary Relational Semantics of System T

Iteration and Recursion in Relational Finiteness Spaces

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Typed λ-calculi

Terms

$$s, t ::= x \mid a \mid \lambda x\, s \mid s\, t$$

Typing

$$\frac{}{\Delta, x : A \vdash x : A}\ \text{(Var)} \qquad \frac{a \in \mathfrak{C}_A}{\Delta \vdash a : A}\ \text{(Const)}$$

$$\frac{\Delta, x : A \vdash s : B}{\Delta \vdash \lambda x\, s : A \to B}\ \text{(Abs)} \qquad \frac{\Delta \vdash s : A \to B \qquad \Delta \vdash t : A}{\Delta \vdash s\, t : B}\ \text{(App)}$$

Conversions

$$(\lambda x\, s)\, t \to s\,[x := t],\ \ldots$$

We could add products and unit type with the corresponding term constructions and conversions.

A category of sets and multi-relations

Definition (The category $\underline{\mathrm{Rel}}$)

Objects: Sets

Morphisms: $\underline{\mathrm{Rel}}(A, B) = \mathfrak{P}\left(A^! \times B\right)$ $\quad (A^! ::= \mathcal{M}_{\mathsf{fin}}(A))$

Identities: $\mathrm{id}_A = \{([\alpha], \alpha);\ \alpha \in A\}$

Composition: If $f \in \underline{\mathrm{Rel}}(A, B)$ and $g \in \underline{\mathrm{Rel}}(B, C)$,

$$g \circ f = \left\{\left(\sum_{i=1}^{n} \overline{\alpha}_i, \gamma\right);\ \exists([\beta_1, \ldots, \beta_n], \gamma) \in g \wedge \forall i,\ (\overline{\alpha}_i, \beta_i) \in f\right\}$$

Intuition: resources

$([\alpha_1, \ldots, \alpha_n], \beta) \in f$ means:

f *can produce result* β *consuming data* $\alpha_1, \ldots, \alpha_n$.

A category of sets and multi-relations

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Objects: Sets

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Folklore

$\underline{\mathrm{Rel}}$ is cartesian closed (with product the disjoint sum of sets).

$\underline{\mathrm{Rel}}$ *can be constructed as the co-Kleisly of the co-monad* $-^!$ *in the category* $\underline{\mathrm{Rel}_0}$ *of sets and relations, which is a model of linear logic.*

The relational model of typed λ-calculi

On types

Fix $[\![X]\!]$ any set. Then $[\![A \to B]\!] = [\![A]\!]^! \times [\![B]\!]$.

On terms

If $x_1 : A_1, \ldots, x_n : A_n \vdash s : A$ we (will soon) define $[\![s]\!] \subseteq {A_1}^! \times \cdots \times {A_n}^! \times A$.

Notation

Write $x_1^{\overline{\alpha}_1} : A_1, \ldots, x_n^{\overline{\alpha}_n} : A_n \vdash s^\alpha : A$ for $(\overline{\alpha}_1, \ldots, \overline{\alpha}_n, \alpha) \in [\![s]\!]$ so that

$$[\![s]\!] = \left\{(\overline{\alpha}_1, \ldots, \overline{\alpha}_n, \alpha);\ x_1^{\overline{\alpha}_1} : A_1, \ldots, x_n^{\overline{\alpha}_n} : A_n \vdash s^\alpha : A\right\}$$

Cf. experiments in linear logic proof nets.

