Chern Simons theory and the volume of three-manifolds

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Volume of representations

Let (G, X) be either the 3-dimensional hyperbolic geometry

$$(\mathrm{PSL}(2;\mathbb{C}), \mathbf{H}^3)$$

or the Seifert geometry

$$(\mathrm{Iso}_0\widetilde{\mathrm{SL}_2(\mathbb{R})}, \widetilde{\mathrm{SL}_2(\mathbb{R})}) \text{ where } \mathrm{Iso}_0\widetilde{\mathrm{SL}_2(\mathbb{R})} = \mathbb{R} \times_{\mathbb{Z}} \widetilde{\mathrm{SL}_2(\mathbb{R})}$$

In either case denote by ω_X the G-invariant volume form on X. Let M be a closed, connected, oriented 3-manifold. For each $\rho\colon \pi_1 M \to G$, there is a *developing map* $D_\rho\colon \widetilde{M} \to X$ which is equivariant. Then a volume can be defined by

$$\mathrm{vol}_G(M, \rho) = \left| \int_M D_\rho^* \omega_X \right|$$

Maximal volumes

According to Reznikov in case $G = \mathrm{PSL}(2;\mathbb{C})$ and Goldman-Brooks in case $G = \mathrm{Iso}_0(\widetilde{\mathrm{SL}_2(\mathbb{R})})$ the set

$$\mathrm{vol}(M, G) = \{\mathrm{vol}_G(M, \rho), \rho\colon\ \pi_1 M \to G\}$$

is always finite. Therefore it makes sense to define the hyperbolic volume

$$\mathrm{HV}(M) = \max \mathrm{vol}(M, \mathrm{PSL}(2;\mathbb{C}))$$

and the Seifert volume

$$\mathrm{SV}(M) = \max \mathrm{vol}(M, \mathrm{Iso}_0\widetilde{\mathrm{SL}_2(\mathbb{R})})$$

Maximal volumes

The hyperbolic and Seifert volume satisfy the following functorial property: for any non-zero degree map $f\colon N \to M$ (and in particular for any finite covering)

$$\mathrm{SV}(N) \geq |\mathrm{deg} f| \mathrm{SV}(M) \text{ and } \mathrm{HV}(N) \geq |\mathrm{deg} f| \mathrm{HV}(M)$$

It is unclear whether these inequalities turn to equalities in the case of finite covering maps.

Volumes of geometric manifolds

Let M be a closed oriented geometric 3-manifold.

Theorem (Reznikov)

If M is hyperbolic then $\mathrm{HV}(M)$ is reached by the volume of a representation iff it is faithful and discrete and $\mathrm{HV}(M)$ is the volume of M for the hyperbolic metric.
If M is geometric but not hyperbolic then $\mathrm{HV}(M) = 0$.

Theorem (Goldman-Brooks)

The same statement is true if M is an $\widetilde{\mathrm{SL}_2(\mathbb{R})}$-manifold and $\mathrm{SV}(M) = 4\pi^2 \frac{\chi^2(\mathcal{O}_M)}{|e(M \to \mathcal{O}_M)|}$.
If M is geometric but neither hyperbolic nor $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ then $\mathrm{SV}(M) = 0$.

Volumes of geometric manifolds

Describing the set $\mathrm{vol}(M, G)$ when M is hyperbolic seems a very hard question. However when M is a Seifert manifold this set can be made explicit, using the works of Goldman-Brooks and Eisendud-Hirsh-Neumann.
Suppose M supports the $\widetilde{\mathrm{SL}_2(\mathbb{R})}$-geometry and that its base 2-orbifold has a positive genus g. Then

$$\mathrm{vol}\left(M, \mathrm{Iso}\widetilde{\mathrm{SL}_2(\mathbb{R})}\right) = \left\{ \frac{4\pi^2}{|e(M)|} \left(\sum_{i=1}^{r} \frac{n_i}{a_i} - n \right)^2 \right\}$$

where $n_1, ..., n_r, n$ are integers such that

$$\sum_{i=1}^{r} \llcorner n_i/a_i \lrcorner - n \leq 2g-2, \;\; \sum_{i=1}^{r} \ulcorner n_i/a_i \urcorner - n \geq 2-2g$$

and $a_1, ..., a_r$ are the indices of the singular points of $\mathcal{O}_M$.

