Heegaard splittings with large subsurface distance

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- *A Heegaard splitting* is a triple: (Σ, H^-, H^+),

- $\Sigma = \partial H^+ = \partial H^- = H^+ \cap H^-$ - *Heegaard surface* so that $M = H^+ \cup_\Sigma H^-$.

- **Curve complexes and Hempel distance:** For Σ - simplicial complex $\mathcal{C}(\Sigma)$ called the *curve complex.*

- i-simplex $\in \mathcal{C}(\Sigma)$ collection $([\gamma_0], \ldots, [\gamma_i])$ of isotopy classes of mutually disjoint essential simple closed curves in Σ.

- On 1-skeleton $\mathcal{C}^1(\Sigma) \subset \mathcal{C}(\Sigma)$ there is a natural path metric d defined by assigning length 1 to every edge.

- Sub-collection $\mathcal{D}(H^\pm)$, of isotopy classes of curves in Σ, that bound disks in $H^\pm$ (called *meridians*) is *handlebody set* associated with $H^\pm$ respectively. $\mathcal{D}(H^\pm)$ are subcomplexes.

- Given (Σ, H^-, H^+) we define the *Hempel distance* $d(\Sigma)$:

$$d(\Sigma) = d_{\mathcal{C}^1(\Sigma)}(\mathcal{D}(H^+), \mathcal{D}(H^-))$$

• If $\partial F \neq \emptyset$ define the *arc and curve complex* $\mathcal{AC}(F)$, by isotopy classes of essential (non-peripheral) s. c. c. and properly embedded arcs.

• If F is an annulus, there are no essential closed curves, and the isotopy classes of essential arcs should be taken rel endpoints.

• An n-simplex is a collection of $n+1$ isotopy classes with disjoint representative loops / arcs. d_F the path metric on the 1-skeleton of $\mathcal{AC}^1(F)$ that assigns length 1 to every edge.

• If F is a connected proper essential subsurface in Σ, there is a map

$$\pi_F : \mathcal{C}^0(\Sigma) \to \mathcal{AC}^0(F) \cup \{\emptyset\}$$

Given a s.c.c. γ in Σ, isotope it to intersect ∂F minimally. If $F \cap \gamma = \emptyset$ set $\pi_F(\gamma) = \emptyset$. Otherwise consider the isotopy classes of components of $F \cap \gamma$ as a simplex in $\mathcal{AC}(F)$, and select (arbitrarily) one vertex to be $\pi_F(\gamma)$.

• If $F \subset \Sigma$ is a connected proper essential subsurface, set $\mathcal{D}_F(H^-) = \pi_F(\mathcal{D}(H^-))$ and $\mathcal{D}_F(H^+) = \pi_F(\mathcal{D}(H^+))$ – the projections to F of all loops that bound essential disks in H^- and H^+, respectively.

- The *F-distance* of Σ ($d_F(\Sigma)$) - distance between these two sets,

$$d_F(\Sigma) = d_{\mathcal{AC}^1(F)}(\mathcal{D}_F(H^+), \mathcal{D}_F(H^-)).$$

Scharlemann and Tomova. If a 3-manifold has a Heegaard surface Σ so that the Hempel distance $d(\Sigma)$ of the splitting is greater than twice its genus $g(\Sigma)$, then any other Heegaard surface Λ of genus $g(\Lambda) < d(\Sigma)/2$ is a stabilization of Σ.

Goal. In this paper, we generalize this theorem to the case where the Heegaard splitting Σ is not necessarily of high distance but has a proper essential subsurface $F \subset \Sigma$ so that the "subsurface distance" measured in F is large:

Theorem 0.1. *Let Σ be a Heegaard surface in a 3-manifold M of genus $g(\Sigma) \geq 2$, and let $F \subset \Sigma$ be a compact essential subsurface. Let Λ be another Heegaard surface for M of genus $g(\Lambda)$. If*

$$d_F(\Sigma) > 2g(\Lambda) + c(F),$$

then, up to ambient isotopy, the intersection $\Lambda \cap \Sigma$ contains F.

Here $c(F) = 0$ unless F is an annulus, 4-holed sphere, or 1 or 2-holed torus, in which case $c(F) = 2$.

This result can be paraphrased as follows:

Result. *If the two disk sets of a Heegaard splitting intersect on a subsurface of the Heegaard surface in a relatively "complicated" way, then any other Heegaard surface whose genus is not too large must contain that subsurface.*

- Idea of the proof:

- To a Heegaard splitting we associate a "sweepout" by parallel surfaces of the manifold minus a pair of spines.

• Given two surfaces Λ and Σ and associated sweepouts, we examine the way in which they interact. In particular under some natural genericity conditions we can assume that one of two situations occur:

(1) Λ *F-spans* Σ, or
(2) Λ *F-splits* Σ.

