On the topology of the space of Kleinian once-punctured torus groups

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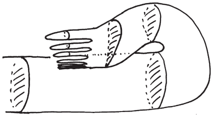
Spaces of Kleinian surface groups

Let Σ be a compact, oriented surface with negative Euler characteristics.

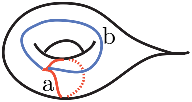
$$
\begin{aligned}
R(\Sigma) &:= \mathrm{Hom}^{TP}(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{C}))/\mathrm{PSL}(2,\mathbb{C}) \quad \text{(with alg. top.)} \\
AH(\Sigma) &:= \{\rho \in R(\Sigma) : \text{faitiful, discrete}\} \\
&= \{\text{hyperbolic manifold} \cong \mathrm{int}(\Sigma \times [0,1])\}
\end{aligned}
$$

- (Quasi-Fuchsian space) $QF(\Sigma) := \mathrm{int}(AH(\Sigma)) \cong T(\Sigma) \times T(\Sigma)$.
- (Density Theorem) $AH(\Sigma) = \overline{QF(\Sigma)}$.
- (McMullen, Anderson-Canary) $QF(\Sigma)$ self-bumps. i.e.
  $\exists \rho \in \partial QF(\Sigma)$, $\forall U$: small nbd. of ρ, $U \cap QF(\Sigma)$ is disconnected.
- (Bromberg, Magid) $AH(\Sigma)$ is not locally connected. i.e.
  $\exists \rho \in \partial AH(\Sigma)$, $\forall U$: small nbd. of ρ, $U \cap AH(\Sigma)$ is disconnected.



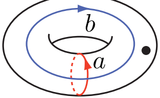
Teichmuller space of once-punctured torus

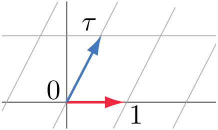
Let S be a once-punctured torus with $\pi_1(S) = \langle a, b \rangle$.
Teichmüller space $T(S)$ of S can be identified with the upper half-plane $\mathbf{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.



$$\begin{array}{rcl} T(S) & \longleftrightarrow & \mathbf{H} \\ \partial T(S) = PML(S) & \longleftrightarrow & \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \\ \mathcal{S} = \{\text{s.c.c. on } S\} & \longleftrightarrow & \hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\} \end{array}$$

Especially $a \leftrightarrow \infty$, $b \leftrightarrow 0$ and $a^{-1}b \leftrightarrow 1$.





The space of Kleinan punctured-torus groups

$$
\begin{aligned}
R(S) &= \{\rho : \pi_1(S) \to PSL(2,\mathbb{C}) \text{ with } \mathrm{tr}\rho([a,b]) = -2\}/PSL(2,\mathbb{C}) \\
AH(S) &= \{\rho \in R(S) : \text{faitiful, discrete}\}
\end{aligned}
$$

Theorem 1 (Minsky's Ending Lamination Theorem)

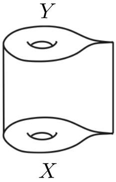
The canonical homeomorphism

$$Q : T(S) \times T(S) \to QF(S)$$

extends continuous bijection

$$Q : \left(\overline{T(S)} \times \overline{T(S)}\right) \setminus \Delta \to AH(S),$$

where $\overline{T(S)} = T(S) \cup \partial T(S)$, and Δ is the diagonal of $\partial T(S) \times \partial T(S)$.

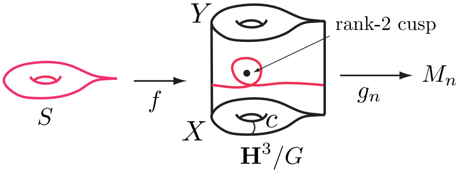


(McMullen, Anderson-Canary) Q^{-1} is not continuous.

Wrapping construction (Anderson-Canary)

Let

- G be a Kleinian group s.t. $\mathbf{H}^3/G \cong S \times (0,1) - c \times \{1/2\}$ for some $c \in \mathcal{S}$,
- $f : S \to \mathbf{H}^3/G$ be the immersed surface as in the figure,
- $g_n : \mathbf{H}^3/G \to M_n$ be the $(1,n)$-Dehn filling map at the rank-2 cusp.



