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On Khovanov's categorification of the Jones polynomial

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Abstract The working mathematician fears complicated words but loves pictures and diagrams. We thus give a no-fancy-anything picture rich glimpse into Khovanov's novel construction of "the categorification of the Jones polynomial". For the same low cost we also provide some computations, including one that shows that Khovanov's invariant is strictly stronger than the Jones polynomial and including a table of the values of Khovanov's invariant for all prime knots with up to 11 crossings.

AMS Classification 57M25

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1 Introduction

In the summer of 2001 the author of this note spent a week at Harvard University visiting David Kazhdan and Dylan Thurston. Our hope for the week was to understand and improve Khovanov's seminal work on the categorification of the Jones polynomial [Kh1, Kh2]. We've hardly achieved the first goal and certainly not the second; but we did convince ourselves that there is something very new and novel in Khovanov's work both on the deep conceptual level (not discussed here) and on the shallower surface level. For on the surface level Khovanov presents invariants of links which contain and generalize the Jones polynomial but whose construction is like nothing ever seen in knot theory before. Not being able to really digest it we decided to just chew some, and then provide our output as a note containing a description of his construction, complete and consistent and accompanied by computer code and examples but stripped of all philosophy and of all the linguistic gymnastics that is necessary for the philosophy but isn't necessary for the mere purpose of having a working construction. Such a note may be more accessible than the original papers. It may lead more people to read Khovanov at the source, and maybe somebody reading such a note will figure out what the Khovanov invariants really are. Congratulations! You are reading this note right now.

1.1 Executive summary In very brief words, Khovanov's idea is to replace the Kauffman bracket $\langle L\rangle$ of a link projection L by what we call "the Khovanov bracket" $[\![L]\!]$, which is a chain complex of graded vector spaces whose graded Euler characteristic is $\langle L\rangle$. The Kauffman bracket is defined by the axioms

$$\langle\emptyset\rangle = 1; \quad \langle\bigcirc L\rangle = (q+q^{-1})\langle L\rangle; \quad \langle\times\rangle = \langle\asymp\rangle - q\langle)(\rangle.$$

Likewise, the definition of the Khovanov bracket can be summarized by the axioms

$$[\![\emptyset]\!] = 0 \to \mathbb{Z} \to 0; \quad [\![\bigcirc L]\!] = V \otimes [\![L]\!]; \quad [\![\times]\!] = \mathcal{F}\left(0 \to [\![\asymp]\!] \xrightarrow{d} [\![)(]\!]\{1\} \to 0\right).$$

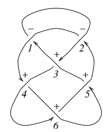
Here V is a vector space of graded dimension $q+q^{-1}$, the operator $\{1\}$ is the "degree shift by 1" operation, which is the appropriate replacement of "multiplication by q", $\mathcal{F}$ is the "flatten" operation which takes a double complex to a single complex by taking direct sums along diagonals, and a key ingredient, the differential d, is yet to be defined.

The (unnormalized) Jones polynomial is a minor renormalization of the Kauffman bracket, $\hat{J}(L) = (-1)^{n_-}q^{n_+-2n_-}\langle L\rangle$. The Khovanov invariant $\mathcal{H}(L)$ is the homology of a similar renormalization $[\![L]\!][-n_-]\{n_+-2n_-\}$ of the Khovanov bracket. The "main theorem" states that the Khovanov invariant is indeed a link invariant and that its graded Euler characteristic is $\hat{J}(L)$. Anything in $\mathcal{H}(L)$ beyond its Euler characteristic appears to be new, and direct computations show that there really is more in $\mathcal{H}(L)$ than in its Euler characteristic.

1.2 Acknowledgements I wish to thank David Kazhdan and Dylan Thurston for the week at Harvard that led to writing of this note and for their help since then. I also wish to thank G. Bergman, S. Garoufalidis, J. Hoste, V. Jones, M. Khovanov, A. Kricker, G. Kuperberg, A. Stoimenow and M. Thistlethwaite for further assistance, comments and suggestions.

