Taylor expansion, β-reduction and normalization

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Models of non-determinism in computation

Denotational semantics…

Give a "meaning" to programs, that is stable under evaluation, *e.g.*:

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… and more

- quantitative parallelism
- probabilistic programs
- quantum stuff

Non-determinism in the λ-calculus

$M, N, \ldots ::= x \mid \lambda x.M \mid M\ N$

$$(\lambda x.M)\ N \to_\beta M\,[N/x]$$

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$M, N, \ldots ::= x \mid \lambda x.M \mid M\,N \mid M + N$

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$$M + N \to_+ M \quad (\text{or } N)$$

Non-determinism in the λ-calculus, contextually

$$M, N, \ldots ::= x \mid \lambda x.M \mid M\,N \mid M + N$$

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$$(M + N)\ P = M\,P + N\,P \qquad \lambda x.\,(M + N) = \lambda x.M + \lambda x.N$$

implicitly call-by-name

Quantitative non-determinism in the λ-calculus

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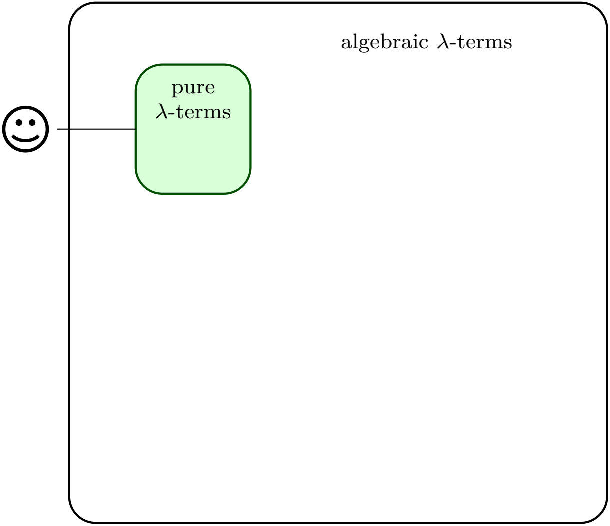
algebraic λ-terms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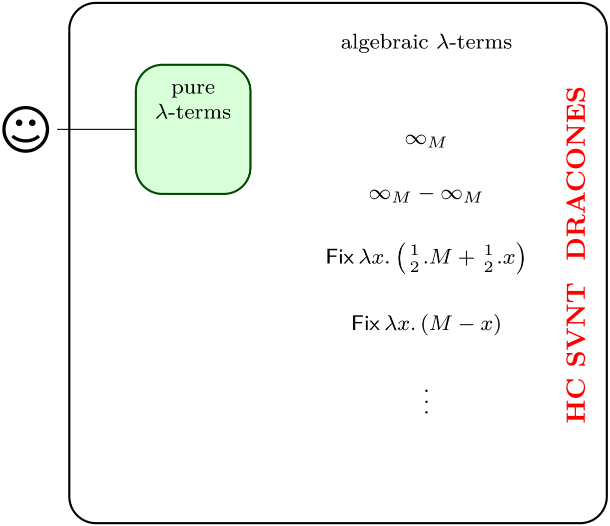
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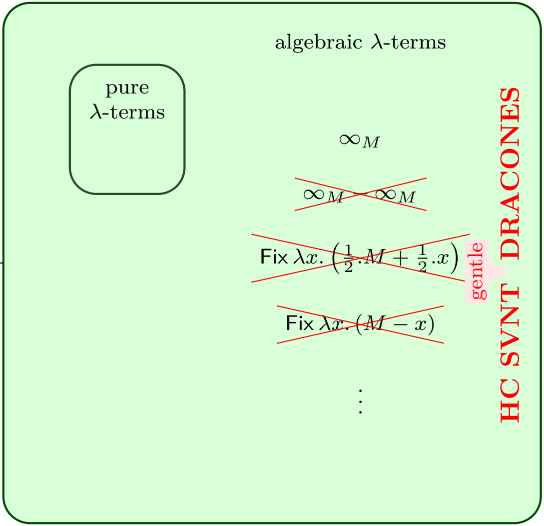
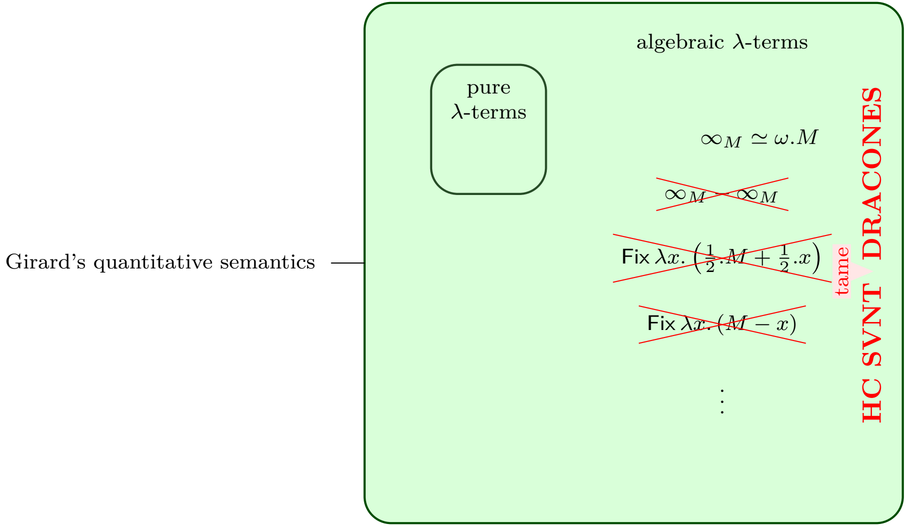
⋮
HC SVNT DRACONES

