On the transport of finiteness structures

Lionel Vaux
Institut de Mathématiques de Luminy, CNRS UMR 6206, Marseille, France
lionel.vaux@univmed.fr

Abstract

Finiteness spaces were introduced by Ehrhard as a model of linear logic, which relied on a finitess property of the standard relational interpretation and allowed to reformulate Girard's quantitative semantics in a simple, linear algebraic setting.

We review recent results obtained in a joint work with Christine Tasson, providing a very simple and generic construction of finiteness spaces: basically, one can *transport* a finiteness structure along any relation mapping finite sets to finite sets. Moreover, this construction is functorial under mild hypotheses, satisfied by the interpretations of all the positive connectives of linear logic.

Recalling that the definition of finiteness spaces follows a standard orthogonality technique, fitting in the categorical framework established by Hyland and Schalk, the question of the possible generalization of transport to a wider setting is quite natural. We argue that the features of transport do not stand on the same level as the orthogonality category construction; rather, they provide a simpler and more direct characterization of the obtained structure, in a webbed setting.

1 Finiteness spaces and finitary relations

Sets and relations. We write $\mathfrak{P}(A)$ for the powerset of A, $\mathfrak{P}_{\mathrm{f}}(A)$ for the set of all finite subsets of A and $!A$ for the set of all finite multisets of elements of A.

Let A and B be sets and f be a relation from A to B: $f \subseteq A \times B$. We then write ${}^t f$ for the transpose relation $\{(\beta, \alpha) \in B \times A;\ (\alpha, \beta) \in f\}$. For all subset $a \subseteq A$, we write $f \cdot a$ for the *direct image* of a by f: $f \cdot a = \{\beta \in B;\ \exists \alpha \in a,\ (\alpha, \beta) \in f\}$. If $\alpha \in A$, we will also write $f \cdot \alpha$ for $f \cdot \{\alpha\}$. We say that a relation f is *quasi-functional* if $f \cdot \alpha$ is finite for all α. If $b \subseteq B$, we define the *division* of b by f as $f \setminus b = \{\alpha \in A;\ f \cdot \alpha \subseteq b\}$. Notice that in general $f \cdot (f \setminus b)$ may be a strict subset of b, and $f \setminus (f \cdot a)$ may be a strict superset of a.

We write $\underline{\mathrm{Rel}}$ for the category of sets and relations. It is a very simple model of linear logic: multiplicatives are given by the compact closed structure associated with cartesian products of sets (linear negation is then the transposition of relations, which is also a dagger); additives are modelled by disjoint union of sets, which gives a biproduct; the exponential modality is that of finite multisets.

Let T and U be two endofunctors of $\underline{\mathrm{Rel}}$, and let f be the data of a relation f^A (which we may also write f) from TA to UA for all set A: we say f is a *lax natural transformation* from T to U if, for all relation g from A to B, $f^B \circ (Tg) \subseteq (Ug) \circ f^A$. As an example, consider the finite multiset functor $!$, and for all A, let σ^A be the only relation from $!A$ to A such that for all $\overline{\alpha} \in !A$, $\sigma^A \cdot \overline{\alpha}$ is the support set of $\overline{\alpha}$. This defines a quasi-functional lax natural transformation from $!$ to the identity functor: notice that in that case, the inclusion $\sigma \circ !g \subseteq g \circ \sigma$ may be strict.

Finiteness spaces. We briefly recall the basic definition of finiteness spaces as given by Ehrhard [Ehr05]. Let A and B be sets, we write $A \perp_{\mathrm{f}} B$ if $A \cap B$ is finite. If $\mathfrak{A} \subseteq \mathfrak{P}(A)$, we define the *predual* of $\mathfrak{A}$ on A as $\mathfrak{A}^{\perp} = \{a' \subseteq A;\ \forall a \in \mathfrak{A},\ a \perp_{\mathrm{f}} a'\}$. A *finiteness structure* on A is a set $\mathfrak{A}$ of subsets of A such that $\mathfrak{A}^{\perp\perp} = \mathfrak{A}$. A *finiteness space* is then a pair $\mathcal{A} = (|\mathcal{A}|, \mathfrak{F}(\mathcal{A}))$ where $|\mathcal{A}|$ is the underlying set, called the *web* of $\mathcal{A}$, and $\mathfrak{F}(\mathcal{A})$ is a finiteness structure on $|\mathcal{A}|$.