Computing points in the relational model

$$\frac{}{\Delta^{\square}, x^{[\alpha]} : A \vdash x^{\alpha} : A}\,[\![\mathrm{Var}]\!] \qquad \frac{a \in \mathfrak{C}_A \quad \alpha \in [\![a]\!]}{\Delta^{\square} \vdash a^{\alpha} : A}\,[\![\mathrm{Const}]\!]$$

$$\frac{\Gamma, x^{\overline{\alpha}} : A \vdash s^{\beta} : B}{\Gamma \vdash \lambda x\, s^{(\overline{\alpha},\beta)} : A \to B}\,[\![\mathrm{Abs}]\!]$$

$$\frac{\Gamma_0 \vdash s^{([\alpha_1,\ldots,\alpha_k],\beta)} : A \to B \qquad \Gamma_1 \vdash t^{\alpha_1} : A \quad \cdots \quad \Gamma_k \vdash t^{\alpha_k} : A}{\sum_{j=0}^{k} \Gamma_j \vdash s\, t^{\beta} : B}\,[\![\mathrm{App}]\!]$$

Notations

$$\Delta^{\square} = x_1^{\square} : A_1, \ldots, x_n^{\square} : A_n$$

$$\Gamma + \Gamma' = x_1^{\overline{\alpha}_1 + \overline{\alpha}'_1} : A_1, \ldots, x_n^{\overline{\alpha}_n + \overline{\alpha}'_n} : A_n$$

Computing points in the relational model

$$\frac{}{\Delta^{[]}, x^{[\alpha]} : A \vdash x^{\alpha} : A}\ [\![\mathrm{Var}]\!] \qquad \frac{a \in \mathfrak{C}_A \quad \alpha \in [\![a]\!]}{\Delta^{[]} \vdash a^{\alpha} : A}\ [\![\mathrm{Const}]\!]$$

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Examples

$$\left[\!\!\left[\lambda x^A\, x\right]\!\!\right] = \mathrm{id}_A = \{([\alpha], \alpha);\ \alpha \in A\}$$

$$\left[\!\!\left[\lambda x^A\, \lambda y^B\, x\right]\!\!\right] = \{([\alpha], [], \alpha);\ \alpha \in A\}$$

Computing points in the relational model

$$\frac{}{\Delta^{[]}, x^{[\alpha]} : A \vdash x^{\alpha} : A}\ [\![\mathrm{Var}]\!] \qquad \frac{a \in \mathfrak{C}_A \quad \alpha \in [\![a]\!]}{\Delta^{[]} \vdash a^{\alpha} : A}\ [\![\mathrm{Const}]\!]$$

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Examples

$$\left[\!\!\left[\lambda x^{A \to A}\, \lambda y^{A}\, (x\,(x\,y))\right]\!\!\right] = \left\{ \left([([\alpha_1, \ldots, \alpha_n], \alpha), (\overline{\alpha}_1, \alpha_1), \ldots, (\overline{\alpha}_n, \alpha_n)], \sum_{i=1}^{n} \overline{\alpha}_i, \alpha \right);\ \ldots \right\}$$

Computing points in the relational model

$$\frac{}{\Delta^{[]}, x^{[\alpha]} : A \vdash x^{\alpha} : A} \llbracket \mathrm{Var} \rrbracket \qquad \frac{a \in \mathfrak{C}_A \quad \alpha \in \llbracket a \rrbracket}{\Delta^{[]} \vdash a^{\alpha} : A} \llbracket \mathrm{Const} \rrbracket$$

$$\frac{\Gamma, x^{\overline{\alpha}} : A \vdash s^{\beta} : B}{\Gamma \vdash \lambda x\, s^{(\overline{\alpha},\beta)} : A \to B} \llbracket \mathrm{Abs} \rrbracket$$

$$\frac{\Gamma_0 \vdash s^{([\alpha_1,\dots,\alpha_k],\beta)} : A \to B \qquad \Gamma_1 \vdash t^{\alpha_1} : A \quad \cdots \quad \Gamma_k \vdash t^{\alpha_k} : A}{\sum_{j=0}^{k} \Gamma_j \vdash s\, t^{\beta} : B} \llbracket \mathrm{App} \rrbracket$$

Invariance

If $s \to t$ then $\llbracket s \rrbracket = \llbracket t \rrbracket$.

Features

Fixpoints

Straightforward: $\mu f = \bigcup_{k \geq 0} f^k \cdot \emptyset^!$.

