On the transport of finiteness structures

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mainly based on joint work with Christine Tasson:
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Sets and relations

Definition

The category $\mathbf{Rel}$ of sets and relations has sets as objects and relations as morphisms: $f \in \mathbf{Rel}(A, B) \iff f \subseteq A \times B$.

Relational composition is given by:

$$(\alpha, \gamma) \in g \circ f \iff \exists \beta,\ (\alpha, \beta) \in f \wedge (\beta, \gamma) \in g.$$

$\mathbf{Rel}$ as a model of linear logic

- compact closed: $\otimes = \times$ and $f^{\perp} = {}^{t}f$;
- cartesian and cocartesian: $\uplus$ is a biproduct;
- exponential structure: given by the comonad $! = \mathfrak{M}_{\mathrm{f}}$.

The relational model of the λ-calculus

Apply the co-Kliesli construction

$\mathbf{Rel}^{!}(A, B) = \mathbf{Rel}(!A, B)$ with composition given by:

$$g \circ^{!} f = \left\{ \left(\sum_{i=1}^{n} \overline{\alpha}_i, \gamma \right) ;\ \exists ([\beta_1, \ldots, \beta_n], \gamma) \in g \wedge \forall i,\ (\overline{\alpha}_i, \beta_i) \in f \right\}$$

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$\mathbf{Rel}^!$ is cartesian closed with product $\uplus$.

Moreover it is cpo-enriched (for set inclusion).

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A key intuition

Morphisms in $\mathbf{Rel}^!$ are the support of power series.

Quantitative semantics

Idea (Girard, pre-LL)

Interpret a term s as a linear combination: $(\!|s|\!) = \sum_{\alpha \in [\![s]\!]} (\!|s|\!)_\alpha \alpha$
so that application is given by:

$$(\!|s\,t|\!)_\beta = \sum_{(\overline{\alpha},\beta) \in [\![s]\!]} (\!|s|\!)_{(\overline{\alpha},\beta)} (\!|t|\!)^{\overline{\alpha}}$$

where $(\!|t|\!)^{[\alpha_1,\ldots,\alpha_k]} = (\!|t|\!)_{\alpha_1} \cdots (\!|t|\!)_{\alpha_k}$.

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Finiteness spaces (Ehrhard, 2000's)

In a typed setting, the sum is always finite.

Led to the introduction of the differential λ-calculus
(Ehrhard–Regnier, 2004):
differentiation as a natural transformation $A \otimes {!A} \multimap {!A}$.

Finiteness spaces

Short version

The category $\mathbf{Fin}$ of finiteness spaces is the tight orthogonality category (in the sense of Hyland–Schalk, 2003) obtained from $\mathbf{Rel}$ by setting:

$$a \perp_A a' \iff a \cap a' \in \mathfrak{P}_{\mathrm{f}}(A)$$

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More explicitly

- A finiteness space is a pair $(|\mathcal{A}|, \mathfrak{F}(\mathcal{A}))$ s.t. $|\mathcal{A}|$ is a set and $\mathfrak{F}(\mathcal{A}) = \mathfrak{F}(\mathcal{A})^{\perp\perp} \subseteq \mathfrak{P}(|\mathcal{A}|)$.
- A finitary relation $f \in \mathbf{Fin}(\mathcal{A}, \mathcal{B})$ is a relation $f \in \mathbf{Rel}(|\mathcal{A}|, |\mathcal{B}|)$ s.t.:
  - $a \in \mathfrak{F}(\mathcal{A})$ implies $f \cdot a \in \mathfrak{F}(\mathcal{B})$;
  - $b' \in \mathfrak{F}(\mathcal{B}^{\perp})$ implies ${}^t f \cdot b' \in \mathfrak{F}(\mathcal{A}^{\perp})$.

Finiteness spaces as a model of linear logic

Short version

All the constructions for multiplicative, additive and exponential structure work out as described by Hyland and Schalk.

Moreover, all this structure is preserved by the forgetful functor $|-| : \mathbf{Fin} \to \mathbf{Rel}$.

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In other words

The relational interpretation of linear logic (or typed λ-calculus) is always finitary.

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All the constructions for multiplicative, additive and exponential structure work out as described by Hyland and Schalk.

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But...

One must prove that these constructions do provide the necessary structure "by hand".