We write $\mathcal{A}^\perp$ for the *dual* finiteness space: $\left|\mathcal{A}^\perp\right| = |\mathcal{A}|$ and $\mathfrak{F}\left(\mathcal{A}^\perp\right) = \mathfrak{F}(\mathcal{A})^\perp$. The elements of $\mathfrak{F}(\mathcal{A})$ are called the *finitary subsets* of $\mathcal{A}$. Standard arguments on closure operators defined by orthogonality apply and in particular $\mathfrak{A}^\perp = \mathfrak{A}^{\perp\perp\perp}$, for all $\mathfrak{A} \subseteq \mathfrak{P}(A)$; hence finiteness structures are exactly preduals. More specific to the orthogonality $\perp_{\mathrm{f}}$, for all finiteness structure $\mathfrak{A}$ on A, we obtain:

(1) $\mathfrak{A}$ is downwards closed for inclusion, i.e. $a \subseteq a' \in \mathfrak{A}$ implies $a \in \mathfrak{A}$;

(2) $\mathfrak{P}_{\mathrm{f}}(A) \subseteq \mathfrak{A}$ and $\mathfrak{A}$ is closed under finite unions, i.e. $a, a' \in \mathfrak{A}$ implies $a \cup a' \in \mathfrak{A}$.

The first property is similar to the one for coherence spaces. The second one is distinctive of finiteness spaces, and is a non-uniformity property: union of finitary subsets models some form of computational non-determinism, which is crucial to interpret the differential λ-calculus [ER03].

Finitary relations. Let $\mathcal{A}$ and $\mathcal{B}$ be two finiteness spaces: we say a relation f from $|\mathcal{A}|$ to $|\mathcal{B}|$ is *finitary* from $\mathcal{A}$ to $\mathcal{B}$ if: for all $a \in \mathfrak{F}(\mathcal{A})$, $f \cdot a \in \mathfrak{F}(\mathcal{B})$, and for all $b' \in \mathfrak{F}\left(\mathcal{B}^\perp\right)$, ${}^t f \cdot b' \in \mathfrak{F}\left(\mathcal{A}^\perp\right)$. The identity relation is finitary from $\mathcal{A}$ to itself, and finitary relations compose: this defines the category $\underline{\mathrm{Fin}}$ whose objects are finiteness spaces and morphisms are finitary relations.

Finitary relations form a finiteness structure: remark that $f \subseteq |\mathcal{A}| \times |\mathcal{B}|$ is finitary iff $f \in \left\{a \times b';\ a \in \mathfrak{F}(\mathcal{A}) \text{ and } b' \in \mathfrak{F}\left(\mathcal{B}^\perp\right)\right\}^\perp$. This reflects the $*$-autonomous structure of $\underline{\mathrm{Fin}}$, with tensor product given by $|\mathcal{A} \otimes \mathcal{B}| = |\mathcal{A}| \times |\mathcal{B}|$ and $\mathfrak{F}(\mathcal{A} \otimes \mathcal{B}) = \{a \times b;\ a \in \mathfrak{F}(\mathcal{A}) \text{ and } b \in \mathfrak{F}(\mathcal{B})\}^{\perp\perp}$, and $*$-functor given by duality on finiteness spaces and transposition on finitary relations: $f \in \underline{\mathrm{Fin}}(\mathcal{A}, \mathcal{B}) \mapsto {}^t f \in \underline{\mathrm{Fin}}(\mathcal{B}^\perp, \mathcal{A}^\perp)$.

2 Transport

Transport of finiteness structures In the following, we present the basic results obtained in recent work with Tasson [TV11]. The starting point is the following lemma, which allows to generate a finiteness structure on a set A, by transporting that of a finiteness space $\mathcal{B}$ along any relation f from A to $|\mathcal{B}|$, provided f maps finite subsets of A to finitary subsets of $\mathcal{B}$.