Least fixpoint operator: $\mathcal{F}ix_A = \bigcup_{k \geq 0} \mathcal{F}ix_A^{(k)}$, where $\mathcal{F}ix_A^{(0)} = \emptyset$ and $\mathcal{F}ix_A^{(k+1)} = \left[\!\left[\lambda f \left(f \left(\mathcal{F}ix_A^{(k)}\, f \right) \right) \right]\!\right] =$

$$\left\{ \left([([\alpha_1, \ldots, \alpha_n], \alpha)] + \textstyle\sum_{i=1}^n \overline{\varphi}_i, \alpha \right);\ \forall i,\ (\overline{\varphi}_i, \alpha_i) \in \mathcal{F}ix_A^{(k)} \right\}.$$

Reflexive objects

Fix $\mathcal{D}_0$ and take $\mathcal{D} = \bigcup_n \mathcal{D}_n$ with $\mathcal{D}_{n+1} = \mathcal{D}_n \cup (\mathcal{D}_n{}^! \times \mathcal{D}_n)$.

Extensional variant: $\left\{ \left(\overline{\delta}_i \right)_{i \in \mathbf{N}} \in \mathcal{D}_n^{\mathbf{N}};\ \overline{\delta}_i = []\ \text{ for almost all } i \right\}$.

Non uniformity

Models some form of intrinsic non-determinism: introduce term constructs 0 and $s + t$ and $[\![0]\!] = \emptyset$ and $[\![s + t]\!] = [\![s]\!] \cup [\![t]\!]$.

Features

Resource

Models Ehrhard–Regnier's differential λ-calculus with
$[\![Ds \cdot t]\!] = \{(\overline{\alpha}, \beta);\ (\overline{\alpha} + [\alpha'], \beta) \in [\![s]\!] \wedge \alpha' \in [\![t]\!]\}$.

Implicit complexity

The size of the semantics is related with execution time in abstract machines (Carvalho, 2007).

An ubiquitous concept

Underlies all web based denotational semantics: coherence, correllation, probabilistic coherence, finiteness...
Very similar to intersection type systems (Carvalho, 2007).

Related with domains à la Scott via an extensional collapse (Ehrhard, 2009).

What about data types ?

System T

Add a base type Nat, with $\mathsf{O} : \mathsf{Nat}$, $\mathsf{S} : \mathsf{Nat} \to \mathsf{Nat}$, and one of:

Iterator

$\mathsf{I}_A : \mathsf{Nat} \to (A \to A) \to A \to A$ with conversions

$$\mathsf{I}\,(\mathsf{O})\,u\,v \to v \qquad \mathsf{I}\,(\mathsf{S}\,\mathrm{t})\,u\,v \to u\,(\mathsf{I}\,\mathrm{t}\,u\,v)$$

Or the tail-recursive variant.

Recursor

R_A of type $\mathsf{Nat} \to (\mathsf{Nat} \to A \to A) \to A \to A$ with conversions

$$\mathsf{R}\,(\mathsf{O})\,u\,v \to v \qquad \mathsf{R}\,(\mathsf{S}\,\mathrm{t})\,u\,v \to u\,\mathrm{t}\,(\mathsf{R}\,\mathrm{t}\,u\,v)$$

Notice $\mathsf{I} \simeq \lambda x\,\lambda y\,\lambda z\,(\mathsf{R}\,x\,(\lambda x'\,y)\,z)$.