Then we have

$$\rho_n := (g_n \circ f)_* = Q(\tau_c^n X, \tau_c^{2n} Y),$$

where τ_c is the Dehn twist around $c \in \mathcal{S}$. Since

$$Q(\tau_c^n X, \tau_c^{2n} Y) \to f_* \in AH(S) \quad \text{and} \quad (\tau_c^n X, \tau_c^{2n} Y) \to (c, c) \in \Delta,$$

Q^{-1} is not continuous.
Simillary, $Q(\tau_c^{pn} X, \tau_c^{(p+1)n} Y)$ converges as $n \to \infty$ for every $p \in \mathbb{Z}$.

Question

Suppose $(X_n, Y_n) \to (X_\infty, X_\infty) \in \Delta$. When does $Q(X_n, Y_n)$ converge or diverge in $AH(S)$?

(Ohshika) If $X_\infty \in PML(S) - \mathcal{S}$, then $Q(X_n, Y_n) \to \infty$.

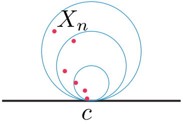
Theorem 2 (I.)

Suppose $(X_n, Y_n) \to (c, c) \in \Delta$ and $c \in \mathcal{S}$.

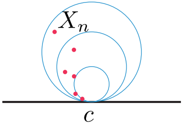
1. If either $\{X_n\}$ or $\{Y_n\}$ converge horocyclically to c, then $Q(X_n, Y_n) \to \infty$.
2. Suppose that both $\{X_n\}$ and $\{Y_n\}$ converge tangentially to c. We symplify the situation as follows: $X_n = \tau_c^{k_n} X$, $Y_n = \tau_c^{l_n} Y$ for some (X, Y). Then $Q(\tau_c^{k_n} X, \tau_c^{l_n} Y)$ converges $\Leftrightarrow$ $\exists p \in \mathbb{Z}$ s.t. $(p+1)k_n - pl_n \equiv$ const. for all large n. (Especially $\frac{k_n}{l_n} \to \frac{p}{p+1}$)

cf. Ohshika generalize this theorem to general hyperbolic surfaces.

$T(S) \cong \mathbf{H}$



$\Leftrightarrow \ l_{X_n}(c) \to 0$



$\Leftrightarrow \ l_{X_n}(c) > \epsilon > 0$

Complex length

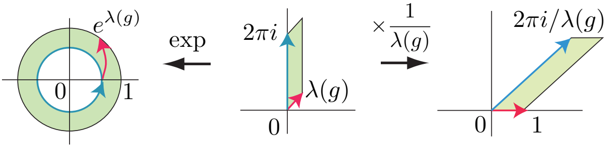
For a loxodromic element $g \in \mathrm{PSL}(2,\mathbb{C})$, its complex length $\lambda(g) \in \mathbb{C}$ is defined as

$$\lambda(g) = l + i\theta \quad (l > 0,\ \theta \in (-\pi, \pi]) \quad \text{s.t. } g \sim (e^{\lambda(g)} z).$$

Note that

$$T_g := (\hat{\mathbb{C}} - \mathrm{fix}(g))/\langle g \rangle \ \cong\ \mathbb{C}^\times/\langle e^{\lambda(g)} z\rangle \ \cong\ \mathbb{C}\Big/\Big\langle z+1,\ z+\frac{2\pi i}{\lambda(g)}\Big\rangle.$$

i.e. $\frac{2\pi i}{\lambda(g)}$ is the Teichmüller parameter of the quotient torus T_g.



Let Γ_n, G be Kleinian groups. We say $\Gamma_n \to G$ geometrically if $\Gamma_n \to G$ in the sence of Hausdorff as closed subsets of $\mathrm{PSL}(2,\mathbb{C})$, or, $(\mathbf{H}^3, *)/\Gamma_n \to (\mathbf{H}^3, *)/G$ in the sense of Gromov.