• F-spanning implies that, up to isotopy, there is a moment in the sweepout corresponding to Σ when the subsurface parallel to F lies in the "upper" half of $M \setminus \Lambda$, and a moment when it lies in the lower half. It then follows that Λ separates the product region between these two copies of F, and it follows fairly easily that it can be isotoped to contain F.

• F-splitting we show that, for each moment in the sweepout corresponding to Σ, the level surface intersects Λ in curves that have essential intersection with F. Studying the way in which these intersections change during the course of the sweepout, we use the topological complexity of Λ to control the subsurface distance of F.

• Combining these two results and imposing the condition that $d_F(\Sigma)$ is greater than a suitable function of $g(\Lambda)$ forces the first, i.e. F-spanning, case to occur.

• The discussion is complicated by some special cases, where F has particularly low complexity, in which the dichotomy between F-spanning and F-splitting doesn't quite hold. In those cases we obtain slightly different bounds.

We will have use for the following fact, which is a variation on a result of Masur-Schleimer:

Lemma 0.2. *Let Σ be the boundary of a handlebody H of genus $g \geq 2$. Let $F \subset \Sigma$ be an essential connected subsurface of Σ. If $\Sigma \setminus F$ is compressible in H then $\pi_F(\mathcal{D}(H))$ comes within distance 2 of every vertex of $\mathcal{AC}(F)$, provided F is not a 4-holed sphere. If it is a 4-holed sphere the distance bound is 3.*

Sweepouts. A *sweepout* of M^3 is a smooth function $f : M \to [-1, 1]$ s.t each $t \in (-1, 1)$ is a regular value, and the level set $f^{-1}(t)$ is a Heegaard surface. Furthermore each of the sets $\Gamma^+ = f^{-1}(1)$ and set $\Gamma^- = f^{-1}(-1)$ are spines of the respective compression bodies. We say the sweepout *represents* the Heegaard splitting associated to each level surface.

Two sweepouts f and h of M determine a smooth function $f \times h : M \to [-1,1] \times [-1,1]$. The differential $D(f \times h)$ has rank 2 (or $\dim Ker(D(f \times h)) = 1$) wherever the level sets of f and h are transverse. Thus we define the *discriminant set* Δ to be the set of points of M for which $\dim Ker(D(f \times h)) > 1$. The discriminant, and its image in $[-1,1] \times [-1,1]$, therefore encode the configuration of tangencies of the level sets of f and h.

The image $(f \times h)(\Delta)$ is a graph in $[-1,1] \times [-1,1]$ with smooth edges, called the *graphic*, or the *Rubinstein-Scharlemann graphic*

We call $f \times h$ *generic* if it is stable away from the spines and each arc $[-1,1] \times \{s\}$ in the square intersects at most one vertex of the graphic.

The following lemma of Kobayashi-Saeki justifies this term:

Lemma 0.3. *Any pair of sweepouts can be isotoped to be generic.*

- Suppose therefore that $f \times h$ is generic.

• Points in the square are denoted by (t, s), and we define the surfaces $\Lambda_s = h^{-1}(s)$ and $\Sigma_t = f^{-1}(t)$.

• If the vertical line $\{t\} \times [-1, 1]$ meets no vertices of the graphic, then $h|_{\Sigma_t}$ is Morse, and its critical points are $\Sigma_t \cap \Delta$.

Above and below. Let f, h sweepouts representing (Σ, H^-, H^+) and (Λ, V^-, V^+), respectively.

• For each $s \in (-1, 1)$, define $V_s^- = h^{-1}([-1, s])$ and $V_s^+ = h^{-1}([s, 1])$. $\Lambda_t = \partial V_s^- = \partial V_s^+$.

• For each $t \in (-1, 1)$, $H_t^- = h^{-1}([-1, t])$, $H_t^+ = h^{-1}([t, 1])$

• $\Sigma_t = \partial H_t^- = \partial H_t^+$, $\Lambda_t = \partial V_s^- = \partial V_s^+$.

Definition 0.4. Σ_t is *mostly above* Λ_s *with respect to* F,

$$\Sigma_t \succ_F \Lambda_s,$$

if $\Sigma_t \cap V_s^-$ is contained in a subsurface of Σ_t that is isotopic into the complement of F_t (or is just contained in a disk, when $F = \Sigma$). Σ_t is *mostly below* Λ_s *with respect to* F, or

$$\Sigma_t \prec_F \Lambda_s,$$

if $\Sigma_t \cap V_s^+$ is contained in a subsurface that is isotopic into the complement of F_t (or contained in a disk).

• Here F is a proper subsurface. $F = \Sigma$ was done by Johnson and the definitions agree.