Using products, one can recover R *from* I*, but only by values.*

A naive interpretation

We are tempted to set:

$$[\![\mathsf{Nat}]\!] = \mathbf{N} \quad [\![\mathsf{O}]\!] = \{0\} \quad [\![\mathsf{S}]\!] = \{([n]\,, n+1);\ n \in \mathbf{N}\}$$

But this would fail: because the successor is linear,

$$\begin{aligned} [\![\mathsf{I}\,(\mathsf{S}\,x)\ (\lambda y'\,y)\ z]\!]_{x:\mathsf{Nat},y:A,z:A} &= [\![y]\!]_{x:\mathsf{Nat},y:A,z:A} \\ &= \{(\square\,, [\alpha]\,, \square\,, \alpha);\ \alpha \in A\} \end{aligned}$$

enforces:

$$[\![\mathsf{I}]\!] \supseteq \{(\square\,, [(\square\,, \alpha)]\,, \square\,, \alpha);\ \alpha \in A\}$$

hence:

$$[\![\mathsf{I}\,\mathsf{O}]\!] \supseteq \{([(\square\,, \alpha)]\,, \square\,, \alpha);\ \alpha \in A\}$$

which would contradict:

$$[\![\mathsf{I}\,\mathsf{O}]\!] = [\![\lambda y\,\lambda z\,z]\!] = \{(\square\,, [\alpha]\,, \alpha);\ \alpha \in |[\![A]\!]|\}$$

Lazy integers

The successor cannot be strict: without looking at x, we should know $\mathsf{S}\,x$ is greater than O.

Greater than...

Take a new copy of Nat: $\mathsf{Nat}^{>} = \{n^{>};\ n \in \mathsf{Nat}\}$.
$n^{>}$ stands for an undetermined integer, greater than n.

Interpretation

Let

$$\begin{aligned} [\![\mathsf{Nat}]\!] &= \mathbf{N} \cup \mathbf{N}^{>} \\ [\![\mathsf{O}]\!] &= \{0\} \\ [\![\mathsf{S}]\!] &= \{([\,], 0^{>})\} \cup \left\{([\nu], \nu^{+});\ \nu \in [\![\mathsf{Nat}]\!]\right\} \end{aligned}$$

with $n^{+} = n + 1$ and $(n^{>})^{+} = (n+1)^{>}$.

Pattern matching lazy natural numbers

Syntax

A new constant $\mathsf{C} : \mathsf{Nat} \to (\mathsf{Nat} \to A) \to A \to A$
with conversions

$$\mathsf{C}\,\mathsf{O}\,t\,u \to u \text{ and } \mathsf{C}\,(\mathsf{S}\,s)\,t\,u \to t\,s$$

Idea:

$$\mathsf{C}\ s\,t\,u = \mathrm{match}\ s\ \mathrm{with}\ \left\{ \begin{array}{ccc} \mathsf{O} & \mapsto & u \\ \mathsf{S}\,s' & \mapsto & t\,s' \end{array} \right.$$

Semantics

$$\begin{array}{rcl} [\![\mathsf{C}]\!] & = & \{([0]\,,[]\,,[\alpha]\,,\alpha);\ \alpha \in |\mathcal{A}|\} \\ & \cup & \{([0^{>}] + \overline{\nu}^{+}, [(\overline{\nu},\alpha)]\,,[]\,,\alpha);\ \overline{\nu} \in [\![\mathsf{Nat}]\!] \wedge \alpha \in A\} \end{array}$$

Interpretation of system T

Recursor

Recall: $\mathsf{R}\,(\mathsf{O})\,u\,v \to v \qquad \mathsf{R}\,(\mathsf{S}\,\mathsf{t})\,u\,v \to u\,\mathsf{t}\,(\mathsf{R}\,\mathsf{t}\,u\,v)$

Then set:

$$[\![\mathsf{R}]\!] = \mathcal{F}ix\left[\!\!\left[\lambda f\,\lambda x^{\mathsf{Nat}}\,\lambda y^{\mathsf{Nat}\to A\to A}\,\lambda z^{A}\left(\mathsf{C}\;x\left(\lambda x'^{A}\,(y\,x'\,(f\,x'\,y\,z))\right)z\right)\right]\!\!\right]$$

Iterator

Take:

$$[\![\mathsf{I}]\!] = \left[\!\!\left[\lambda x^{\mathsf{Nat}}\,\lambda y^{\mathsf{Nat}\to A\to A}\,\lambda z^{A}\,(\mathsf{R}\,x\,(\lambda x'\,y)\,z)\right]\!\!\right]$$

Theorem

$\underline{\mathrm{Rel}}$ *is a model of system* T.