Geometric limits of cyclic groups

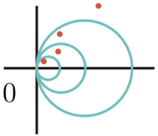
Lemma

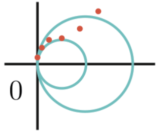
Suppose loxodromic γ_n *converges to* $\delta = (z+1)$ *in* $PSL(2,\mathbb{C})$*.*

1. *If* $\lambda(\gamma_n) \to 0$ *horocyclically*, $\langle \gamma_n \rangle \to \langle \delta \rangle$ *geometrically.*
2. *If* $\lambda(\gamma_n) \to 0$ *tangeitially, and if* $\exists m_n \in \mathbb{Z}$ *and* $\exists \xi \in \mathbb{C}$ *s.t.*

$$\frac{2\pi i}{\lambda(\gamma_n)} - m_n \to \xi \quad (n \to \infty),$$

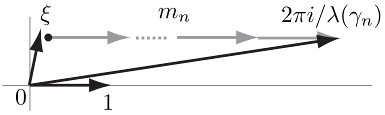
then $\lim_{n\to\infty} \gamma_n^{-m_n} = \hat{\delta} = (z+\xi)$ *and* $\langle \gamma_n \rangle \to \langle \delta, \hat{\delta} \rangle$ *geometrically.*





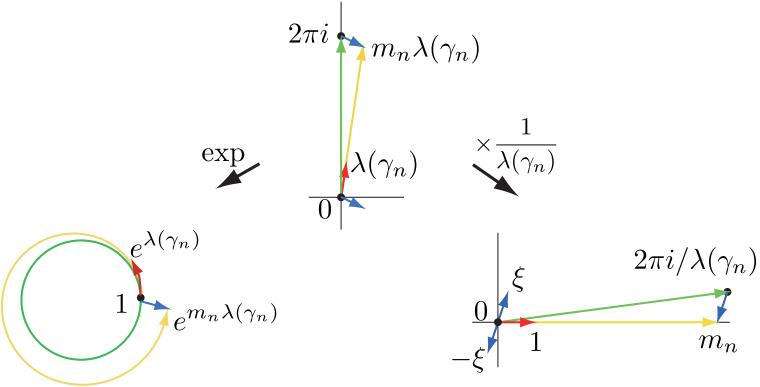
Note that

1. $\lambda(\gamma_n) \to 0$ horocyclically
   $\Leftrightarrow \mathrm{Im}(2\pi i/\lambda(\gamma_n)) \to \infty$.
2. $\lambda(\gamma_n) \to 0$ tangentially
   $\Leftrightarrow \mathrm{Im}(2\pi i/\lambda(\gamma_n)) < \exists C$ and
   $\mathrm{Re}(2\pi i/\lambda(\gamma_n)) \to \infty$.



Proof of Lemma

Since $\frac{2\pi i}{\lambda(\gamma_n)} - m_n \to \zeta$ and $|m_n| \to \infty$, we have $\frac{2\pi i}{m_n \lambda(\gamma_n)} \to 1$.
Let $g_n := (e^{\lambda(\gamma_n)} z) \sim \gamma_n$. Then $g_n(1) = e^{\lambda(\gamma_n)} \approx 1$ and $g_n^{m_n}(1) = e^{m_n \lambda(\gamma_n)} \approx 1$.
From the figure below, we see that if $\gamma_n \to (z+1)$ then $\gamma_n^{m_n} \to (z - \xi)$. □



Theorem 2 (I.)

Let $X, Y \in T(S)$ and $k_n, l_n \in \mathbb{Z}$ be divergent sequences. Then $Q(\tau_c^{k_n}X, \tau_c^{l_n}Y)$ converges $\Leftrightarrow \exists p \in \mathbb{Z}$ s.t. $(p+1)k_n - pl_n \equiv$ const. $(\forall n \gg 0)$

proof

We regard $T(S) \cong \mathbf{H}$ and assume $c = a \leftrightarrow \infty \in \partial\mathbf{H}$. Then $\tau_a : X \mapsto X + 1$ in $\mathbf{H}$.
Let $m_n := l_n - k_n$. Then $Q(\tau_a^{k_n}X, \tau_a^{l_n}Y) = \tau_a^{k_n} \cdot Q(X, \tau_a^{m_n}Y)$.
By setting $\rho_n := Q(\tau_a^{k_n}X, \tau_a^{l_n}Y)$ and $\eta_n := Q(X, \tau_a^{m_n}Y)$, we have

$$\rho_n(a) = \eta_n(a) \quad \text{and} \quad \rho_n(b) = \eta_n(a)^{k_n}\eta_n(b).$$

Since $\eta_n \to Q(X, a)$, ρ_n converges $\Leftrightarrow \eta_n(a)^{k_n}$ converges.
On the other hand, by Minsky's Pivot Theorem, we have

$$\frac{2\pi i}{\lambda(\eta_n(a))} \underset{\mathbf{H}}{\sim} X - \overline{\tau_a^{m_n}Y} + i \quad (= X - \overline{Y} - m_n + i).$$

