Unified quantum invariants of integral homology spheres associated to simple Lie algebras

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The Witten–Reshetikhin–Turaev invariants

Let $\mathfrak{g}$ be a finite dimensional, simple Lie algebra over $\mathbb{C}$.

Let $\mathcal{Z} \subset \mathbb{C}$ denote the set of roots of unity.

For a closed 3-manifold M, the WRT invariant of M associated to $\mathfrak{g}$ is a partial map

$$\tau_M^{\mathfrak{g}} \colon \mathcal{Z} \dashrightarrow \mathbb{C}.$$

Set $\mathcal{Z}_{\mathfrak{g}} = \{\zeta \in \mathcal{Z} \mid \tau_M^{\mathfrak{g}}(\zeta) \text{ is defined}\}$. We have a map

$$\tau_M^{\mathfrak{g}} \colon \mathcal{Z}_{\mathfrak{g}} \longrightarrow \mathbb{C}.$$

Remark

In the construction of $\tau_M^{\mathfrak{g}}(\zeta)$, we have to choose a certain root of ζ, but we ignore this choice in this talk for simplicity. Our result implies that, for $\mathbb{Z}$HS's, $\tau_M^{\mathfrak{g}}(\zeta)$ does not depend on this choice.

Quantum link invariants

Let $L = L_1 \cup \cdots \cup L_n \subset S^3$ be a framed link of n-components.

Let $V_1, \ldots, V_n$ be finite dimensional representations of the quantum group $U_h(\mathfrak{g})$.

Let $J_L(V_1, \ldots, V_n) \in \mathbb{Z}[q^{1/2\mathcal{D}}, q^{-1/2\mathcal{D}}]$ denote the quantum invariant of the framed link L colored by $V_1, \ldots, V_n$.

This notation extends to $J_L(x_1, \ldots, x_n)$, where each x_i is a *color*, i.e., a linear combination (with coefficients in a certain ring) of finite dimensional representations.

Kirby colors

A color Ω is called a *Kirby color* at a root of unity $\zeta \in \mathcal{Z}$ if

- $J_L(\Omega, \ldots, \Omega)|_{q=\zeta}$ is invariant under handle slides,
- $J_{U_\pm}(\Omega)|_{q=\zeta} \neq 0$.

If Ω is a Kirby color at ζ, then it is well-known that for a closed 3-manifold $M = S^3_L$, surgery on S^3 along a framed link L

$$\tau_M^{\mathfrak{g},\Omega}(\zeta) = \left. \frac{J_L(\Omega, \ldots, \Omega)}{J_{U_+}(\Omega)^{\sigma_+} J_{U_-}(\Omega)^{\sigma_-}} \right|_{q=\zeta}$$

is an invariant of M. Here $\sigma_\pm$ is the number of eigenvalues of the linking matrix of L of sign $\pm$.

The WRT invariant

At $\zeta \in \mathcal{Z}$ with $r = \text{order}(\zeta)$ sufficiently large ($r > d(h^\vee - 1)$), there is a Kirby color either $\Omega^{\mathfrak{g}}$ or $\Omega^{P\mathfrak{g}}$, which gives the $\mathfrak{g}$ WRT invariant or the $P\mathfrak{g}$ WRT invariant, respectively.

$$\tau_M^{\mathfrak{g}}(\zeta) = \tau_M^{\mathfrak{g},\Omega^{\mathfrak{g}}}(\zeta) \quad \text{for } \zeta \in \mathcal{Z}_{\mathfrak{g}},$$
$$\tau_M^{P\mathfrak{g}}(\zeta) = \tau_M^{\mathfrak{g},\Omega^{P\mathfrak{g}}}(\zeta) \quad \text{for } \zeta \in \mathcal{Z}_{P\mathfrak{g}}.$$

(The $P\mathfrak{g}$ WRT invariant $\tau_M^{P\mathfrak{g}}(\zeta)$ is the "projective version" of the WRT invariant.)