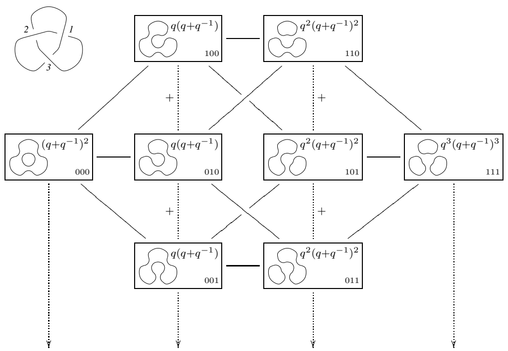
2 The Jones polynomial

All of our links are oriented links in an oriented Euclidean space. We will present links using their projections to the plane as shown in the example on the right. Let L be a link projection, let $\mathcal{X}$ be the set of crossings of L, let $n=|\mathcal{X}|$, let us number the elements of $\mathcal{X}$ from 1 to n in some arbitrary way and let us write $n=n_++n_-$ where n_+ (n_-) is the number of right-handed (left-handed) crossings in $\mathcal{X}$. (again, look to the right).



$n_+=4$; $n_-=2$

Recall that the Kauffman bracket [Ka] of L is defined by the formulas[1] $\langle\emptyset\rangle = 1$, $\langle\bigcirc L\rangle = (q+q^{-1})\langle L\rangle$ and $\langle\times\rangle = \langle\asymp\rangle - q\langle\rangle\!\langle\rangle$, that the unnormalized Jones polynomial is defined by $\hat{J}(L) = (-1)^{n_-}q^{n_+-2n_-}\langle L\rangle$, and that the Jones polynomial of L is simply $J(L) := \hat{J}(L)/(q+q^{-1})$. We name $\asymp$ and $)($ the 0- and 1-smoothing of $\times$, respectively. With this naming convention each vertex $\alpha \in \{0,1\}^{\mathcal{X}}$ of the n-dimensional cube $\{0,1\}^{\mathcal{X}}$ corresponds in a natural way to a "complete smoothing" S_α of L where all the crossings are smoothed and the result is just a union of planar cycles. To compute the unnormalized Jones polynomial, we replace each such union S_α of (say) k cycles with a term of the form $(-1)^r q^r (q+q^{-1})^k$, where r is the "height" of a smoothing, the number of 1-smoothings used in it. We then sum all these terms over all $\alpha \in \{0,1\}^{\mathcal{X}}$ and multiply by the final normalization term, $(-1)^{n_-}q^{n_+-2n_-}$. Thus the whole procedure (in the case of the trefoil knot) can be depicted as in the diagram below. Notice that in this diagram we have split the summation over the vertices of $\{0,1\}^{\mathcal{X}}$ to a summation over vertices of a given height followed by a summation over the possible heights. This allows us to factor out the $(-1)^r$ factor and turn the final summation into an alternating summation:



$$(q+q^{-1})^2 \quad - \quad 3q(q+q^{-1}) \quad + \quad 3q^2(q+q^{-1})^2 \quad - \quad q^3(q+q^{-1})^3 \tag{1}$$

$$= q^{-2}+1+q^2-q^6 \xrightarrow[\text{(with } (n_+,n_-)=(3,0))]{\cdot(-1)^{n_-}q^{n_+-2n_-}} q+q^3+q^5-q^9 \xrightarrow{\cdot(q+q^{-1})^{-1}} J(\text{trefoil}) = q^2+q^6-q^8.$$

[1] Our slightly unorthodox conventions follow [Kh1]. At some minor regrading and renaming cost, we could have used more standard conventions just as well.

3 Categorification

3.1 Spaces

Khovanov's "categorification" idea is to replace polynomials by graded vector spaces[2] of the appropriate "graded dimension", so as to turn the Jones polynomial into a homological object. With the diagram (1) as the starting point the process is straight forward and essentially unique. Let us start with a brief on some necessary generalities:

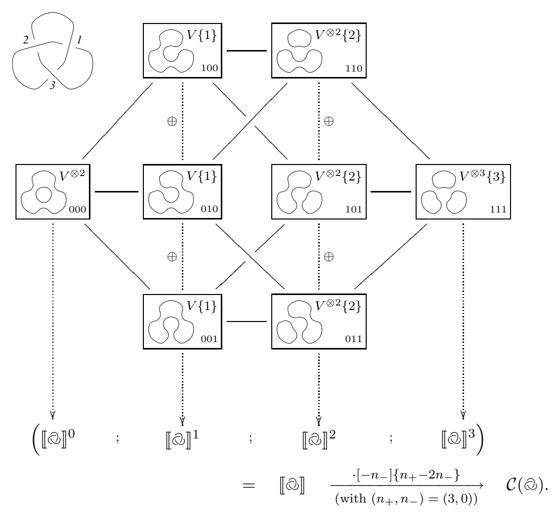
Definition 3.1 Let $W = \bigoplus_m W_m$ be a graded vector space with homogeneous components $\{W_m\}$. The graded dimension of W is the power series $q\dim W := \sum_m q^m \dim W_m$.