Thus we may assume that $\frac{2\pi i}{\lambda(\eta_n(a))} + m_n \to {}^{\exists}\xi$. Therefore $\lim \eta_n(a)^{m_n}$ is primitive in the geometric limit of $\langle \eta_n(a) \rangle = \langle \rho_n(a) \rangle$.
$\eta_n(a)^{k_n}$ converges $\Leftrightarrow$
$\exists p \in \mathbb{Z}$ s.t. $k_n \equiv pm_n +$ const. $\equiv p(k_n - l_n) +$ consts. $(\forall n \gg 0)$.

Self-bumping of $QF(S)$

Theorem 3

Given $\rho \in \partial AH(S)$, the followings are equivalent:

1. $QF(S)$ self-bumps at $\rho \in \partial AH(S)$.
2. ρ is the limit of the sequence

$$Q(\tau_c^{pn} X, \tau_c^{(p+1)n} Y)$$

for some $c \in \mathcal{S}$, $X, Y \in \overline{T(S)} \setminus \{c\}$, and $p \in \mathbb{Z}$.

(2) $\Rightarrow$ (1) is due to McMullen for general surfaces.
I don't know whether (1) $\Rightarrow$ (2) is true or not for general surfaces. (It seems not to be true...)

Bromberg's theory (1)

Bromberg's theory tells us the local topology of $AH(S)$ near the Maskit slice

$$\mathbf{M} := \{\rho \in AH(S) : \rho(a) \text{ is parabolic}\}.$$

The Maskit slice $\mathbf{M}$ can be enbedded into $\mathbb{C}$ as follows:

Let us define $\sigma_\mu : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ by

$$\sigma_\mu(a) = (z + 2), \quad \sigma_\mu(b) = (1/z + \mu).$$

Then the map

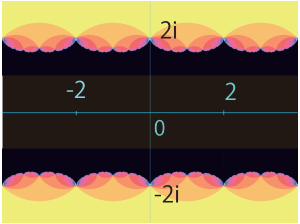
$$f : \mathbb{C} \to \{\rho \in R(S) : \rho(a) \text{ is parabolic}\}, \quad \mu \mapsto \sigma_\mu$$

is bijective.

Set

$$\mathcal{M} := \{\mu \in \mathbb{C} : \sigma_\mu \in AH(S)\} = f^{-1}(\mathbf{M}).$$

- $\mathcal{M}$ is also called the Maskit slice.
- $\mathcal{M}$ is $\langle z + 2 \rangle$-invariant.

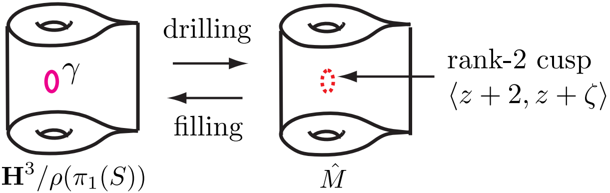


Bromberg's theory (2)

Let $\rho \in AH(S)$ with $\lambda(\rho(a)) \approx 0$. Then the geodesic γ in $\mathbf{H}^3/\rho(\pi_1(S))$ associated to $\rho(a)$ is short. By Drilling Theorem, there exist a complete hyp 3-mfd $\hat{M}$ and a $(1+\epsilon)$- bi-Lipschitz map

$$\mathbf{H}^3/\rho(\pi_1(S)) - \mathcal{N}(\gamma) \quad \longrightarrow \quad \hat{M} - \mathcal{N}(\text{rank-2 cusp})$$

where
- $\hat{M}$ is homeomorphic to $\hat{N} := S \times (0,1) - \{a\} \times \{1/2\}$.
- $\hat{M} = \mathbf{H}^3/\langle z+2,\ 1/z+\mu,\ z+\zeta\rangle \quad (\mu, \zeta \in \mathbb{C})$.
- $\rho(a) \leftrightarrow z+2, \quad \rho(b) \leftrightarrow 1/z+\mu, \quad id \leftrightarrow z+\zeta$.