Geometric decomposition of 3-manifolds

Let M be a closed oriented and irreducible 3-manifold.
The geometrization of 3-manifolds implies that M can be decomposed along a family of tori and Klein bottles $\mathcal{T}_M$ such that each component of $M \setminus \mathcal{T}_M$ is geometric.
When M is non-geometric, i.e. $\mathcal{T}_M \neq \emptyset$, we write

$$M^* = M \setminus \mathcal{T}_M = \mathcal{S}(M) \cup \mathcal{H}(M)$$

where the components of $\mathcal{S}(M)$ are $\mathbf{H}^2 \times \mathbb{R}$ and those of $\mathcal{H}(M)$ are hyperbolic. When $\mathcal{H}(M) = \emptyset$, then M is a graph manifold.
Denote by $\tau \colon \partial M^* \to \partial M^*$ the sewing involution such that $M \simeq M^*/\tau$.

Seifert volume of graph manifolds

Proposition (DW)

Any closed oriented non-geometric graph manifold has a virtually positive Seifert Volume.

Remark

It is not clear whether the hypothesis "virtually" is needed because we don't know if there are examples of non-geometric graph manifolds with $\mathrm{SV} = 0$*.*
The geometric pieces of M^* *are virtual product so they don't contribute to the volume. The volume is positive rather because they are glued together in a non-trivial way (so that their geometries do not extend). Therefore one can expect that the volume of representations detects the sewing involution of* M*.*

Hyperbolic volume of 3-manifolds

Conjecture (M. Boileau)

A closed 3-manifold has a virtually positive HV *iff* $\|M\| > 0$*.*

By Reznikov $\mathrm{HV}(M) \leq \mu_3 \|M\|$ therefore manifolds with virtually positive hyperbolic volume must contain some hyperbolic pieces in their geometric decomposition.

Proposition (DW)

There are one-edged manifolds M *with* $\|M\| > 0$ *but* $\mathrm{HV}(M) = 0$*.*

Meanwhile

Proposition (DW)

If the dual graph of M *is a tree (in particular if* M *is a* $\mathbf{Q}HS$*) with* $\|M\| > 0$ *then* M *has a virtually positive hyperbolic volume.*

Notice that in this case the term "virtually" cannot be dropped.

Basic gauge theory

Let M be a closed oriented 3-manifold and denote by $p\colon \widetilde{M} \to M$ its universal covering.
Given a representation $\rho\colon \pi_1 M \to G$ we get a principal G-bundle

$$\xi\colon P = \widetilde{M} \times G/\pi_1 M \to M$$

defined by

$$\xi([x, g]) = p(x)$$

and endowed with a G-invariant horizontal foliation $\mathcal{F}_\rho$ which is the image of the trivial foliation on $\widetilde{M} \times G$.
Denote by $A \in \Omega^1(P; \mathfrak{g})$ a 1-form connection such that $\ker A = T\mathcal{F}_\rho$. Chern and Simons associated to any connection A a closed 3-form $Tf(A) \in \Omega^3(P; \mathbb{C})$ based on a degree 2 invariant polynomial $f\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$. If moreover $P \to M$ is trivial, the Chern-Simons invariant of A is defined after choosing a section δ of $P \to M$ by

$$\mathfrak{cs}_M(A, \delta) = \int_M \delta^* Tf(A)$$

The volume as a Chern-Simons invariant

When $G = \mathrm{PSL}(2;\mathbb{C})$ choose

$$f(A \otimes A) = P_1(A \otimes A) = -\frac{1}{8\pi^2}\mathrm{tr}A^2 \in \mathbb{C}$$

and using a computation of Yoshida we get

$$\Im \mathfrak{cs}_M(A,\delta) = -\frac{1}{\pi^2}\mathrm{vol}_G(M,\rho)$$

when $G = \mathrm{Iso}_0\widetilde{\mathrm{SL}(2;\mathbb{R})} = \mathbb{R} \times_{\mathbb{Z}} \widetilde{\mathrm{SL}_2(\mathbb{R})}$ choose

$$f(A \otimes A) = \mathrm{tr}X^2 + t^2 \in \mathbb{R}$$

where $A = X + t$ in the Lie algebra $\mathfrak{sl}(2;\mathbb{R}) \oplus \mathbb{R}$ and we get

$$\mathfrak{cs}_M(A,\delta) = \frac{2}{3}\mathrm{vol}_G(M,\rho)$$

Additivity Principle

Suppose $M = S \cup_{\partial S = \partial Q} Q$ with base points $x \in \mathrm{int} S$ and $y \in \mathrm{int} Q$. For each component T_i of $\partial S \cap \partial Q$ fix a basis (λ_i, μ_i) for $H_1(T_i; \mathbb{Z})$. Let $\varphi \colon \pi_1(S, x) \to G$ and $\psi \colon \pi_1(Q, y) \to G$ denote two representations such that