At $\zeta = 1$, the color $\Omega = 1$ (the trivial 1-dim. rep.) is a Kirby color. The associated invariant is trivial:

$$\tau_M^{\mathfrak{g},1}(1) = 1.$$

The ring $\widehat{\mathbb{Z}[q]}$

Definition (The completion ring $\widehat{\mathbb{Z}[q]}$)

Define a completion $\widehat{\mathbb{Z}[q]}$ of the polynomial ring $\mathbb{Z}[q]$ by

$$\widehat{\mathbb{Z}[q]} = \varprojlim_n \mathbb{Z}[q]/((q-1)(q^2-1)\cdots(q^n-1)).$$

Definition (Evaluation maps)

For each root of unity $\zeta \in \mathcal{Z}$, the evaluation map

$$\mathrm{ev}_\zeta \colon \mathbb{Z}[q] \longrightarrow \mathbb{Z}[\zeta], \quad f(q) \mapsto f(\zeta)$$

induces a ring homomorphism

$$\mathrm{ev}_\zeta \colon \widehat{\mathbb{Z}[q]} \longrightarrow \mathbb{Z}[\zeta].$$

Main result

Theorem (H ($\mathfrak{g} = sl_2$), Le–H (general $\mathfrak{g}$))

Let $\mathfrak{g}$ be a finite dimensional, simple complex Lie algebra. Then there is a (unique) invariant $J_M \in \widehat{\mathbb{Z}[q]}$ of an integral homology sphere M such that, for each $\zeta \in \mathcal{Z}_{\mathfrak{g}}$, we have

$$\mathrm{ev}_\zeta(J_M) = \tau_M^{\mathfrak{g}}(\zeta).$$

Remark

For $\mathfrak{g} = sl_2$, the above result has been generalized to rational homology spheres by Beliakova-Blanchet-Le, Le, Beliakova-Le, Beliakova-Bühler-Le.

Problem

Generalize the theorem to rational homology spheres for general $\mathfrak{g}$.

Corollaries

Corollary

J_M gives an extension of $\tau_M^{\mathfrak{g}}$ to the whole $\mathcal{Z}$. I.e., we may extend $\tau_M^{\mathfrak{g}}$ by

$$\tau_M^{\mathfrak{g}}(\zeta) = \mathrm{ev}_\zeta(J_M) \text{ for all } \zeta \in \mathcal{Z}.$$

Corollary

For all $\zeta \in \mathcal{Z}$, we have $\tau_M^{\mathfrak{g}}(\zeta) = \mathrm{ev}_\zeta(J_M) \in \mathbb{Z}[\zeta]$.

Remark

Some special cases (with $\mathrm{order}(\zeta)$ being prime) of the above corollary has been obtained by H. Murakami, Masbaum–Roberts, Masbaum–Wenzl, Takata–Yokota and Le.

Determination of J_M by WRT invariants

Definition (A topology on $\mathcal{Z}$)

Define a topology on the set $\mathcal{Z}$ as follows. For a subset $S \subset \mathcal{Z}$, a point $\zeta \in \mathcal{Z}$ is a limit point of S in $\mathcal{Z}$ if and only if there are infinitely many elements $\zeta' \in S$ such that order$(\zeta' \zeta^{-1})$ is a prime power.

Examples

(1) The set $\{\exp \frac{2\pi\sqrt{-1}}{p} \mid p = 2, 3, 5, 7, \ldots\}$ has a limit point 1.

(2) If p is a prime, then the set $\{\exp \frac{2\pi\sqrt{-1}}{p^e} \mid e = 0, 1, 2, \ldots\}$ has limit points $\exp \frac{2\pi\sqrt{-1}a}{p^e}$, $a \in \mathbb{Z}$, $e \geq 0$.

(3) The set $\{\exp \frac{2\pi\sqrt{-1}}{6^e} \mid e = 0, 1, 2, \ldots\}$, has no limit points.

Determination of J_M

Proposition (H)

If $S \subset \mathcal{Z}$ has at least one limit point in $\mathcal{Z}$, then the homomorphism

$$\widehat{\mathbb{Z}[q]} \longrightarrow \prod_{\zeta \in S} \mathbb{Z}[\zeta], \quad f(q) \mapsto (f(\zeta))_{\zeta \in S}$$

is injective.

Corollary

The invariant J_M is uniquely determined by the values of $\tau_M^{\mathfrak{g}}(\zeta)$ for ζ in a subset $S \subset \mathcal{Z}$ with a limit point in $\mathcal{Z}$.