Let us define $\sigma_{\mu,\zeta} : \pi_1(\hat{N}) = \langle a, b, c : [a, c] = id\rangle \to \mathrm{PSL}(2, \mathbb{C})$ by

$$\sigma_{\mu,\zeta}(a) = (z+2), \quad \sigma_{\mu,\zeta}(b) = (1/z+\mu), \quad \sigma_{\mu,\zeta}(c) = (z+\zeta).$$

Bromberg's theory (3)

Bromberg' idea is to use (μ, ζ) as parameters of $AH(S)$ near the Maskit slice. Let

$$\mathcal{B} := \{(\mu, \zeta) \in \mathbb{C}^2 : \sigma_{\mu,\zeta} : \text{discrete, faithful}\}.$$

It is known by Bromberg that

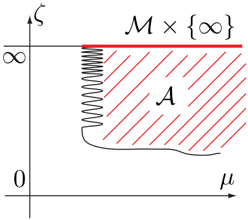
$$(\mu, \zeta) \in \mathcal{B} \iff \mu + k\zeta \in \mathcal{M} \text{ for all } k \in \mathbb{Z}.$$

Set

$$\mathcal{A} := \{(\mu, \zeta) \in \mathcal{B} : \mathrm{Im}(\zeta) > 0\} \cup \{(\mu, \infty) : \mu \in \mathcal{M}\}$$

and define a map $\Phi : \mathcal{A} \to AH(S)$ by

$$\Phi(\mu, \zeta) = \begin{cases} \rho = (\zeta\text{–filling of } \sigma_{\mu,\zeta}) & (\text{ if } \zeta \neq \infty), \\ \sigma_\mu & (\text{ if } \zeta = \infty). \end{cases}$$



Theorem 4 (Bromberg)

$\Phi : \mathcal{A} \to AH(S)$ is a local homeomorphism at (μ, ∞) for any $\mu \in \mathrm{int}(\mathcal{M})$.

Bromberg's theory (4)

Theorem 5 (Bromberg)

$AH(S)$ is not locally connected.

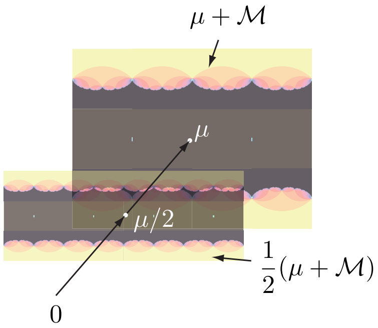
Sketch of proof There is an open set $U \subset \mathcal{M}$ such that vertical slices

$$\mathcal{A}_\mu := \{\zeta : (\mu, \zeta) \in \mathcal{A}\}$$

of $\mathcal{A}$ is not locally conected at ∞ for all $\mu \in U$. This can be seen from

$$\zeta \in \mathcal{A}_\mu \;\Leftrightarrow\; \zeta \in \bigcap_{k\in\mathbb{Z}, k\neq 0} \frac{1}{k}(\mu + \mathcal{M})$$

and the figure on the right.
Therefore $\mathcal{A}$, and hence $AH(S)$, is not locally connected. □



Linear slices

Given $\lambda \in \mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}$, we define the linear slice $\mathbf{L}(\lambda) \subset AH(S)$ for λ by

$$\mathbf{L}(\lambda) = \{\rho \in AH(S) : \lambda(\rho(a)) = \lambda\}.$$

Note that $\mathbf{L}(0)$ is nothing but the Maskit slice; i.e.

$$\mathbf{L}(0) = \mathbf{M} = \{\rho \in AH(S) : \rho(a) \text{ is palabolic}\}.$$

Question

When $\lambda_n \in \mathbb{C}_+$, $\lambda_n \to 0$, does $\mathbf{L}(\lambda_n)$ converge to $\mathbf{L}(0) = \mathbf{M}$?

Trace coordinate

Let consider the trace coordinate map

$$\Psi : R(S) \to \mathbb{C}_+ \times \mathbb{C}, \quad \rho \mapsto (\lambda(\rho(a)), \mathrm{tr}(\rho(b))).$$

This map is well defined and is a homeomorphism on a domain of $R(S)$ consists of ρ with small $\lambda(\rho(a))$.
Let us define a subset $\mathcal{L}(\lambda) \subset \mathbb{C}$ (which is also called the Linear slice for λ) by

$$\mathcal{L}(\lambda) := \{\mathrm{tr}(\rho(b)) : \rho \in \mathbf{L}(\lambda)\}.$$

Then $\Psi|_{\mathbf{L}(\lambda)} : \mathbf{L}(\lambda) \to \{\lambda\} \times \mathcal{L}(\lambda)$ is a homeomorphism.