Plain n.d. choice ($\mathbf{S} = \mathbf{B}$):

plenty of models:
e.g., De'Liguoro-Piperno's trees
or the relational model



Choice as sum ($\mathbf{S} = \mathbf{N}$):



Quantitative semantics

Normal functors (Girard, '80s, before LL)

λ-terms $\rightsquigarrow$ set-valued power series (cf. Joyal's analytic functors)

Interprets non-deterministic choice *quantitatively*:

$$[\![M \oplus N]\!] = [\![M]\!] + [\![N]\!] \qquad \text{(disjoint sum of sets)}$$

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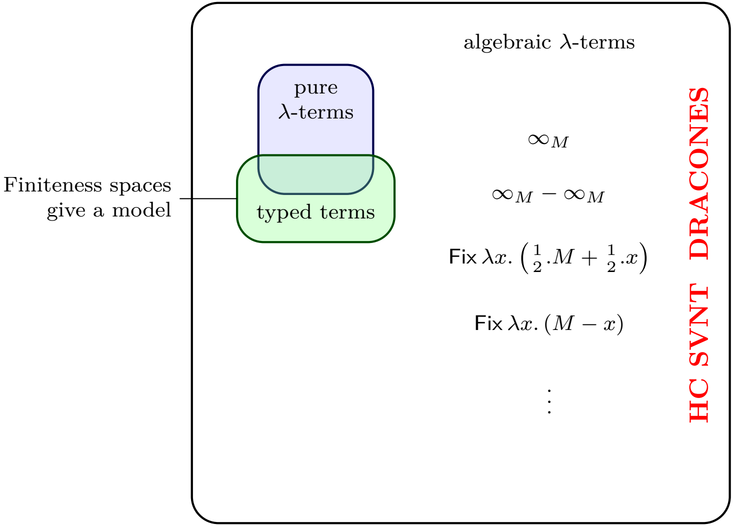
Reformulate q.s. for linear logic in standard algebra:

- types $\rightsquigarrow$ particular topological vector spaces (or semimodules):
  $[\![A]\!] \subseteq \mathbf{S}^{|A|}$ + some additional structure
- function terms $\rightsquigarrow$ power series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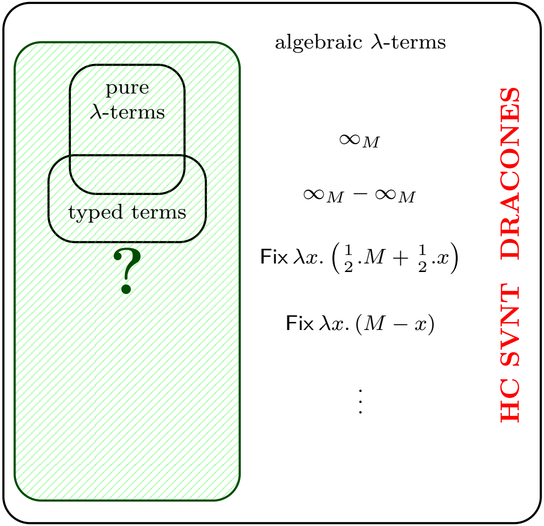
Normalizing Taylor expansions



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Differentiation of λ-terms (Ehrhard-Regnier, 2003-2004)

- differential λ-calculus
- a finitary fragment: resource λ-calculus
  = the target of Taylor expansion

Resource λ-calculus

$$\begin{array}{rcccll} \Delta & \ni & s, t, \ldots & ::= & x \mid \lambda x.s \mid \langle s \rangle\ \bar{t} & \text{(terms)} \\ !\Delta & \ni & \bar{s}, \bar{t}, \ldots & ::= & [s_1, \ldots, s_n] & \text{(monomials)} \end{array}$$

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Resource reduction

$$\langle \lambda x.s \rangle\ \bar{t} \to_\partial \partial_x s \cdot \bar{t} \quad \text{(anywhere)}$$

Semantically: *(in a typed setting)*

$$\partial_x s \cdot [s_1, \ldots, s_n] = \left(\frac{\partial^n s}{\partial x^n}\right)_{x=0} \cdot (s_1, \ldots, s_n)$$

Syntactically:

$$\partial_x s \cdot \bar{t} = \begin{cases} \sum_{f \in \mathfrak{S}_n} s\left[t_{f(1)}, \ldots, t_{f(n)} / x_1, \ldots, x_n\right] & \text{if } \mathbf{n}_x(s) = \#\bar{t} = n \\ 0 & \text{otherwise} \end{cases}$$

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- Linearity: $\lambda x.0 = 0$, $\langle s \rangle\ [t_1 + t_2, u] = \langle s \rangle\ [t_1, u] + \langle s \rangle\ [t_2, u]$, ...
- Resource reduction preserves free variables, is size-decreasing, strongly confluent and strongly normalizing.

Taylor expansion of λ-terms

Many models related with LL validate:

$$(M)\ N = \sum_{n \in \mathbf{N}} \frac{1}{n!} \langle M \rangle\ N^n \text{ where } N^n = [N, \ldots, N]$$

In those models $[\![M]\!] = [\![\Theta\,(M)]\!]$:

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Taylor expansion: $\Theta(M)$

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Normalizing Taylor expansions

We want to set

$$\mathsf{NF}\left(\sum_{i\in I} a_i.s_i\right) = \sum_{i\in I} a_i.\mathsf{NF}\left(s_i\right)$$

Normalizing vectors fails in general!

$\mathsf{NF}\left(\sum_{n\in\mathbf{N}} \langle\lambda x.x\rangle^n [y]\right) = ?$ (with $\langle\lambda x.x\rangle^n [y] = \langle\lambda x.x\rangle\ [\langle\lambda x.x\rangle\ [\cdots [y]\cdots]]$)

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Theorem (Ehrhard-Regnier, 2004 (published in TCS in 2008))

For all $M \in \Lambda$ and $t \in \Delta$, there is at most one $s \in \mathsf{support}\left(\Theta\left(M\right)\right)$ such that $\mathsf{NF}\left(s\right)_t \neq 0$.