Lemma 2.1 (Transport)**.** *Let A be a set, $\mathcal{B}$ a finiteness space and f a relation from A to $|\mathcal{B}|$ such that $f \cdot \alpha \in \mathfrak{F}(\mathcal{B})$ for all $\alpha \in A$. Then $\mathfrak{F} = \{a \subseteq A;\ f \cdot a \in \mathfrak{F}(\mathcal{B})\}$ is a finiteness structure on A and, more precisely, $\mathfrak{F} = \{f \setminus b;\ b \in \mathfrak{F}(\mathcal{B})\}^{\perp\perp}$.*

The proof of this *transport lemma* [TV11, Lemma 3.4] is very similar to that of the characterization of the exponential modality, given in Ehrhard's paper [Ehr05, Lemma 4]. Actually, we obtain this characterization as a straightforward application of transport, through the support relation, which is quasi-functional: if $\mathcal{A}$ is a finiteness space, then for all $\overline{\alpha} \in {!}\,|\mathcal{A}|$, $\sigma \cdot \overline{\alpha} \in \mathfrak{P}_{\mathrm{f}}(A) \subseteq \mathfrak{F}(\mathcal{A})$; moreover, if $a \subseteq |\mathcal{A}|$, then $\sigma \setminus a = {!}a$. We thus obtain a finiteness space ${!}\mathcal{A}$ such that $|{!}\mathcal{A}| = {!}\,|\mathcal{A}|$ and $\mathfrak{F}({!}\mathcal{A}) = \{\overline{a} \subseteq {!}|\mathcal{A}|;\ \sigma \cdot \overline{a} \in \mathfrak{F}(\mathcal{A})\} = \{{!}a;\ a \in \mathfrak{F}(\mathcal{A})\}^{\perp\perp}$.

Corollary 2.2. *Let A be a set, $(\mathcal{B}_i)_{i \in I}$ a family of finiteness spaces and $(f_i)_{i \in I}$ a family of relations such that, for all $\alpha \in A$ and all $i \in I$, $f_i \cdot \alpha \in \mathfrak{F}(\mathcal{B}_i)$. Then $\mathfrak{F} = \{a \subseteq A;\ \forall i \in I,\ f_i \cdot a \in \mathfrak{F}(\mathcal{B}_i)\}$ is a finiteness structure on A and, more precisely, $\mathfrak{F} = \left\{\bigcap_{i \in I}(f_i \setminus b_i);\ \forall i \in I,\ b_i \in \mathfrak{F}(\mathcal{B}_i)\right\}^{\perp\perp}$.*

Again, we obtain the following characterization of the tensor product, by applying this generalized transport lemma: denoting π_1 and π_2 the two obvious projection relations we obtain $\mathfrak{F}(\mathcal{A} \otimes \mathcal{B}) = \{c \subseteq |\mathcal{A}| \times |\mathcal{B}|\,;\ \pi_1 \cdot c \in \mathfrak{F}(\mathcal{A}) \text{ and } \pi_2 \cdot c \in \mathfrak{F}(\mathcal{B})\}$. Similarly, the direct product of an arbitrary family of finiteness spaces is given by $\left|\&_{i\in I}\mathcal{A}_i\right| = \biguplus_{i\in I}|\mathcal{A}_i| = \bigcup_{i\in I}\{i\}\times|\mathcal{A}|_i$ and $\mathfrak{F}\left(\&_{i\in I}\mathcal{A}_i\right) = \left\{\biguplus_{i\in I}a_i;\ \forall i\in I,\ a_i\in\mathfrak{F}(\mathcal{A}_i)\right\}$: this is otained by transport through the restrictions $\rho_i = \{((i,\alpha),\alpha);\ \alpha\in|\mathcal{A}_i|\}$. It turns out that the transport lemma is very versatile: for any sensible notion of datatype (lists, trees, graphs, *etc.*), it allows to form a finiteness spaces of such objects, with finiteness given by that of the elements (or nodes), possibly with an additional finiteness condition on the shape (e.g., bounded length).

