$(\forall M \in \Lambda_+)\ M \in \mathsf{SN} \iff \mathcal{T}(M) \in \mathfrak{F}$

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Outline

Everything is in the title:

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We characterize the strong normalizability (SN)
of non-deterministic λ-terms (Λ_+)
as a finiteness structure ($\mathfrak{F}$)
via Taylor expansion ($\mathcal{T}$).

The end

Thanks for your attention.

Quantitative semantics

An oldish idea (Girard, '80s)

- types $\rightsquigarrow$ particular topological vector spaces:
  $[\![A]\!] \subseteq \mathbf{k}^{|A|}$ + possibly some additional structure
- terms $\rightsquigarrow$ analytic functions defined by power series:

$$\begin{array}{rcl} |A \to B| & \subseteq & |A|^! \times |B| \\ ((M)\,N)_\beta & = & \sum_{(\overline{\alpha},\beta)} M_{(\overline{\alpha},\beta)} N^{\overline{\alpha}} \end{array}$$

  where $|A|^!$ is the set of finite multisets over $|A|$, and for all $\overline{\alpha} = [\alpha_1, \ldots, \alpha_n] \in |A|^!$, $N^{\overline{\alpha}} = \prod_{\alpha \in |A|} N_\alpha^{\overline{\alpha}(a)} = \prod_i N_{\alpha_i}$
- this was the origin of linear logic (via coherence spaces)

How to ensure the convergence of the series?

Originally, $\mathbf{k} = \mathsf{Sets}$.

Finiteness structures

Definition

- If $a, a' \subseteq A$, write $a \perp a'$ iff $a \cap a'$ is finite.
- If $\mathfrak{S} \subseteq \mathfrak{P}(A)$, let $\mathfrak{S}^\perp := \{a' \subseteq A;\ \forall a \in \mathfrak{S},\ a \perp a'\}$.
- A finiteness structure is any $\mathfrak{F} = \mathfrak{S}^\perp$.

Then you can build a denotational model of linear logic where

$$[\![A]\!] = \left\{a \in \mathbf{k}^{|A|};\ |a| \in \mathfrak{Fin}(A)\right\}$$

with $\mathfrak{Fin}(A)$ a finiteness structure on $|A|$ so that for all $a \in \mathfrak{Fin}(A)$, $\beta \in |B|$ and all $f \in \mathfrak{Fin}(A \to B)$,

$$\{\overline{\alpha};\ (\overline{\alpha}, \beta) \in f\} \perp a^!.$$

Short version: the sum of the previous slide is always finite.
Motto: finiteness structures enforce finite interactions/reductions/cut elimination.

λ-terms as analytic functions

So we can *differentiate* (typed) λ-terms, and compute their Taylor expansion!

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So we can *differentiate* (typed) λ-terms, and compute their Taylor expansion!
And one can mimick that in the syntax:

- differential λ-calculus (Ehrhard-Regnier 2003)
- a finitary fragment: resource λ-calculus (Ehrhard-Regnier 2004)
  this is the target of Taylor expansion

Resource λ-calculus

Resource terms

$$\begin{array}{cccc} \Delta & \ni & s, t, \ldots & ::= & x \mid \lambda x.s \mid \langle s \rangle\ \bar{t} \\ \Delta^! & \ni & \bar{s}, \bar{t}, \ldots & ::= & [s_1, \ldots, s_n] \end{array}$$

Resource reduction

$$\langle \lambda x.s \rangle\ \bar{t} \to_\rho \partial_x s \cdot \bar{t} \quad \text{(anywhere)}$$

Multilinear substitution

$$\partial_x s \cdot \bar{t} = \begin{cases} \sum_{f \in \mathfrak{S}_n} s\left[t_{f(1)}, \ldots, t_{f(n)} / x_1, \ldots, x_n\right] & \text{if } \deg_x(s) = \#\bar{t} = n \\ 0 & \text{otherwise} \end{cases}$$

So we have formal sums of resource terms: $S, T, \ldots := \sum_{i=1}^{n} t_i$.
Everything is linear, e.g.: $\lambda x.0 = 0$, $\langle s \rangle\ [t_1 + t_2, u] = \sum_i \langle s \rangle\ [t_i, u]$.

Theorem

Resource reduction is strongly confluent and terminating.

Taylor expansion of λ-terms

Semantically, $(M)\,N = \sum_{n\in\mathbf{N}} \frac{1}{n!}\,\langle M\rangle\ N^n$ where $N^n = [N,\ldots,N]$.

Taylor expansion

$\vec{\mathcal{T}}\,(M) \in {\mathbf{Q}^+}^{\Delta}$ is given by

$$\vec{\mathcal{T}}\,((M)\,N) = \sum_{n\in\mathbf{N}} \frac{1}{n!}\left\langle \vec{\mathcal{T}}\,(M)\right\rangle \vec{\mathcal{T}}\,(N)^n$$

and $\vec{\mathcal{T}}\,(x) = x$, $\vec{\mathcal{T}}\,(\lambda x.M) = \lambda x.\vec{\mathcal{T}}\,(M)$.