A uniformity property of iteration

Definition

If $k \in \mathbf{N}$, $\underline{k} ::= \mathcal{S}_l^k \mathcal{O} = \{l^>;\ l < k\} \cup \{k\} \in \mathfrak{F}(\mathcal{N}_l)$.
Say $n \in \mathfrak{F}(\mathcal{N}_l)$ is uniform if $n \subseteq \underline{k}$ for some k.

Lemma

$[\![\mathrm{I}]\!] = \bigcup_{k\geq 0} \mathcal{I}^{(k)}$, *where* $\mathcal{I}^{(0)} = \{([0]\,, []\,, [\alpha]\,, \alpha);\ \alpha \in |\mathcal{A}|\}$ *and*
$\mathcal{I}^{(k+1)} = \mathcal{I}^{(0)} \cup$

$$\left\{ \begin{array}{r} \left([0^>] + \sum_{i=0}^{n} \overline{\nu}_i^+, [([\alpha_1, \ldots, \alpha_n]\,, \alpha)] + \sum_{i=1}^{n} \overline{\varphi}_i, \sum_{i=1}^{n} \overline{\alpha}_i, \alpha\right); \\ \forall i,\ (\overline{\nu}_i, \overline{\varphi}_i, \overline{\alpha}_i, \alpha_i) \in \mathcal{I}^{(k)} b \end{array} \right\}$$

Theorem

If $(\overline{\nu}, \overline{\varphi}, \overline{\alpha}, \alpha) \in \mathcal{I}^{(k)} \setminus \mathcal{I}^{(k-1)}$ $(\mathcal{I}^{(-1)} = \emptyset)$ *then* $\mathrm{Supp}(\nu) \subseteq \underline{k}$.

Polar construction: coherence spaces

For $a, a' \subseteq A$, write $a \perp a'$ when $\#(a \cap a') \leq 1$

Polar

If $\mathfrak{F} \subseteq \mathfrak{P}(A)$, let $\mathfrak{F}^\perp = \{a' \subseteq A;\ \forall a \in \mathfrak{F},\ a \perp a'\}$

For all $\mathfrak{F} \subseteq \mathfrak{P}(A)$:

- $\mathfrak{F} \subseteq \mathfrak{F}^{\perp\perp}$;
- if $\mathfrak{G} \subseteq \mathfrak{F}$, $\mathfrak{F}^\perp \subseteq \mathfrak{G}^\perp$;
- $\mathfrak{F}^\perp = \mathfrak{F}^{\perp\perp\perp}$.

A coherence on A is $\mathfrak{F}$ such that $\mathfrak{F}^{\perp\perp} = \mathfrak{F}$.

Definition

A coherence space is a pair $\mathcal{A} = (|\mathcal{A}|, \mathfrak{F}(\mathcal{A}))$
where $\mathfrak{F}(\mathcal{A})$ is a coherence on $|\mathcal{A}|$.
The elements of $\mathfrak{F}(\mathcal{A})$ are called the cliques of $\mathcal{A}$.

Polar construction

For $a, a' \subseteq A$, write $a \perp a'$ when …

Polar

If $\mathfrak{F} \subseteq \mathfrak{P}(A)$, let $\mathfrak{F}^{\perp} = \{a' \subseteq A;\ \forall a \in \mathfrak{F},\ a \perp a'\}$

For all $\mathfrak{F} \subseteq \mathfrak{P}(A)$:

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A … on A is $\mathfrak{F}$ such that $\mathfrak{F}^{\perp\perp} = \mathfrak{F}$.

Definition

A … is a pair $\mathcal{A} = (|\mathcal{A}|, \mathfrak{F}(\mathcal{A}))$
where $\mathfrak{F}(\mathcal{A})$ is a … on $|\mathcal{A}|$.
The elements of $\mathfrak{F}(\mathcal{A})$ are called the … of $\mathcal{A}$.