$$\varphi(\lambda_i) \stackrel{G}{\sim} \psi(\lambda_i) \text{ and } \varphi(\mu_i) \stackrel{G}{\sim} \psi(\mu_i) \tag{0.1}$$

$$\ker \varphi | H_1(T_i; \mathbb{Z}) = \ker \psi | H_1(T_i; \mathbb{Z}) \simeq \mathbb{Z} \tag{0.2}$$

and generated by a primitive curve denoted m_i in T_i. Denote $m = (m_1, ..., m_p)$.

Let A, resp. B, be a (flat) connection over S, resp. over Q, corresponding to φ, resp. ψ. By condition (0.1) there exists a gauge transformation g over S, resp. h over Q, such that $g * A$ and $h * B$ smoothly match over the T_i's giving rise to a global smooth and flat connection $C = g * A \cup h * B$ and therefore to a representation $\rho \colon \pi_1 M \to G$.

By condition (0.2) φ and ψ extend to $\widehat{\varphi}\colon\ \pi_1 S(m) \to G$ and $\widehat{\psi}\colon\ \pi_1 Q(m) \to G$ where

$$S(m) = S \bigcup_{\partial \mathbf{D}^2 = m_i} (\cup_i \mathbf{D}^2 \times \mathbf{S}^1) \text{ and } Q(m) = Q \bigcup_{\partial \mathbf{D}^2 = m_i} (\cup_i \mathbf{D}^2 \times \mathbf{S}^1)$$

In the same way the connections A and B extend to

$$\widehat{A} = A \cup (\cup_i A_i) \text{ and } \widehat{B} = B \cup (\cup_i A_i)$$

$$\begin{aligned} \mathfrak{cs}_M(C) &\equiv \mathfrak{cs}_S(A) + \mathfrak{cs}_Q(B) \\ &\equiv \mathfrak{cs}_{S(m)}(\widehat{A}) + \mathfrak{cs}_{Q(m)}(\widehat{B}) \end{aligned}$$

Applying the correspondence with the volume we get

$$\mathrm{vol}_G(M, \rho) = \mathrm{vol}_G(S(m), \widehat{\varphi}) + \mathrm{vol}_G(Q(m), \widehat{\psi})$$

Seifert volume of graph manifolds

In this section $G = \mathrm{Iso}_0\widetilde{\mathrm{SL}(2;\mathbb{R})}$.
Let M be a non-geometric closed graph manifold. Up to a finite covering we may assume:
each component of $\mathcal{T}_M$ is shared by two distinct Seifert pieces,
each component Q of $M^* = M \setminus \mathcal{T}_M$ is a product $F \times \mathbf{S}^1$, where F is a hyperbolic surface of large genus.
We fix a Seifert piece $S = F_g \times \mathbf{S}^1$ of M and we suppose for simplicity that $M = S \cup Q$, where $Q = F \times \mathbf{S}^1$. If S consists of p boundary components $T_1, ..., T_p$ we denote by μ_i, λ_i the generators of $H_1(T_i;\mathbb{Z})$ such that $\lambda_i = \partial F \cap T_i$ and $\mu_i = \mathbf{S}^1$.
The space $S(\mu) = S(\mu_1, ..., \mu_p)$ is a Seifert manifold with geometry $\mathbf{H}^2 \times \mathbb{R}$ or $\widetilde{\mathrm{SL}(2;\mathbb{R})}$ depending on whether $e(S(\mu)) = 0$ or $e(S(\mu)) \neq 0$.
Assume $e(S(\mu)) \neq 0$.

By the Eisendud-Hirsh-Neumann generalization of the Milnor-Wood inequality, if $\underbrace{g \gg e}(S(\mu))$ there exists a representation $\underbrace{\varphi\colon \pi_1 S} \to \mathrm{SL}(2;\mathbb{R}) \subset G$ factorizing through $\widehat{\varphi}\colon \pi_1 S(\mu) \to \mathrm{SL}(2;\mathbb{R})$ such that

$$\varphi(\mathrm{fibre}) = \mathrm{sh}1 \text{ and } \varphi(\lambda_i) \sim \mathrm{sh}\alpha_i$$