Definition 3.2 Let $\cdot\{l\}$ be the "degree shift" operation on graded vector spaces. That is, if $W = \bigoplus_m W_m$ is a graded vector space, we set $W\{l\}_m := W_{m-l}$, so that $q\dim W\{l\} = q^l\, q\dim W$.

Definition 3.3 Likewise, let $\cdot[s]$ be the "height shift" operation on chain complexes. That is, if $\bar{\mathcal{C}}$ is a chain complex $\ldots \to \bar{\mathcal{C}}^r \xrightarrow{d^r} \bar{\mathcal{C}}^{r+1} \ldots$ of (possibly graded) vector spaces (we call r the "height" of a piece $\bar{\mathcal{C}}^r$ of that complex), and if $\mathcal{C} = \bar{\mathcal{C}}[s]$, then $\mathcal{C}^r = \bar{\mathcal{C}}^{r-s}$ (with all differentials shifted accordingly).

Armed with these three notions, we can proceed with ease. Let L, $\mathcal{X}$, n and $n_\pm$ be as in the previous section. Let V be the graded vector space with two basis elements $v_\pm$ whose degrees are ± 1 respectively, so that $q\dim V = q+q^{-1}$. With every vertex $\alpha \in \{0,1\}^{\mathcal{X}}$ of the cube $\{0,1\}^{\mathcal{X}}$ we associate the graded vector space $V_\alpha(L) := V^{\otimes k}\{r\}$, where k is the number of cycles in the smoothing of L corresponding to α and r is the height $|\alpha| = \sum_i \alpha_i$ of α (so that $q\dim V_\alpha(L)$ is the polynomial that appears at the vertex α in the cube at (1)). We then set the rth chain group $[\![L]\!]^r$ (for $0 \le r \le n$) to be the direct sum of all the vector spaces at height r: $[\![L]\!]^r := \bigoplus_{\alpha:r=|\alpha|} V_\alpha(L)$. Finally (for this long paragraph), we gracefully ignore the fact that $[\![L]\!]$ is not yet a complex, for we have not yet endowed it with a differential, and we set $\mathcal{C}(L) := [\![L]\!][-n_-]\{n_+-2n_-\}$. Thus the diagram (1) (in the case of the trefoil knot) becomes:

[2] Everything that we do works just as well (with some linguistic differences) over $\mathbb{Z}$. In fact, in [Kh1] Khovanov works over the even more general ground ring $\mathbb{Z}[c]$ where $\deg c = 2$.



(2)

The graded Euler characteristic $\chi_q(\mathcal{C})$ of a chain complex $\mathcal{C}$ is defined to be the alternating sum of the graded dimensions of its homology groups, and, if the degree of the differential d is 0 and all chain groups are finite dimensional, it is also equal to the alternating sum of the graded dimensions of the chain groups. A few paragraphs down we will endow $\mathcal{C}(L)$ with a degree 0 differential. This granted and given that the chains of $\mathcal{C}(L)$ are already defined, we can state and prove the following theorem:

Theorem 1 *The graded Euler characteristic of $\mathcal{C}(L)$ is the unnormalized Jones polynomial of L:*

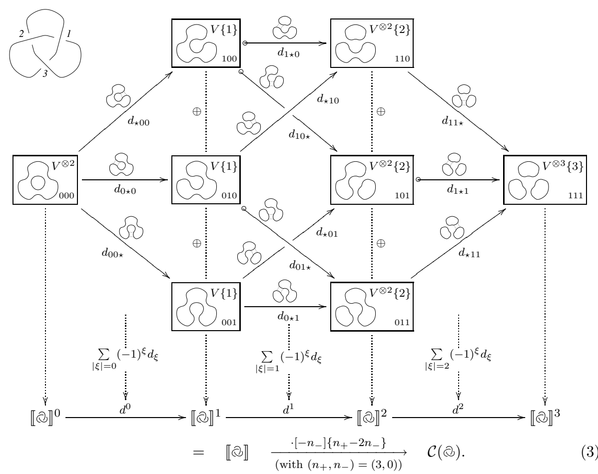
$$\chi_q(\mathcal{C}(L)) = \hat{J}(L).$$