Corollary

The invariant J_M is uniquely determined by the values of $\tau_M^{\mathfrak{g}}(\zeta)$ for $\zeta \in \mathcal{Z}_{\mathfrak{g}}$.

The WRT invariant at roots of unity at the same order

The WRT invariant at two roots of unity $\zeta, \zeta' \in \mathcal{Z}$ of the same order are related as follows. Note that there is a unique ring automorphism $\alpha\colon \mathbb{Z}[\zeta] \longrightarrow \mathbb{Z}[\zeta]$ such that $\alpha(\zeta) = \zeta'$.

Proposition

Let $\zeta \in \mathcal{Z}$, and let $\alpha\colon \mathbb{Z}[\zeta] \longrightarrow \mathbb{Z}[\zeta]$ be a ring automorphism. Then we have

$$\tau_M^{\mathfrak{g}}(\alpha(\zeta)) = \alpha(\tau_M^{\mathfrak{g}}(\zeta))$$

Proof.

$$\tau_M^{\mathfrak{g}}(\alpha(\zeta)) = \mathrm{ev}_{\alpha(\zeta)}(J_M) = \alpha(\mathrm{ev}_\zeta(J_M)) = \alpha(\tau_M^{\mathfrak{g}}(\zeta)).$$

□

Relation between $\tau_M^{\mathfrak{g}}$ at two different roots of unity

Let $\zeta, \zeta' \in \mathcal{Z}$. For $f(q) \in \widehat{\mathbb{Z}[q]}$, we have

$$f(q) = f(\zeta) + (q - \zeta)g(q),$$

for some $g(q) \in \widehat{\mathbb{Z}[\zeta][q]}$.

Here $\widehat{\mathbb{Z}[\zeta][q]} = \varprojlim_n \mathbb{Z}[\zeta][q]/((q-1)(q^2-1)\cdots(q^n-1))$.

Evaluating with $q = \zeta'$, we obtain

$$f(\zeta') = f(\zeta) + (\zeta' - \zeta)g(\zeta') \quad \text{in } \mathbb{Z}[\zeta, \zeta'].$$

Hence we have the following result.

Proposition

Let $\zeta, \zeta' \in \mathcal{Z}$. Then we have

$$\tau_M^{\mathfrak{g}}(\zeta) \equiv \tau_M^{\mathfrak{g}}(\zeta') \pmod{(\zeta - \zeta')}.$$

Remark

We have $(\zeta - \zeta') \subsetneq (1)$ if and only if $\mathrm{order}(\zeta'\zeta^{-1})$ *is a prime power.*

$\tau_M^{\mathfrak{g}}(1) = 1$

Proposition

For every integral homology sphere M, we have
$\tau_M^{\mathfrak{g}}(1) = \mathrm{ev}_1(J_M) = 1$.

Corollary

For $\zeta \in \mathcal{Z}$, we have

$$\tau_M^{\mathfrak{g}}(\zeta) \equiv 1 \pmod{(\zeta - 1)}.$$

Remark

The above corollary has been known for the special case where $\mathfrak{g} = sl_2$ *and* $\mathrm{order}(\zeta)$ *a prime by H. Murakami.*

Remark

Presumably, one can prove that $\tau_M^{\mathfrak{g}}(\zeta) = 1$ for some other $\zeta \in \mathcal{Z}$.
It is well-known that $\tau_M^{sl_2}(\zeta) = 1$ if $\mathrm{order}(\zeta) = 1, 2, 3, 6$.

Taylor expansions

Let $\zeta \in \mathcal{Z}$. The inclusion $\mathbb{Z}[q] \longrightarrow \mathbb{Z}[\zeta][q]$ induces

$$T_\zeta : \widehat{\mathbb{Z}[q]} \longrightarrow \mathbb{Z}[\zeta][[q-\zeta]]$$

since $(q-\zeta)^n$ divides $(q-1)(q^2-1)\cdots(q^{nr}-1)$, $r = \mathrm{order}(\zeta)$.
For $f(q) \in \widehat{\mathbb{Z}[q]}$, $T_\zeta(f(q))$ may be regarded as the *Taylor expansion* of $f(q)$ at ζ.