and by integration along the fiber just like in the Goldman Brooks paper we get

$$\mathrm{vol}_G(S(\mu), \widehat{\varphi}) = 2\pi^2 |e(S(\mu))|$$

We have to find a representation $\psi\colon\ \pi_1 Q \to G$ factorizing through $\widehat{\varphi}\colon\ \pi_1 Q(\mu) \to \widetilde{\mathrm{SL}(2;\mathbb{R})}$ such that $\psi(\lambda_i) \sim \mathrm{sh}\alpha_i$. But since the product $\lambda_1 .... \lambda_p$ is a commutator in F of length $l = g(F)$ it is sufficient to know if

$$\mathrm{sh}\alpha_1 .... \mathrm{sh}\alpha_p$$

can be written as a commutator of length $\leq l$. By a result of Eisendud-Hirsh-Neumann this is true provided $l \gg |\alpha_1 + ... + \alpha_p|$. Again this condition may be assumed up to a finite covering. Notice that

$$\psi\colon\ \pi_1 Q \to \widetilde{\mathrm{SL}(2;\mathbb{R})}$$

factors through $\pi_1 F$ and therefore

$$\mathrm{vol}_G(Q(h), \widehat{\psi}) = 0$$

by a cohomological dimension argument.

Thurston Hyperbolic Dehn filling theorem

Denote by Q a compact manifold whose interior admits a complete finite volume hyperbolic metric. Denote by $T_1, ..., T_p$ the boundary components of M and assume each T_i is endowed with a homological basis (μ_i, λ_i).
Using the Thurston's theorem we deform the faithful and discrete representation to $\varphi\colon \pi_1 Q \to \mathrm{PSL}(2;\mathbb{C})$ sending λ_i to a hyperbolic isometry and μ_i to an order q rotation (for q big enough).
Next consider a $q \times q$-characteristic covering $p\colon \widetilde{Q} \to Q$. The induced representation $\rho = \varphi|\pi_1 \widetilde{Q}$ factorizes through $\widehat{\rho}\colon \pi_1 \widetilde{Q}(\widetilde{\mu}) \to \mathrm{PSL}(2;\mathbb{C})$ such that

$$\mathrm{vol}_{\mathrm{PSL}(2;\mathbb{C})}(\widetilde{Q}(\widetilde{\mu}), \widehat{\rho}) > 0$$

Hyperbolic volume of 3-manifolds

Suppose $M = Q \cup S$ and Q is hyperbolic.
Suppose $\partial S = \partial Q = T$ is connected. Then
$\mathrm{Rk}(H_1(\partial S;\mathbb{Z}) \to H_1(S;\mathbb{Z})) = 1$ and we choose the homological basis (λ, μ) in T such that μ is torsion.
Fix a prime number q (big enough) and a representation $\rho = \varphi|\pi_1\widetilde{Q}$ as above. Denote by (x, x^{-1}) the eigenvalues of $\varphi(\lambda)$.
By construction the elements $\rho(\widetilde{\lambda}_1), ..., \rho(\widetilde{\lambda}_p)$ are all conjugated to $\begin{pmatrix} x^q & 0 \\ 0 & x^{-q} \end{pmatrix}$ whereas $\widetilde{\mu}_1, ..., \widetilde{\mu}_p$ are sent trivially.
Next construct a finite covering $q \times q$-characteristic covering $r\colon \widetilde{S} \to S$ that can be glued to $p\colon \widetilde{Q} \to Q$ to define a global finite covering $\widetilde{M} \to M$ and we construct an abelian representation $\psi\colon \pi_1\widetilde{S} \to \mathrm{PSL}(2;\mathbb{C})$ defined by
$\psi(\widetilde{\lambda}_1) = ... = \psi(\widetilde{\lambda}_p) = \begin{pmatrix} x^q & 0 \\ 0 & x^{-q} \end{pmatrix}$ whereas $\widetilde{\mu}_1, ..., \widetilde{\mu}_p$ are sent trivially. This representation factorizes through $\widehat{\psi}\colon \pi_1\widetilde{S}(\widetilde{\mu}) \to \mathrm{PSL}(2;\mathbb{C})$ such that $\mathrm{vol}_{\mathrm{PSL}(2;\mathbb{C})}(\widetilde{S}(\widetilde{\mu}), \widehat{\psi}) = 0$.