Recall that $\mathbf{L}(0) = \mathbf{M}$, and for $\sigma_\mu \in \mathbf{M}$, we have $\mathrm{tr}(\sigma_\mu(b)) = \mathrm{tr}(1/z + \mu) = i\mu$.
Thus we have

$$\mathcal{L}(0) = i\mathcal{M} = \{i\mu : \mu \in \mathcal{M}\}.$$

Question

When $\lambda_n \in \mathbb{C}_+$, $\lambda_n \to 0$, does $\mathcal{L}(\lambda_n)$ converge to $\mathcal{L}(0) = i\mathcal{M}$?

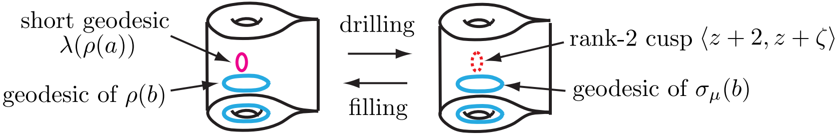
Relation between Bromberg's coordinate and trace coordinate

Let us consider the map $\Psi \circ \Phi$:

$$\mathcal{A} \xrightarrow{\Phi} AH(S) \xrightarrow{\Psi} \mathbb{C}_+ \times \mathbb{C}, \qquad (\mu, \zeta) \mapsto \rho \mapsto (\lambda, \beta).$$

Then we have

$$4\pi i/\lambda \approx \zeta, \quad \beta \approx i\mu.$$



Especially λ fix $\approx \zeta$ fix. Let consider horizontal slices of $\mathcal{A}$;

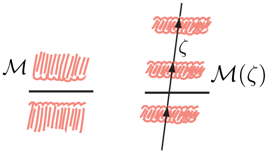
$$\mathcal{M}(\zeta) := \{\mu \in \mathbb{C} : (\mu, \zeta) \in \mathcal{A}\}.$$

Then we can expect that $\mathcal{L}(\lambda) \approx i\mathcal{M}(\zeta)$.

Hausdorff limits of linear slices

Since $\mu \in \mathcal{M}(\zeta) \Leftrightarrow \mu \in k\zeta + \mathcal{M}$ $(\forall k \in \mathbb{Z})$,
we obtain the picture of $\mathcal{M}(\zeta)$.

- $\mathcal{M}(\zeta)$ is $\langle z+2, z+\zeta \rangle$-invariant.
- $\mathcal{M}(\zeta + 2k) = \mathcal{M}(\zeta)$ for all $k \in \mathbb{Z}$.



Theorem 6 (I.)

Suppose that $\lambda_n \in \mathbb{C}_+$, $\lambda_n \to 0$.

- If $\lambda_n \to 0$ horocyclically, then

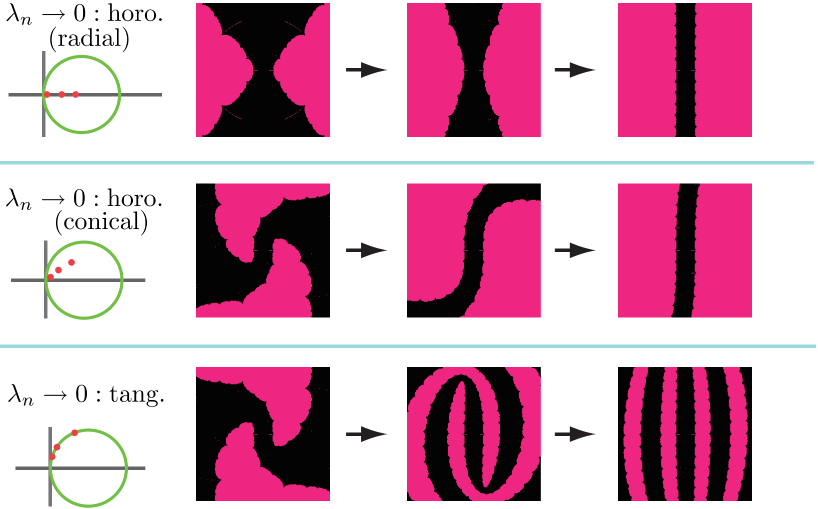
$$\mathcal{L}(\lambda_n) \to \mathcal{L}(0) = i\mathcal{M} \qquad (\textit{Hausdorff})$$