For $\zeta = 1$, we have

$$T_1 : \widehat{\mathbb{Z}[q]} \longrightarrow \mathbb{Z}[[q-1]].$$

Proposition

The Taylor expansion $T_1(J_M) \in \mathbb{Z}[[q-1]]$ is equal to the $\mathfrak{g}$ Ohtsuki series of M.

Remark

The existence of the Ohtsuki series of integral homology spheres are proved by for $\mathfrak{g} = sl_2$, and by Le for the general $\mathfrak{g}$.

Injectivity of the Taylor expansion

Proposition (H)

For $\zeta \in \mathcal{Z}$, the homomorphism

$$T_\zeta \colon \widehat{\mathbb{Z}[q]} \longrightarrow \mathbb{Z}[\zeta][[q-\zeta]]$$

is injective.

Corollary

The unified WRT invariant J_M, and hence the WRT invariants $\tau_M^{\mathfrak{g}}(\zeta)$, $\zeta \in \mathcal{Z}_{\mathfrak{g}}$, determined by the Ohtsuki series.

Corollary

The unified WRT invariant J_M, and hence the WRT invariants $\tau_M^{\mathfrak{g}}(\zeta)$, $\zeta \in \mathcal{Z}_{\mathfrak{g}}$ is determined by the Le–Murakami–Ohtsuki invariant.

Outline of proof

- The first step is to construct the invariant $J_M \in \widehat{\mathbb{Z}[q]}$ using the universal quantum invariant of bottom tangles associated to the quantum group $U_h = U_h(\mathfrak{g})$.
  Here we use neither the definition of $\tau_M^{\mathfrak{g}}(\zeta)$ nor the quantum link invariants associated to finite-dimensional representations of U_h.
- The second step is to show that $\mathrm{ev}_\zeta(J_M) = \tau_M^{\mathfrak{g}}(\zeta)$ for $\zeta \in \mathcal{Z}_{\mathfrak{g}}$.

Fact

Every integral homology sphere M can be obtained from S^3 by surgery along an algebraically-split, ± 1-framed link L.

Theorem (H)

Two algebraically-split ± 1-framed links L and L' in S^3 gives the same result of surgery if and only if they are related by a sequence of Hoste moves.

Here a *Hoste move* on an algebraically-split, ± 1-framed link is either surgery on an unknotted component or its inverse move.

Universal quantum invariants of bottom tangles

Definition

A *bottom tangle* is a framed tangle in a cube cosisiting of arc components, the endpoints of each of whose component are located side-by-side on a line in the bottom face of the cube.

Definition

Using the ribbon Hopf algebra structure of the quantum group U_h, one can define the universal quantum invariant J_T of an n-component bottom tangle T, which takes values in the n-fold completed tensor power $U_h^{\hat{\otimes} n}$ of U_h.

Full-twist forms

To define J_M, we need *full-twist forms* on U_h, which are partially-defined linear functionals on U_h

$$t_\pm \colon\ U_h \dashrightarrow \mathbb{C}[[h]]$$

which play the role of "performing ± 1-framed surgery on the closure of the component".

The full-twist form $t_\pm$ is defined by

$$t_\pm(x) = \langle x, r^{\pm 1} \rangle.$$

Here $\mathbf{r}^{\pm 1} \in U_h$ is the ribbon element, which is the universal invariant of the "twist tangle".

$\langle , \rangle$ is a partially defined bilinear map

$$\langle , \rangle \colon\ U_h \hat{\otimes} U_h \dashrightarrow \mathbb{C}[[h]],$$

called the quantum Killing form, which may be regarded as the adjoint of the universal invariant of the "clasp tangle".

Definition of J_M

Let M be an integral homology sphere. Let L be an n-component, algebraically-split, ± 1-framed link in S^3 such that $S^3_L \cong M$. Take a 0-framed bottom tangle T whose closure is isotopic to L, ignoring the framings.

Set

$$J_M = (t_{\epsilon_1} \otimes \cdots \otimes t_{\epsilon_n})(J_T) \in \mathbb{C}[[h]].$$

We can prove that

- J_M is well defined,
- J_M is invariant under the Hoste moves, hence gives an invariant of an integral homology sphere,
- $J_M \in \widehat{\mathbb{Z}[q]} (\subset \mathbb{Z}[[q-1]] \subset \mathbb{C}[[h]])$, where $q = \exp h$.