For instance, the associativity of $\otimes$ follows from the fact that

$$\{a \times b;\ a \in \mathfrak{F}(\mathcal{A}), b \in \mathfrak{F}(\mathcal{B})\}^{\perp\perp} = \{c \subseteq |\mathcal{A} \otimes \mathcal{B}|;\ c_1 \in \mathfrak{F}(\mathcal{A}), c_2 \in \mathfrak{F}(\mathcal{B})\}.$$

Transporting a finiteness structure

Theorem (Transport [Tasson–V. 2011])

Let f be a relation from A to $|\mathcal{B}|$ such that

$$f \cdot \alpha \in \mathfrak{F}(\mathcal{B}) \text{ for all } \alpha \in A.$$

Then

$$\mathfrak{F} = \{a \subseteq A;\ f \cdot a \in \mathfrak{F}(\mathcal{B})\}$$

is a finiteness structure on A.

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Remark

This means f maps finite subsets to finitary subsets, which is necessary for $\mathfrak{F}$ to contain all finite subsets of A.

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More precisely:

$$\mathfrak{F} = \{f \setminus b;\ b \in \mathfrak{F}(\mathcal{B})\}^{\perp\perp}.$$

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Definition

$$f \setminus b = \bigcup \{a \subseteq A;\ f \cdot a \subseteq b\}$$

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Example

Consider the set $\mathfrak{M}_{\mathrm{f}}(|\mathcal{B}|)$ and the support relation σ.
Then $\sigma \cdot \overline{b} = supp\left(\overline{b}\right)$, $f \setminus b = b^! = \mathfrak{M}_{\mathrm{f}}(b)$ and

$$\mathfrak{F}(!\mathcal{B}) = \left\{b^!;\ b \in \mathfrak{F}(\mathcal{B})\right\}^{\perp\perp} = \left\{\overline{b} \subseteq |!B|;\ supp\left(\overline{b}\right) \in \mathfrak{F}(\mathcal{B})\right\}.$$

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Sketch of proof.

Take $a \in \{f \setminus b;\ b \in \mathfrak{F}(\mathcal{B})\}^{\perp\perp}$ and $b' \in \mathfrak{F}\left(\mathcal{B}^{\perp}\right)$,
and find (using AC) $a' \subseteq_{\mathrm{f}} A$ s.t. $f \cdot a \cap b' \subseteq f \cdot a'$. $\square$

(Very similar to the characterization of $!\mathcal{A}$ in Ehrhard's paper.)

Transporting a finiteness structure

Theorem (Transport [Tasson–V. 2011])

Let f_i be a relation from A to $|\mathcal{B}_i|$ such that

$$f_i \cdot \alpha \in \mathfrak{F}(\mathcal{B}_i) \text{ for all } \alpha \in A.$$

Then

$$\mathfrak{F} = \{a \subseteq A;\ f_i \cdot a \in \mathfrak{F}(\mathcal{B}_i)\,,\ \forall i \in I\}$$

is a finiteness structure on A.

More precisely:

$$\mathfrak{F} = \left\{ \bigcap_{i\in I} f \setminus b_i;\ b_i \in \mathfrak{F}(\mathcal{B}_i)\,,\ \forall i \in I \right\}^{\perp\perp}.$$

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Example

Consider the set $|\mathcal{A}| \times |\mathcal{B}|$ and the projection relations. Then

$$\{a \times b;\ a \in \mathfrak{F}(\mathcal{A}), b \in \mathfrak{F}(\mathcal{B})\}^{\perp\perp} = \{c \subseteq |\mathcal{A} \otimes \mathcal{B}|;\ c_1 \in \mathfrak{F}(\mathcal{A}), c_2 \in \mathfrak{F}(\mathcal{B})\}.$$

Transport is functorial

Theorem (Transport functors [Tasson–V. 2011])

Assume $T : \mathbf{Rel} \to \mathbf{Rel}$ *is a functor on relations, and* $\phi : T \Rightarrow 1_{\mathbf{Rel}}$ *is an almost-functional lax natural transformation.*

Then the following defines a functor $\mathcal{T} : \mathbf{Fin} \to \mathbf{Fin}$ *with web* T*:*

- *for all* $\mathcal{A} \in \mathbf{Fin}$, $|\mathcal{T}\mathcal{A}| = T\,|\mathcal{A}|$ *and* $\mathfrak{F}\,(\mathcal{T}\mathcal{A})$ *is transported from* $\mathfrak{F}\,(\mathcal{A})$ *along* $\phi_{|\mathcal{A}|}$*;*
- *for all* $f \in \mathbf{Fin}(\mathcal{A}, \mathcal{B})$, $\mathcal{T} f = T f$.