Transport functors. We say an endofunctor $\mathcal{T}$ of $\underline{\mathrm{Fin}}$ *has a web* if there exists an endofunctor T of $\underline{\mathrm{Rel}}$, such that $|\mathcal{T}\mathcal{A}| = T|\mathcal{A}|$ for all finiteness space $\mathcal{A}$, and $\mathcal{T}f = Tf$ for all $f\in\underline{\mathrm{Fin}}(\mathcal{A},\mathcal{B})\subseteq\underline{\mathrm{Rel}}(|\mathcal{A}|,|\mathcal{B}|)$. We then say T is the web of $\mathcal{T}$ and write $T=|\mathcal{T}|$. Notice that in that case, if $f\subseteq A\times B$, Tf must be finitary from $\mathcal{T}(A,\mathfrak{A})$ to $\mathcal{T}(B,\mathfrak{B})$ for all finiteness structures $\mathfrak{A}$ and $\mathfrak{B}$ making f finitary from $(A,\mathfrak{A})$ to $(B,\mathfrak{B})$. We show that, under mild hypotheses, the transport lemma allows to define such functors.

Let T be a functor in $\underline{\mathrm{Rel}}$. We call *ownership relation* on T the data of a quasi-functional lax natural transformation ϵ from T to the identity functor. Given such an ownership relation, we can transport the finiteness structure of any space $\mathcal{A}$ to the web $T|\mathcal{A}|$: indeed, $\epsilon^{|\mathcal{A}|}$ then satisfies the condition of Lemma 2.1 because it is quasi-functional and finite subsets are always finitary. In such a situation, we write $T_\epsilon\mathcal{A}$ for the finiteness space $(T|\mathcal{A}|, \{\widetilde{a}\subseteq T|\mathcal{A}|;\ \epsilon\cdot\widetilde{a}\in\mathfrak{F}(\mathcal{A})\})$. If $f\in\underline{\mathrm{Rel}}(|\mathcal{A}|,|\mathcal{B}|)$, we also write $T_\epsilon f = Tf$: then T_ϵ defines a functor on $\underline{\mathrm{Fin}}$ (with web T) iff Tf is finitary from $T_\epsilon\mathcal{A}$ to $T_\epsilon\mathcal{B}$ as soon as f is finitary from $\mathcal{A}$ to $\mathcal{B}$. In that case, we say T_ϵ is the *transport functor* deduced from the *transport situation* (T,ϵ).

We now provide sufficient conditions for a transport situation to give rise to a transport functor. A *shape relation* on (T,ϵ) is the data of a fixed set M of *shapes* and a quasi-functional lax natural transformation μ from T to the constant functor which sends every set to M and every relation to the identity, subject to the following additional condition: for all $\widetilde{a}\subseteq TA$, if $\mu\cdot\widetilde{a}$ and $\epsilon\cdot\widetilde{a}$ are both finite, then $\widetilde{a}$ is finite.

Lemma 2.3. *Let (T,ϵ) be a transport situation. If T is symmetric (i.e. ${}^t(Tf) = T\,{}^tf$ for all f) and there exists a shape relation on (T,ϵ), then T_ϵ is an endofunctor in $\underline{\mathrm{Fin}}$.*

The symmetry of T is essential in the proof, since it allows ϵ and μ to interact with ${}^tTf$ as well as with Tf (the definition of finitary relations is related with both directions). Moreover, the existence of a shape relation is also crucial, since some transport situations on symmetric functors do not preserve finitary relations. This is in particular the case of a would-be infinitary tensor: although we can apply the transport lemma to define $\bigotimes_{i\in I}\mathcal{A}_i$ for all family $(\mathcal{A}_i)_{i\in I}$ of finiteness spaces (consider the projections $(\pi_i)_{i\in I}$), the tensor of finitary relations is not necessarily finitary. It is however important to note that the shape relation plays no rôle in the definition of T_ϵ: its existence is a mere side condition ensuring functoriality.

A direct consequence is the functoriality of the exponential !: the shape of a finite multiset is its size. Lemma 2.3 is easily generalized to functors of arbitrary arity, such as the direct product of finiteness spaces, given by disjoint union of webs and finitary subsets: the shape of an element $(j,\alpha)\in\biguplus_{i\in I}|\mathcal{A}_i|$ is the index j. The functoriality of binary tensor product also follows, this time with no need of an additional shape relation: the binary cartesian product of finite sets is always finite.

The properties of transport functors are further studied in [TV11]: we show that, under additional hypotheses, transport functors are Scott-continuous, which allows to take fixed points of such; this is put to use by giving an account of recursive algebraic datatypes in $\underline{\mathrm{Fin}}$.