Theorem (Ehrhard-Regnier 2004 (pub. TCS 2008))

If $M \in \Lambda$ has a normal form, then $\vec{\mathcal{T}}\,(M)$ normalizes to $\vec{\mathcal{T}}\,(\mathsf{NF}\,(M))$.

Theorem (Ehrhard-Regnier, CiE 2006)

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Normalizing Taylor expansions: uniformity to the rescue

But how can $\vec{\mathcal{T}}(M)$ even normalize? Take $S \in \mathbf{k}^{\Delta}$: we want to set

$$\mathsf{NF}(S) = \sum_{t \in \Delta} S_t . \mathsf{NF}(t)$$

but this means infinite sums (and in general we might consider all kinds of coefficients).

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Theorem (Ehrhard-Regnier 2004)

Write $\mathcal{T}(M) = \left|\vec{\mathcal{T}}(M)\right|$. *Then for all* $t \in \Delta$, *there is at most one* $s \in \mathcal{T}(M)$ *such that* $\mathsf{NF}(s)_t \neq 0$.

Proof.

λ-terms are uniform (aka deterministic). $\square$

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This fails for arbitrary linear combinations: consider $\sum_{n \in \mathbf{N}} \langle \lambda x.x \rangle^n [y]$ where $\langle \lambda x.x \rangle^n [y] = \langle \lambda x.x \rangle [\langle \lambda x.x \rangle [\cdots [y] \cdots]]$.

What about non-deterministic λ-calculi?

A minimalistic non-uniform calculus

$$\Lambda_+ \ni M, N, \ldots ::= x \mid \lambda x.M \mid (M)\,N \mid M + N$$

$$(\lambda x.M)\,N \to_\beta M\,[N/x] \quad \text{(anywhere)}$$

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Then $\infty_M \to_\beta^* M + \infty_M$!

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Taylor expansion in a non uniform setting

$$\vec{\mathcal{T}}\,(M + N) = \vec{\mathcal{T}}\,(M) + \vec{\mathcal{T}}\,(N)$$

We would like to set:

$$\mathsf{NF}\left(\vec{\mathcal{T}}\,(M)\right)_t = \sum_{s \in \Delta} \vec{\mathcal{T}}\,(M)_s\, \mathsf{NF}\,(s)$$

but of course, normalizing $\vec{\mathcal{T}}\,(\infty_M)$ leads to infinite sums.

Finiteness structures to the rescue

When is $\mathsf{NF}(\vec{\mathcal{T}}(M))$ defined?

- Write $s \geq t$ if $s \to_\rho^* t + \cdots$.
- Then let $\uparrow t = \{s \in \Delta;\ s \geq t\}$.
- We want: for all normal $t \in \Delta$, $\mathcal{T}(M) \perp \uparrow t$.

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Let system F_+ be system F plus $\dfrac{\Gamma \vdash M : A \quad \Gamma \vdash N : A}{\Gamma \vdash M + N : A}$.

Theorem (Ehrhard, LICS 2010)

If $M \in \Lambda_+$ is typable in system F_+, then $\mathcal{T}(M) \in \{\uparrow t\ ;\ t \in \Delta\}^\perp$.

Proof.

Manage sets of resource terms as if they were λ-terms, and follow the usual reducibility technique, associating a finiteness structure $\mathfrak{Fin}(A) \subseteq \{\uparrow t\ ;\ t \in \Delta\}^\perp$ with each type A.

$\square$

A remark

In the previous theorem, "tests" are not restricted to normal terms.
This rules out looping terms, e.g., $\Omega = (\Delta)\,\Delta$ with $\Delta = \lambda x.\,(x)\,x$:

- consider $\delta_n = \lambda x.\,\langle x \rangle\ [x^n]$;
- then for all $n \in \mathbf{N}$, $\langle \delta_n \rangle\ [\delta_0, \delta_0, \delta_1 \ldots, \delta_{n-1}] \geq \langle \delta_0 \rangle\ [] \to_\rho 0$.

Our results

- Typability in F can be relaxed to strong normalizability.
- Then the implication

$$M \in \mathsf{SN} \Rightarrow \mathcal{T}(M) \in \{\uparrow t \ ; \ t \in \Delta\}^{\perp}$$

  can be reversed…
- provided the finiteness $\{\uparrow t \ ; \ t \in \Delta\}^{\perp}$ is refined to a tighter one.