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Theorem (Ehrhard-Regnier, CiE 2006)

$\mathsf{NF}\left(\Theta\left(M\right)\right) \simeq \mathsf{BT}\left(M\right)$ *(in particular* $\mathsf{NF}\left(\Theta\left(\Omega\right)\right) = 0 \simeq \bot$*)*

Normalizable resource vectors

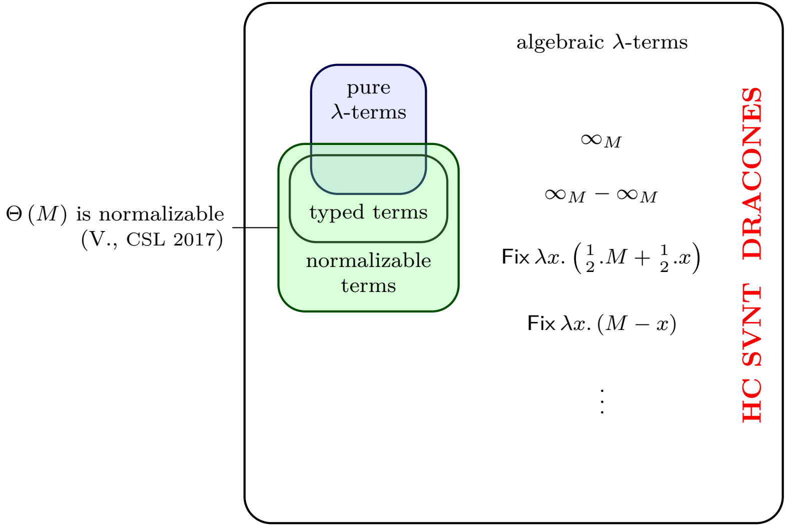
Definition

Say $\sigma \in \mathbf{S}^{\Delta}$ *is normalizable if, for all* $t \in \Delta$*, there are finitely many* $s \in \mathsf{support}\,(\sigma)$ *such that* $\mathsf{NF}\,(s)_t \neq 0$.

Lemma (V., CSL 2017)

$\Theta\,(M)$ *is normalizable as soon as* M *is.*

Proof. Generalize (Ehrhard, LICS 2010) and (Pagani–Tasson–V., FoSSaCS 2016): introduce a *finiteness structure* on resource terms and show it is closed under anti-left-β-reduction. □



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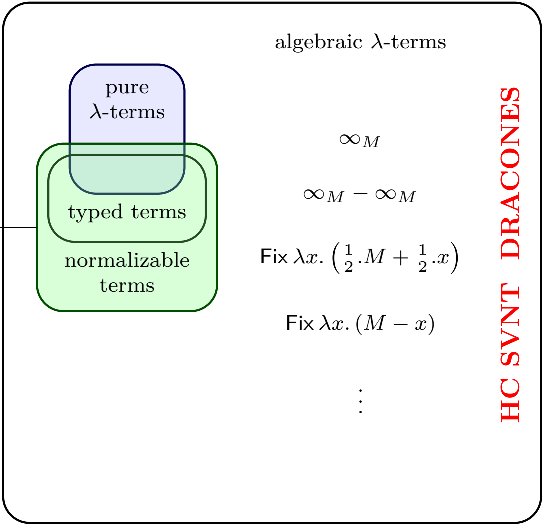

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We design a reduction relation $\widetilde{\Rrightarrow_\partial}$ on $\mathbf{S}^\Delta$ such that:

- If $M \to_\beta N$ then $\Theta(M) \widetilde{\Rrightarrow_\partial} \Theta(N)$.
- If $\sigma \in \mathbf{S}^\Delta$ is normalizable and $\sigma \widetilde{\Rrightarrow_\partial} \sigma'$ then σ' is normalizable and $\mathsf{NF}(\sigma) = \mathsf{NF}(\sigma')$.

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Parallel reduction on resource vectors

$$\sum_{i \in i} a_i.s_i \widetilde{\Rightarrow_\partial} \sum_{i \in I} a_i.\sigma'_i$$

whenever $s_i \Rightarrow_\partial \sigma'_i$ for all $i \in I$, where $\Rightarrow_\partial$ is the parallel version of $\to_\partial$.