- Suppose that $\lambda_n \to 0$ tangentially, and that ${}^{\exists}m_n \in \mathbb{Z}$ s.t. $\dfrac{2\pi i}{\lambda_n} - m_n \to {}^{\exists}\xi \in \mathbb{C}$. Then

$$\mathcal{L}(\lambda_n) \to i\mathcal{M}(2\xi) \qquad (\textit{Hausdorff})$$

cf. The case of $\lambda_n \in \mathbb{R}_+$, $\lambda_n \to 0$ was observed by Parker-Parkkonen.

Linear slices $\mathcal{L}(\lambda_n)$ as $\lambda_n \to 0$

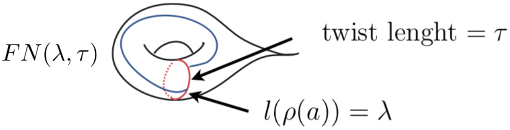


(Drawn by K. Sakugawa)

Complex Fenchel-Nielsen coordinate

Real Fencel-Nielsen coordinate is:

$$FN : \mathbb{R}_{>0} \times \mathbb{R} \to \{\rho \in R(S) : \rho(\pi_1(S)) \text{ is Fuchsian}\} \; (\subset R(S) \xrightarrow{\Psi} \mathbb{C}_+ \times \mathbb{C})$$



We have

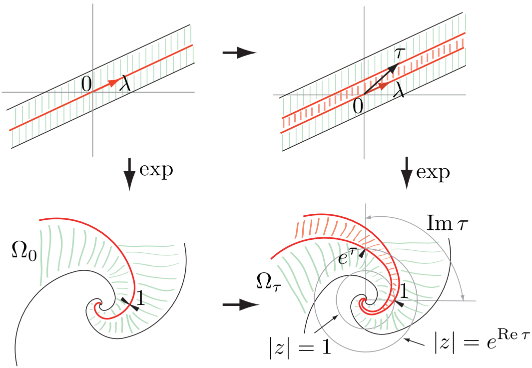
$$\Psi \circ FN(\lambda, \tau) = \left(\lambda, \frac{2\cosh(\tau/2)}{\tanh(\lambda/2)}\right).$$

By analitic continuation, we obtain the complex Fencel-Nielsen coordinate map

$$FN : (\mathbb{C} \setminus 2\pi i\mathbb{Z}) \times \mathbb{C} \to R(S).$$

Geometric meaning of complex FN deformation

Assume that $\eta_0 := FN(\lambda, 0) \in QF(S)$. Then $\eta_\tau := FN(\lambda, \tau)$ for $\tau \in \mathbb{C}$ is obtained as follows (In the case of $\lambda \in \mathbb{R}_+$, this deformation is known as complex earthquake) :
Let Ω_0 be a component of $\Omega(\eta_0(\pi_1(S)))$. Then a component Ω_τ of $\Omega(\eta_\tau(\pi_1(S)))$ is obtained from Ω_0 by cutting and sliding along the axis of $\eta_0(a)$ (and its all conjugations) and inserting domains at cut loci.



Linear slices in τ-plane

Given $\lambda \in \mathbb{C}_+$, we set

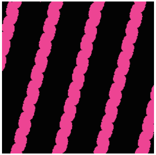
$$\widetilde{\mathcal{L}}(\lambda) := \{\tau \in \mathbb{C} : FN(\lambda, \tau) \in AH(S)\}.$$

Then $\widetilde{\mathcal{L}}(\lambda)$ is a (branched) covering of $\mathcal{L}(\lambda)$:

$$\begin{array}{ccccc} \mathbb{C}_+ \times \mathbb{C} & \xrightarrow{FN} & R(S) & \xrightarrow{\Psi} & \mathbb{C}_+ \times \mathbb{C}, \\ \{\lambda\} \times \widetilde{\mathcal{L}}(\lambda) & \longrightarrow & \mathbf{L}(\lambda) & \longrightarrow & \{\lambda\} \times \mathcal{L}(\lambda). \end{array}$$