$M \in \mathsf{SN} \Rightarrow \mathcal{T}(M) \in \{\uparrow t \; ; \; t \in \Delta\}^{\perp}$

In the ordinary λ-calculus:

- SN = typability in system D (simple types + $\cap$)
- "any" proof by reducibility for simple types is valid for D

So we:

- introduce a system D_+ of intersection types for non uniform terms
- prove that $M \in \mathsf{SN}$ implies $\Gamma \vdash M : A$ in D_+
- adapt Ehrhard's proof to D_+

System D_+

System D uses the rules:

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash M : B}{\Gamma \vdash M : A \cap B} \qquad \frac{\Gamma \vdash M : A \cap B}{\Gamma \vdash M : A} \qquad \frac{\Gamma \vdash M : A \cap B}{\Gamma \vdash M : B}$$

This is not sufficient here, due to constraints for typing sums:

- observe that $(x + y)\, z = (x)\, z + (y)\, z$
- let $\Gamma = x : A \to B \cap B', y : A \to B \cap B'', z : A$,
- then $\Gamma \vdash (x + y)\, z : B$
- but $x + y$ is not typable in Γ.

We need (a limited amount of) subtyping:

- $A \cap B \preceq A$ and $A \cap B \preceq B$;
- $(A \to B) \cap (A \to C) \preceq A \to (B \cap C)$;
- $A \to B \preceq A' \to B'$ as soon as $A' \preceq A$ and $B \preceq B'$.

$$\frac{\Gamma \vdash M : A \quad A \preceq B}{\Gamma \vdash M : B}$$

Then the proofs go *almost* as usual.

$\mathcal{T}(M) \in \{\uparrow t \,;\, t \in \Delta\}^{\perp} \Rightarrow M \in \mathsf{SN}$

Fails!

Let $\Delta_3 := \lambda x.\,(x)\,x\,x$ and $\Omega_3 := (\Delta_3)\,\Delta_3$, then $\mathcal{T}(\Omega_3) \perp \uparrow s$ for all s.

Why?

We ruled out loops, but the divergence of Ω_3 is of another nature.
A diverging λ-term either loops or reduces to terms of arbitrary height.

Fix: add more tests

- Consider a structure $\mathfrak{S} \subseteq \mathfrak{P}(\Delta)$ and let $\mathfrak{F}_{\mathfrak{S}} = \{\uparrow a \,;\, a \in \mathfrak{S}\}^{\perp}$ with $\uparrow a = \bigcup_{s \in a} \uparrow s$.
- Of course, not all $\mathfrak{S}$ are acceptable, otherwise we reject too many terms (consider $\mathfrak{S} = \mathfrak{P}(\Delta)$).
- We need to rule out unbounded height: it suffices to test against linear terms.

$\mathcal{T}(M) \in \mathfrak{F}_{\mathfrak{S}} \Rightarrow M \in \mathsf{SN}$

… as soon as $\mathfrak{S}$ contains all sets of linear terms and all singletons.

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We prove the contraposition: given an infinite reduction sequence from M, we find $a \in \mathfrak{S}$ such that $\mathcal{T}(M) \not\perp \uparrow a$.

Lemma

If $M \to_\beta^* N$ then for all $t \in \mathcal{T}(N)$ there is $s \in \mathcal{T}(M)$ such that $s \geq t$.

Proof that $\mathcal{T}(M) \in \mathfrak{F}_{\mathfrak{S}} \Rightarrow M \in \mathsf{SN}$.

- if M reduces to terms of unbounded height:
  - take M_i any term of height $\geq i$ with $M \to_\beta^* M_i$;
  - take $a = \{s_i;\ i \in \mathbf{N}\}$ with $s_i \in \mathcal{T}(M_i)$ a linear resource term
- otherwise M (in fact $\mathcal{T}(M)$) loops and we can follow a looping reduction path backwards (with some care)

$\square$

Glueing everything together

We can adapt the reducibility proof provided $\mathfrak{S}$ satisfies:

- for all $n \in \mathbf{N}$, for all $a \in \mathfrak{S}$, $\{s \in a;\ \mathbf{h}(s) \leq n\}$ is finite.
- some additional, purely technical conditions.

One interesting example:

$$\mathfrak{B} = \{a \subseteq \Delta;\ \#(a) \text{ is bounded}\}$$

where $\#(a) = \{\#(s);\ s \in a\}$ and $\#(s)$ is the maximum size of a bag of arguments in s.
Clearly $\mathfrak{B}$ contains all singletons and all sets of linear terms.

Theorem (Pagani-Tasson-V.)

The following three properties are equivalent:

- $M \in \mathsf{SN}$;
- M *is typable in system* D_+;
- $\mathcal{T}(M) \in \mathfrak{F}_{\mathfrak{B}}$.

Conclusion

We are happy.

We have established a nice and novel characterization of SN.

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Are we?

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the Taylor expansion of a strongly normalizable term is normalizable

which is a bit frustrating (why strongly?).

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Yes we are: plenty of future work!

- Our machinery is modular enough that it can be adapted to weak normalizability and head-reduction: just change the reduction order $\geq$ on resource terms.
- Normalizability of Taylor expansion is the difficult part: that $\vec{\mathcal{T}}(\mathsf{NF}(M)) = \mathsf{NF}(\vec{\mathcal{T}}(M))$ follows easily.
- Paves the way for a semantically founded notion of Böhm trees for various non uniform settings (quantitative non-determinism, probabilistic stuff, *etc.*).

The end

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Questions?