Proof The theorem is trivial by design; just compare diagrams (1) and (2) and all the relevant definitions. Thus rather than a proof we comment on the statement and the construction preceding it: If one wishes our theorem to hold,

everything in the construction of diagram (2) is forced, except the height shift $[-n_-]$. The parity of this shift is determined by the $(-1)^{n_-}$ factor in the definition of $\hat{J}(L)$. The given choice of magnitude is dictated within the proof of Theorem 2. □

3.2 Maps

Next, we wish to turn the sequence of spaces $\mathcal{C}(L)$ into a chain complex. Let us flash the answer upfront, and only then go through the traditional ceremony of formal declarations:



(3)

This diagram certainly looks threatening, but in fact, it's quite harmless. Just hold on tight for about a page! The chain groups $[\![L]\!]^r$ are, as we have already seen, direct sums of the vector spaces that appear in the vertices of the cube along the columns above each one of the $[\![L]\!]^r$ spaces. We do the same for the arrows d^r — we turn each edge ξ of the cube to map between the vector spaces at its ends, and then we add up these maps along columns as shown above.

The edges of the cube $\{0,1\}^{\mathcal{X}}$ can be labeled by sequences in $\{0,1,\star\}^{\mathcal{X}}$ with just one $\star$ (so the tail of such an edge is found by setting $\star \to 0$ and the head by setting $\star \to 1$). The height $|\xi|$ of an edge ξ is defined to be the height of its tail, and hence if the maps on the edges are called d_ξ (as in the diagram), then the vertical collapse of the cube to a complex becomes $d^r := \sum_{|\xi|=r} (-1)^\xi d_\xi$.

It remains to explain the signs $(-1)^\xi$ and to define the per-edge maps d_ξ. The former is easy. To get the differential d to satisfy $d \circ d = 0$, it is enough that all square faces of the cube would anti-commute. But it is easier to arrange the d_ξ's so that these faces would (positively) commute; so we do that and then sprinkle signs to make the faces anti-commutative. One may verify that this can be done by multiplying d_ξ by $(-1)^\xi := (-1)^{\sum_{i<j}\xi_i}$, where j is the location of the $\star$ in ξ. In diagram (3) we've indicated the edges ξ for which $(-1)^\xi = -1$ with little circles at their tails. The reader is welcome to verify that there is an odd number of such circles around each face of the cube shown.

It remains to find maps d_ξ that make the cube commutative (when taken with no signs) and that are of degree 0 so as not to undermine Theorem 1. The space V_α on each vertex α has as many tensor factors as there are cycles in the smoothing S_α. Thus we put these tensor factors in V_α and cycles in S_α in bijective correspondence once and for all. Now for any edge ξ, the smoothing at the tail of ξ differs from the smoothing at the head of ξ by just a little: either two of the cycles merge into one (see say $\xi = 0\star 0$ above) or one of the cycles splits in two (see say $\xi = 1\star 1$ above). So for any ξ, we set d_ξ to be the identity on the tensor factors corresponding to the cycles that don't participate, and then we complete the definition of ξ using two linear maps $m : V \otimes V \to V$ and $\Delta : V \to V \otimes V$ as follows:

 $\longrightarrow \left(V \otimes V \xrightarrow{m} V\right) \qquad m : \begin{cases} v_+ \otimes v_- \mapsto v_- & v_+ \otimes v_+ \mapsto v_+ \\ v_- \otimes v_+ \mapsto v_- & v_- \otimes v_- \mapsto 0 \end{cases}$ (4)

 $\longrightarrow \left(V \xrightarrow{\Delta} V \otimes V\right) \qquad \Delta : \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- \mapsto v_- \otimes v_- \end{cases}$ (5)

We note that because of the degree shifts in the definition of the V_α's and because we want the d_ξ's to be of degree 0, the maps m and Δ must be of degree -1. Also, as there is no canonical order on the cycles in S_α (and hence on the tensor factors of V_α), m and Δ must be commutative and co-commutative respectively. These requirements force the equality $m(v_+ \otimes v_-) = m(v_- \otimes v_+)$ and force the values of m and Δ to be as shown above up to scalars.