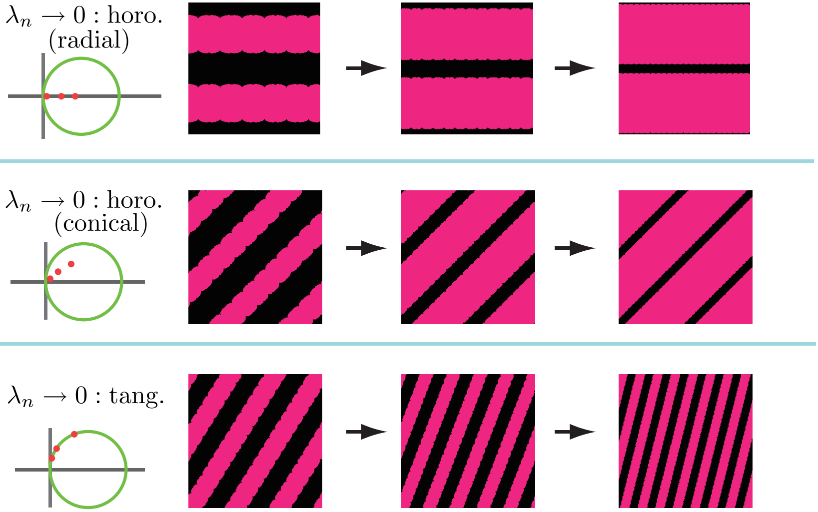
$\widetilde{\mathcal{L}}(\lambda)$ is $\langle z + \lambda, z + 2\pi i \rangle$-invariant, where the action $z \mapsto z + \lambda$ corresponds to the Dehn twist about a.

$\widetilde{\mathcal{L}}(\lambda)$



$\mathcal{L}(\lambda)$

Linear slices $\widetilde{\mathcal{L}}(\lambda_n)$ as $\lambda_n \to 0$



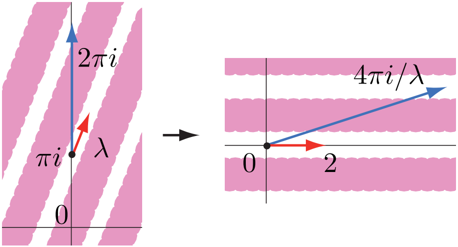
center $= \pi i$, width $= 2\pi$.

(Drawn by K. Sakugawa)

Let us consider the normalization

$$\frac{2}{\lambda}(\widetilde{\mathcal{L}}(\lambda) - \pi i)$$

of $\widetilde{\mathcal{L}}(\lambda)$, which is $\langle z+2, z+4\pi i/\lambda \rangle$-invariant.



Corollary of Theorem 6 (I.)

Suppose that $\lambda_n \in \mathbb{C}_+, \lambda_n \to 0$ as $n \to \infty$.

1. If $\lambda_n \to 0$ horocyclically, then

$$\frac{2}{\lambda_n}(\widetilde{\mathcal{L}}(\lambda_n) - \pi i) \to \mathcal{M}. \quad (\textit{Hausdorff})$$

2. If $\lambda_n \to 0$ tangentially and $2\pi i/\lambda_n - {}^{\exists}m_n \to {}^{\exists}\xi \in \mathbb{C}$, then

$$\frac{2}{\lambda_n}(\widetilde{\mathcal{L}}(\lambda_n) - \pi i) \to \mathcal{M}(2\xi). \quad (\textit{Hausdorff})$$

The previous corollary can be rephrased as follows:

Corollary of Theorem 6 (I.)

Suppose that $\lambda_n \in \mathbb{C}_+, \lambda_n \to 0$ as $n \to \infty$.

1. If $\lambda_n \to 0$ horocyclically, then
$$(\mathbb{C}, \widetilde{\mathcal{L}}(\lambda_n), \pi i)/\langle z+\lambda, z+2\pi i\rangle \to (\mathbb{C}, \mathcal{M}, 0)/\langle z+2\rangle.$$
2. If $\lambda_n \to 0$ tangentially and $2\pi i/\lambda_n - {}^{\exists}m_n \to {}^{\exists}\xi \in \mathbb{C}$, then
$$(\mathbb{C}, \widetilde{\mathcal{L}}(\lambda_n), \pi i)/\langle z+\lambda, z+2\pi i\rangle \to (\mathbb{C}, \mathcal{M}(2\xi), 0)/\langle z+2, z+2\xi\rangle.$$