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Definition

$\phi : T \Rightarrow U$ is lax natural if $\phi_B \circ T\,f \subseteq U\,f \circ \phi_A$

Example

The support relation $\sigma : \mathfrak{M}_{\mathrm{f}} \Rightarrow 1_{\mathbf{Rel}}$.

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Definition

$f : A \to B$ is almost-functional if $\alpha \cdot \in \mathfrak{P}_{\mathrm{f}}\,(B)$ for all $\alpha \in A$.
In other words: f preserves finite sets.

Remark

This ensures the transport theorem always applies.

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Remark

Preservation of identities and composition is trivially deduced from that of T.

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Sketch of proof.

It only remains to prove $T f \in \mathbf{Fin}(\mathcal{T}\mathcal{A}, \mathcal{T}\mathcal{B})$, i.e.:

- $\overline{a} \in \mathfrak{F}(\mathcal{T}\mathcal{A})$ implies $T f \cdot \overline{a} \in \mathfrak{F}(\mathcal{T}\mathcal{B})$: by lax naturality;
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Counter-example The functor $-^\infty$ of streams, equipped with the obvious support relation, does not preserve finitary relations!

E.g. the total endorelation is finitary on $\mathbf{2}$, but not on $\mathbf{2}^\infty$.

Transport is functorial (when it contains finite data)

Theorem (Transport functors [Tasson–V. 2011])

Assume $T : \mathbf{Rel} \to \mathbf{Rel}$ is a symmetric functor on relations, and $\phi : T \Rightarrow 1_{\mathbf{Rel}}$ is an almost-functional lax natural transformation. Assume moreover that there exists a shape relation on (T, ϕ). Then the following defines a functor $\mathcal{T} : \mathbf{Fin} \to \mathbf{Fin}$ with web T:

- *for all $\mathcal{A} \in \mathbf{Fin}$, $|\mathcal{T}\mathcal{A}| = T\,|\mathcal{A}|$ and $\mathfrak{F}(\mathcal{T}\mathcal{A})$ is transported from $\mathfrak{F}(\mathcal{A})$ along $\phi_{|\mathcal{A}|}$;*
- *for all $f \in \mathbf{Fin}(\mathcal{A}, \mathcal{B})$, $\mathcal{T} f = T f$.*

Definition

A shape relation on (T, ϕ) is an almost-functional lax natural transformation μ from T to a constant functor Z such that:

for all $\overline{a} \subseteq TA$, $\overline{a}$ is finite as soon as $\phi_A \cdot \overline{a}$ and $\mu_A \cdot \overline{a}$ are.

T is symmetric if $T\,{}^t f = {}^t T f$.

What is transport good for?

Constructing finiteness spaces

e.g., the finiteness space of binary trees with nodes in $|\mathcal{A}|$ and leaves in $|\mathcal{B}|$, with finitess structure given by bounded height, finitary $\mathcal{A}$-support and finitary $\mathcal{B}$-support.

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Higher order linear logic ?

Transport of other structures

Coherence spaces

Let f be a relation from A to $|\mathcal{B}|$ such that $f \cdot \alpha \in \mathfrak{C}(\mathcal{B})$ for all $\alpha \in A$. Then

$$\mathfrak{C} = \{a \subseteq A;\ f \cdot a \in \mathfrak{C}(\mathcal{B})\}$$

is a coherence on A.

More precisely: $\mathfrak{C} = \{f \setminus b;\ b \in \mathcal{C}(\mathcal{B})\}^{\perp\perp}$.

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Very easy.

Transport of other structures

Coherence spaces

Let f be a relation from A to $|\mathcal{B}|$ such that $f \cdot \alpha \in \mathfrak{C}(\mathcal{B})$ for all $\alpha \in A$. Then

$$\mathfrak{C} = \{a \subseteq A;\ f \cdot a \in \mathfrak{C}(\mathcal{B})\}$$

is a coherence on A.

More precisely: $\mathfrak{C} = \{f \setminus b;\ b \in \mathcal{C}(\mathcal{B})\}^{\perp\perp}$.

Very easy.

Totality spaces

Fail ???

Transport and orthogonality

They play complementary roles:

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Towards a more general notion of transport?

- on top of orthogonality;
- restricted to webbed models ($\mathbf{Rel}$) or in an enriched setting.

Fin