Polar construction: sets

For $a, a' \subseteq A$, write $a \perp a'$ always

Polar

If $\mathfrak{F} \subseteq \mathfrak{P}(A)$, let $\mathfrak{F}^{\perp} = \{a' \subseteq A;\ \forall a \in \mathfrak{F},\ a \perp a'\}$

For all $\mathfrak{F} \subseteq \mathfrak{P}(A)$:

- $\mathfrak{F} \subseteq \mathfrak{F}^{\perp\perp}$;
- if $\mathfrak{G} \subseteq \mathfrak{F}$, $\mathfrak{F}^{\perp} \subseteq \mathfrak{G}^{\perp}$;
- $\mathfrak{F}^{\perp} = \mathfrak{F}^{\perp\perp\perp}$.

The powerset of A is the only $\mathfrak{F}$ such that $\mathfrak{F}^{\perp\perp} = \mathfrak{F}$.

Definition

A set is a pair $\mathcal{A} = (|\mathcal{A}|, \mathfrak{F}(\mathcal{A}))$
where $\mathfrak{F}(\mathcal{A})$ is the powerset of $|\mathcal{A}|$.
The elements of $\mathfrak{F}(\mathcal{A})$ are called the subsets of $\mathcal{A}$.

Polar construction: finiteness spaces

For $a, a' \subseteq A$, write $a \perp a'$ when $a \cap a'$ is finite

Polar

If $\mathfrak{F} \subseteq \mathfrak{P}(A)$, let $\mathfrak{F}^{\perp} = \{a' \subseteq A;\ \forall a \in \mathfrak{F},\ a \perp a'\}$

For all $\mathfrak{F} \subseteq \mathfrak{P}(A)$:

- $\mathfrak{F} \subseteq \mathfrak{F}^{\perp\perp}$;
- if $\mathfrak{G} \subseteq \mathfrak{F}$, $\mathfrak{F}^{\perp} \subseteq \mathfrak{G}^{\perp}$;
- $\mathfrak{F}^{\perp} = \mathfrak{F}^{\perp\perp\perp}$.

A finiteness structure on A is $\mathfrak{F}$ such that $\mathfrak{F}^{\perp\perp} = \mathfrak{F}$.

Definition

A finiteness space is a pair $\mathcal{A} = (|\mathcal{A}|, \mathfrak{F}(\mathcal{A}))$
where $\mathfrak{F}(\mathcal{A})$ is finiteness structure on $|\mathcal{A}|$.
The elements of $\mathfrak{F}(\mathcal{A})$ are called the finitary subsets of $\mathcal{A}$.

A category of finiteness spaces

Finitary multi-relations

If $\mathcal{A}$ and $\mathcal{B}$ are finiteness spaces, define $\mathcal{A} \Rightarrow \mathcal{B}$ by $|\mathcal{A} \Rightarrow \mathcal{B}| = |\mathcal{A}|^{!} \times |\mathcal{B}|$ and $f \in \mathfrak{F}(\mathcal{A} \Rightarrow \mathcal{B})$ iff

$$\begin{array}{rrl} \forall a \in \mathfrak{F}(\mathcal{A}), & & f \cdot a^{!} \in \mathfrak{F}(\mathcal{B}) \\ & \text{and} & \forall \beta \in |\mathcal{B}|,\ (f^{\perp} \cdot \{\beta\}) \cap a^{!} \text{ is finite} \end{array}$$

Theorem

Finitary multi-relations compose and $\mathrm{id}_{|\mathcal{A}|} \in \mathfrak{F}(\mathcal{A} \Rightarrow \mathcal{A})$*. Hence we can define the category* $\underline{\mathrm{Fin}}$ *of finiteness spaces and finitary multi-relations.*

A finiteness property of the relational model

Theorem

$\underline{\mathrm{Fin}}$ *is cartesian closed, with the same constructions as* $\underline{\mathrm{Rel}}$*, with product* $\mathcal{A} \times \mathcal{B}$ *such that* $|\mathcal{A} \times \mathcal{B}| = |\mathcal{A}| \uplus |\mathcal{B}|$ *and* $\mathfrak{F}(\mathcal{A} \times \mathcal{B}) = \{a \uplus b;\ a \in \mathfrak{F}(\mathcal{A}) \wedge b \in \mathfrak{F}(\mathcal{B})\}$.