Suppose $\partial S = \partial Q$ is no longer connected and $S = F \times \mathbf{S}^1$ where F is a surface with positive genus. Denote by h the $\mathbf{S}^1$-factor of S and by $d_1, ..., d_r$ the boundary component of F.
Set $(\lambda_i, \mu_i) = (d_i, h)$ for each $i = 1, ..., r$.
Passing to a finite covering we may assume there is a representation $\rho = \varphi|\pi_1 Q \to \mathrm{PSL}(2;\mathbb{C})$ which factorizes through $\widehat{\rho}\colon \pi_1 Q(\mu) \to \mathrm{PSL}(2;\mathbb{C})$ such that

$$\mathrm{vol}_{\mathrm{PSL}(2;\mathbb{C})}(Q(\mu), \rho) > 0$$

Since $d_1 ... d_r$ is a commutator in F and since any element of $\mathrm{PSL}(2;\mathbb{C})$ is a commutator then there exists a representation $\psi\colon \pi_1 S \to \mathrm{PSL}(2;\mathbb{C})$ defined by $\psi(d_1) = \rho(d_1), ..., \psi(d_r) = \rho(d_r)$ whereas h is sent trivially. This representation factorizes through $\widehat{\psi}\colon \pi_1 S(\mu) \to \mathrm{PSL}(2;\mathbb{C})$ such that

$$\mathrm{vol}_{\mathrm{PSL}(2;\mathbb{C})}(S(\mu), \widehat{\psi}) = 0$$

because $\|S(\mu)\| = 0$.

Detecting the sewing involution

The sewing involution can be made explicit by fixing a homological basis for each component of ∂M^*.
Suppose for instance $M = Q_1 \cup_\tau Q_2$ where ∂Q_i is connected.
In this case we say M is one-edged.
Fix a basis (λ_i, μ_i) for $H_1(\partial Q_i; \mathbb{Z})$ and we denote by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the matrix of τ in $\mathrm{SL}(2; \mathbb{Z})$ where

$$\tau_*(\mu_2) = a\mu_1 + c\lambda_1 \text{ and } \tau_*(\lambda_2) = b\mu_1 + d\lambda_1$$

When Q_i is a Seifert manifold, we fix a fibration with fibre h_i, we denote by $\mathcal{O}_i$ the 2-orbifold Q_i/h_i and the basis (λ_i, μ_i) is chosen s.t. $\lambda_i = h_i$ and μ_i is a section of $\partial\mathcal{O}_i$ in ∂Q_i.

Proposition (DW)

Let $M = Q_1 \cup_\tau Q_2$ be an one-edged graph manifold and denote by G the group $\mathrm{Iso}_e\widetilde{\mathrm{SL}_2(\mathbb{R})}$. There exists a n-fold $q \times q$-characteristic covering $\widetilde{M} \to M$, where n, q depend only on Q_1 and Q_2, and a representation $\varphi\colon \pi_1\widetilde{M} \to G$ such that

$$
\begin{aligned}
\frac{\mathrm{vol}_G(\widetilde{M}, \varphi)}{n} &= \frac{8\pi^2}{q^2} \text{ if } a = d = 0 \\
&= \frac{4\pi^2}{q^2|b|} \text{ if } c = 0 \\
&= \frac{4\pi^2}{q^2|ac|} \text{ if } ac \neq 0 \\
&= \frac{4\pi^2}{q^2|cd|} \text{ if } cd \neq 0.
\end{aligned}
$$

If Q is a 3-manifold with connected boundary and hyperbolic interior we denote by z_0 the shape of the cusp of $\mathrm{int}\, Q$.

Proposition (DW)

Let $M = Q \cup_\tau S$ be an one-edged manifold, where S is Seifert and Q hyperbolic. Then there exists a n-fold q-characteristic covering $\widetilde{M} \to M$, where n, q depend only on Q and S, and a representation $\varphi\colon \pi_1 \widetilde{M} \to \mathrm{PSL}(2;\mathbb{C})$ such that for any $\|(a,c)\|_2 > 2\pi(1+|z_0|^2)/\Im z_0$ then

$$\frac{\mathrm{vol}_{\mathrm{PSL}(2;\mathbb{C})}(\widetilde{M},\varphi)}{n} = \mathrm{vol} Q_+(a,c) + \frac{\pi(q-1)}{2q}\mathrm{length}(\gamma)$$

where γ is the geodesic added to Q_+ to complete the cusp with respect to the (a,c)-Dehn filling. The same statement is true for (b,d).

Using the computation of Neumann and Zagier we get

$$\frac{\mathrm{vol}_{\mathrm{PSL}(2;\mathbb{C})}(\widetilde{M},\varphi)}{n} = \mathrm{vol} Q_+ - \pi^2 \frac{\Im z_0}{q|a+z_0 c|^2} + O\left(\frac{1}{a^4+c^4}\right)$$