3 On possible generalizations of transport

The orthogonality category of finiteness spaces. The category $\underline{\mathrm{Fin}}$ is the tight orthogonality category associated with $\perp_{\mathrm{f}}$ on $\underline{\mathrm{Rel}}$, following the theory of Hyland and Schalk [HS03]. The transport lemma can be used to establish the self-stability of $\perp_{\mathrm{f}}$ easily. More generally, it provides simple and concrete characterizations of the abstract structure generated by double-glueing. In fact, the very merit of transport lies precisely in making the bidual closure typical of the orthogonality construction almost trivial, since it simply amounts to the downwards closure for inclusion.

This difference in approach shows in the formulation of transport. Key ingredients seem to rely strongly on the fact that we consider a webbed model (interpretations of proofs are particular subsets of their types), and in particular on the order enrichment of the category given by inclusion of relations. We can only remark that the condition "f sends finite subsets to finitary subsets" can be rephrased as f being negative from $(A, \mathfrak{P}(A), \mathfrak{P}_{\mathrm{f}}(A))$ to $(|\mathcal{B}|, \mathfrak{F}(\mathcal{B}), \mathfrak{F}(\mathcal{B}^{\perp}))$. The possible generalization of transport to a wider setting is nonetheless an appealing perspective. As a first step, we turn our attention to other models of linear logic related with the relational model.

Transport in other webbed models. Recall that a coherence space $\mathcal{A}$ is the data of a set $|\mathcal{A}|$ and a reflexive binary relation $\frown_{\mathcal{A}}$ on $|\mathcal{A}|$ (its *coherence*). Equivalently, $\mathcal{A}$ can be characterized by the set $\mathfrak{C}(\mathcal{A}) \subseteq \mathfrak{P}(|\mathcal{A}|)$ of its *cliques*, i.e. sets of pairwise coherent elements. A relation $f \subseteq |\mathcal{A}| \times |\mathcal{B}|$ is said to be linear if, for all $a \in \mathfrak{C}(\mathcal{A})$, $f \cdot a \in \mathfrak{C}(\mathcal{B})$ and for all $b' \in \mathfrak{C}(\mathcal{B})^{\perp}$, ${}^{t}f \cdot b' \in \mathfrak{C}(\mathcal{A})^{\perp}$, where $\mathfrak{C}(\mathcal{A})^{\perp}$ denotes the dual for the *partial orthogonality*: $a \perp_{\mathrm{p}} a'$ iff $a \cap a'$ has at most one element.

The transport lemma is easily adapted to coherence spaces: just replace "finiteness structure" with "clique", and observe that, if $f \cdot \alpha$ is always a clique, then $f \cdot a$ is a clique iff $f \cdot \{\alpha, \alpha'\}$ is a clique for all $\alpha, \alpha' \in a$, which defines a new coherence. The technique we used to establish the functoriality of transport, however, does not apply directly: if (T, ϵ) is a transport situation and $f \in \mathfrak{C}(\mathcal{A} \multimap \mathcal{B})$, then Tf sends cliques to cliques by lax-naturality of ϵ, but establishing the reverse direction (inverse images of anticliques are anticliques) will require to tweak the notion of shape relation to accomodate coherence rather than finiteness. This is the subject of ongoing work.

Transport does not seem to be meaningful for the webbed model obtained from the *total orthogonality*: $a \perp_{\mathrm{t}} a'$ iff $a \cap a'$ is a singleton. This defines Loader's totality spaces: intuitively, total subsets represent maximal cliques. This maximality property is not compatible with the fact that, by construction, the structures obtained by transport are downwards closed for inclusion.

References

[Ehr05] Thomas Ehrhard. Finiteness spaces. *Mathematical Structures in Computer Science*, 15(4):615–646, 2005.

[ER03] Thomas Ehrhard and Laurent Regnier. The differential lambda-calculus. *Theoretical Computer Science*, 309:1–41, 2003.

[HS03] Martin Hyland and Andrea Schalk. Glueing and orthogonality for models of linear logic. *Theor. Comput. Sci.*, 294(1/2):183–231, 2003.

[TV11] Christine Tasson and Lionel Vaux. Transport of finiteness structures and applications. *Mathematical Structures in Computer Science*, 2011. To appear.