$\underline{\mathrm{Fin}}$ *is also the co-Kleisly of a co-monad* $!$ *in the category* $\underline{\mathrm{Fin}_0}$ *of finiteness spaces and finitary relations:*

$$f \in \underline{\mathrm{Fin}_0}(\mathcal{A}, \mathcal{B}) \textit{ iff } \forall a \in \mathfrak{F}(\mathcal{A}),\ f \cdot a \in \mathfrak{F}(\mathcal{B}) \textit{ and } \forall b \in \mathfrak{F}\left(\mathcal{B}^{\perp}\right),\ f^{\perp} \cdot b \in \mathfrak{F}\left(\mathcal{A}^{\perp}\right);$$

$$|!\mathcal{A}| = |\mathcal{A}|^{!} \textit{ and } \mathfrak{F}(!\mathcal{A}) = \left\{u;\ \exists a \in \mathfrak{F}(\mathcal{A}),\ u \subseteq a^{!}\right\}.$$

Interpretation of pure typed λ-calculus

For all base type A, fix $\mathfrak{F}(A)$ a finiteness structure on $[\![A]\!]$. Then set $\mathfrak{F}(A \to B) = ([\![A]\!], \mathfrak{F}(A)) \Rightarrow ([\![B]\!], \mathfrak{F}(B))$.

Theorem

Assume, for all $a \in \mathfrak{C}_A$, $a \in \mathfrak{F}(A)$. *If* $x_1 : A_1, \dots, x_n : A_n \vdash s : A$ *then* $[\![s]\!] \in \mathfrak{F}(A_1 \to \cdots \to A_n \to A)$.

Finiteness of system T

Finitary lazy integers

Let $n \in \mathfrak{F}(\mathsf{Nat})$ iff n is finite.
Then $[\![\mathsf{O}]\!] = \{0\} \in \mathfrak{F}(\mathsf{Nat})$ and
$[\![\mathsf{S}]\!] = \{([\,], 0^{>})\} \cup \{([\nu], \nu^{+});\ \nu \in [\![\mathsf{Nat}]\!]\} \in \mathfrak{F}(\mathsf{Nat} \to \mathsf{Nat})$.

Theorem

We also have: $[\![\mathsf{I}]\!] \in \mathfrak{F}(\mathsf{Nat} \to (A \to A) \to A \to A)$ *and* $[\![\mathsf{R}]\!] \in \mathfrak{F}(\mathsf{Nat} \to (\mathsf{Nat} \to A \to A) \to A \to A)$.

Not immediate because, in general, $\mathcal{F}ix_{|\mathcal{A}|}$ is not finitary in $(\mathcal{A} \Rightarrow \mathcal{A}) \Rightarrow \mathcal{A}$. For instance $\mathcal{F}ix_{[\![\mathsf{Nat}]\!]}\,[\![\mathsf{S}]\!] = \mathsf{Nat}^{>}$.

Proof (i) *the image of a finitary subset is finitary*

Notation

Write $[\![\mathsf{R}]\!] = \bigcup_{k\geq 0} \mathcal{R}^{(k)}$, where $\mathcal{R}^{(0)} = \{([0], [\,], [\alpha], \alpha);\ \alpha \in [\![A]\!]\}$ and $\mathcal{R}^{(k)} = \Phi^k\left(\mathcal{R}^{(0)}\right)$ with $\Phi = \ldots$

For all k, $\mathcal{R}^{(k)} \in \mathfrak{F}(\mathsf{Nat} \to (\mathsf{Nat} \to A \to A) \to A \to A)$.