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$$\sum_{i \in i} a_i.s_i \widetilde{\Rrightarrow_\partial} \sum_{i \in I} a_i.\sigma'_i$$

whenever $s_i \Rrightarrow_\partial \sigma'_i$ for all $i \in I$, where $\Rrightarrow_\partial$ is the parallel version of $\to_\partial$.

Technical issues

- Given $\sigma = \sum_{i \in i} a_i.s_i$ and a family of reductions $(s_i \Rrightarrow_\partial \sigma'_i)_{i \in I}$, $\sum_{i \in I} a_i.\sigma'_i$ might not converge.
- Actually need an extra condition on the family of reductions to avoid inconsistencies (if $-1 \in \mathbf{S}$).

$\mathsf{NF}(\Theta(M)) = \Theta(\mathsf{NF}(M))$

We design a reduction relation $\widetilde{\Rrightarrow_\partial}$ on $\mathbf{S}^\Delta$ such that:

- If $M \to_\beta N$ then $\Theta(M) \widetilde{\Rrightarrow_\partial} \Theta(N)$.
- If $\sigma \in \mathbf{S}^\Delta$ is normalizable and $\sigma \widetilde{\Rrightarrow_\partial} \sigma'$ then σ' is normalizable and $\mathsf{NF}(\sigma) = \mathsf{NF}(\sigma')$.

Then it is sufficient to follow a reduction from M to its normal form.

Parallel reduction on resource vectors

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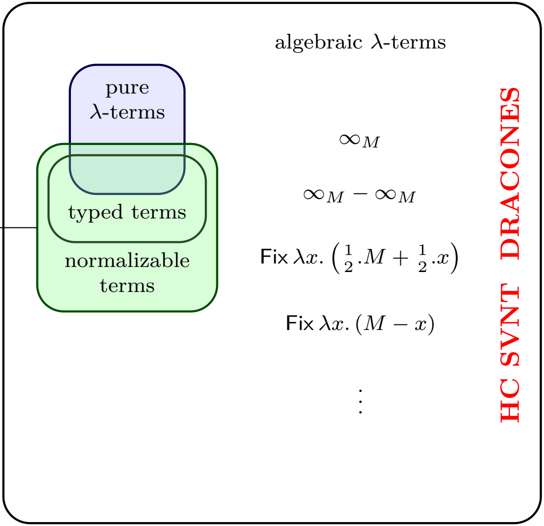
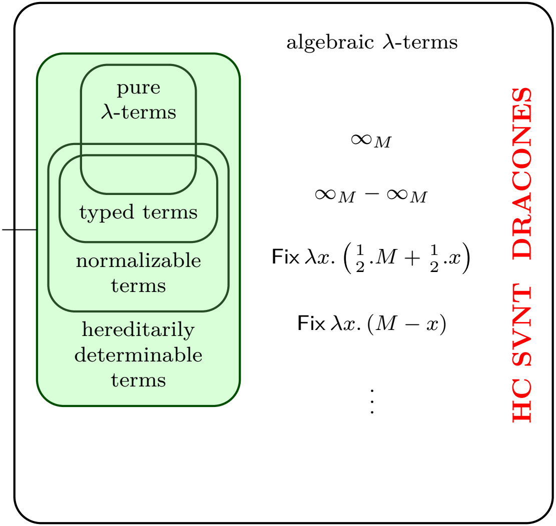
But it is always OK when we follow β-reductions.

$\mathsf{NF}(\Theta(M)) = \Theta(\mathsf{NF}(M))$
(V., CSL 2017)



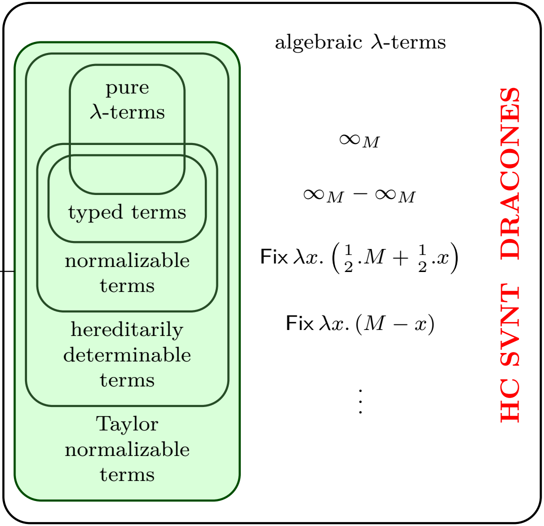
$\Theta(\mathsf{NA}_n(M)) \underset{n\to\infty}{\longrightarrow} \mathsf{NF}(\Theta(M))$

(V., `arXiv:1706.04700 [cs.LO]`)



$\mathsf{NF}(\Theta(-))$ is stable under β

(V., CSL 2017)



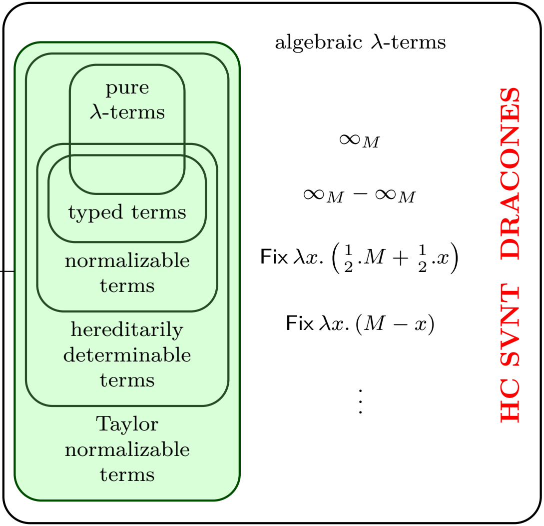
Normalizing Taylor expansions: living alongside dragons

$\mathsf{NF}(\Theta(-))$ is stable under β

(V., CSL 2017)

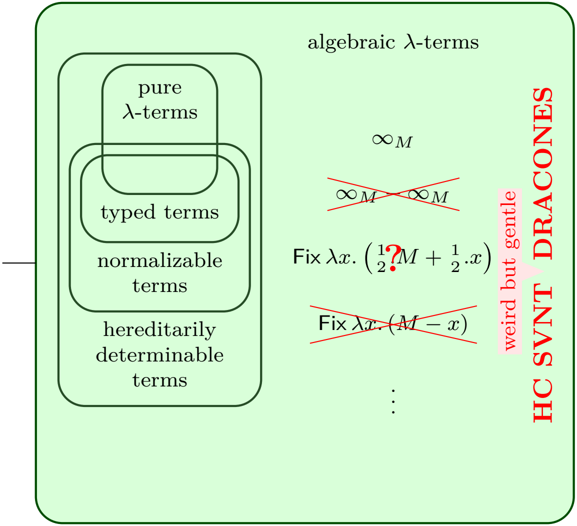
Proposal

$\mathsf{BT}\,(M) := \mathsf{NF}\,(\Theta\,(M))$



If $\mathbf{S}$ *is complete (say* $\mathbf{S} = [0, +\infty]$*):*

$\mathsf{NF}(\Theta(M)) = \Theta(\mathsf{BT}(M))$
(Tsukada-Asada-Ong, LICS 2017)



Conclusion

Normalization and Taylor expansion commute

provided it makes sense to normalize

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Further work

- unify with TAO's results
- does $\mathsf{NF}(\Theta(M))$ coincide with existing notions of (non extensional) Böhm trees?
- when is Taylor expansion injective on normal forms? $\rightsquigarrow$ might lead to injectivity results for a class of quantitative denotational models
- adapt those results to proof nets (WIP within the GDRI-LL)
- generalization to infinitary λ-calculi?

The end

Questions?