Lemma

For all $\gamma = (\overline{\nu}, \overline{\varphi}, \overline{\alpha}, \alpha) \in [\![\mathsf{R}]\!]$, $\gamma \in \mathcal{R}^{(\max(\overline{\nu}))}$.

Corollary

If $n \in \mathfrak{F}(\mathsf{Nat})$, *then* $[\![\mathsf{R}]\!] \cdot n^! \subseteq \mathcal{R}^{(\max(n))} \cdot n^!$ *which is finitary.*

Proof (ii) *the preimage of a singleton is antifinitary*

Notation

$\#([\alpha_1, \ldots, \alpha_k]) ::= k$

$\#\#([(\overline{\nu}_1, \overline{\alpha}_1, \alpha_1), \ldots, (\overline{\nu}_k, \overline{\alpha}_k, \alpha_k)]) ::= \sum_{j=1}^{k} \#(\overline{\nu}_j)$.

Lemma

For all $\gamma = (\overline{\nu}, \overline{\varphi}, \overline{\alpha}, \alpha) \in [\![R]\!]$, $\#(\nu) = \#(\overline{\alpha}) + \#(\overline{\varphi}) + \#\#(\overline{\varphi})$.

Corollary

For all $n \in \mathfrak{F}([\![\mathrm{Nat}]\!])$, *and all* $\delta = (\overline{\varphi}, \overline{\alpha}, \alpha)$,

$\left([\![R]\!]^{\perp} \cdot \{\delta\}\right) \cap n^{!} = \left\{\overline{\nu} \in n^{!};\ \#(\nu) = \#(\overline{\alpha}) + \#(\overline{\varphi}) + \#\#(\overline{\varphi})\right\}$

is finite.

Quantitative semantics

Idea

Interpret s by a linear combination: $(\!|s|\!) = \sum_{\alpha \in [\![s]\!]} (\!|s|\!)_\alpha \alpha$
so that application is given by:

$$(\!|s\ t|\!) = \sum_{(\overline{\alpha}, \beta) \in [\![s]\!]} (\!|s|\!)_{(\overline{\alpha},\beta)} (\!|t|\!)^{\overline{\alpha}} \beta$$

where $(\!|t|\!)^{[\alpha_1,\dots,\alpha_k]} ::= (\!|t|\!)_{\alpha_1} \cdots (\!|t|\!)_{\alpha_k}$.

We need some notion of convergence !

In finiteness spaces

Because $[\![s]\!] \in \mathfrak{F}(A \Rightarrow B)$ and $[\![t]\!] \in \mathfrak{F}(A)$, we have: for all $\beta \in [\![B]\!]$, there are finitely many $\overline{\alpha} \in [\![t]\!]^!$ such that $(\overline{\alpha}, \beta) \in [\![s]\!]$.

Such a model grounded the introduction of differential λ-calculus.

Towards a quantitative semantics of system T

Roadmap

- Find a correct quantitative semantics of C (easy).
- See that it defines a quantitative semantics of R and I (this is not automatic, because fixpoints are not finitary).

A differential system T

What's the point ?
Maybe interesting expressivity results.
Think of a language where you can define

$$\lambda x^{\mathsf{Nat}\to\mathsf{Nat}}\,\lambda y^{\mathsf{Nat}}\,\lambda z^{\mathsf{Nat}}\,(\mathrm{D}x\cdot y)\,z.$$

Type fixpoints in finiteness spaces

Ongoing work with Christine Tasson (PPS, Paris).

Data types

The finitary relational interpretation should be feasible for arbitrary datatype, at least when expressed as fixpoints of positive functors.
This is not trivial, because we must generalize laziness.
Idea: use $\mu X\, F(1 \,\&\, X)$ rather than $\mu X\, F(X)$.

Good tempered morphisms

An interesting problem: characterize those finitary morphisms which admit finitary fixpoints (all of them, or at least a large class).
Recursors for arbitrary data types should belong there.

The end.