• Define R_a^F (respectively R_b^F) in $(-1,1) \times (-1,1)$ to be the set of all values (t,s) such that $\Sigma_t \succ_F \Lambda_s$ (respectively $\Sigma_t \prec_F \Lambda_s$).

• R_a^F and R_b^F are disjoint, open and bounded by arcs of the graphic, so that all interior vertices appearing in ∂R_a^F or ∂R_b^F have valence 4. R_a^F and R_b^F intersect each vertical line in a pair of intervals:

Relative spanning and splitting. Here we extend the notion of spanning and splitting relative to a subsurface F.

Definition 0.5. We say that h *F-spans* f if there is a horizontal arc $[-1,1] \times \{s\}$ in $(-1,1) \times (-1,1)$ that intersects both R_a^F and R_b^F.

The complementary situation is the following:

Definition 0.6. We say that h *weakly F-splits* f if there is no horizontal arc $[-1,1] \times \{s\}$ that meets both R_a^F and R_b^F.

A stronger condition:

Definition 0.7. *h F-splits f* if, for some $\{s\} \in (-1, 1)$, the arc $[-1, 1] \times \{s\}$ is disjoint from the closures of both R_a^F and R_b^F.

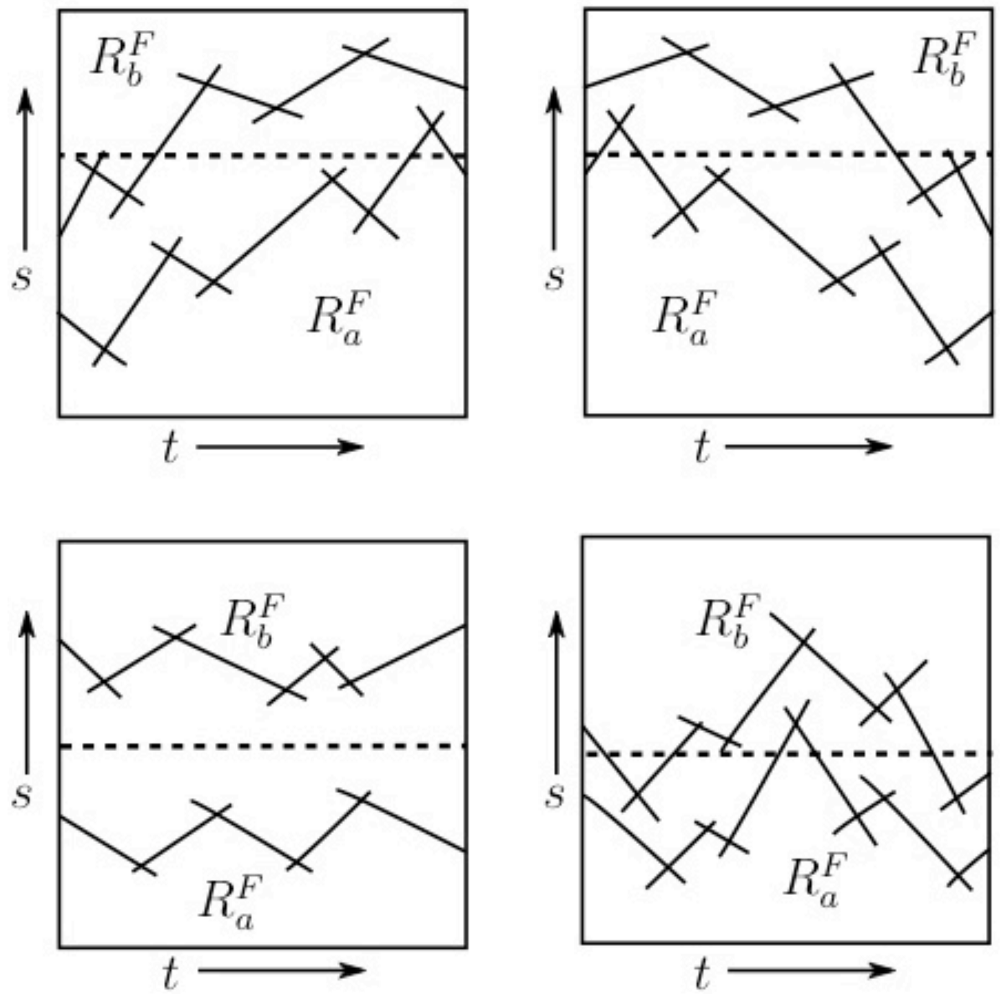

FIGURE 1

• F-spanning and weak F-splitting are complementary conditions.

• In the other direction: If F is not an annulus, 4-holed sphere or a 1-holed or 2-holed torus, then

$$h \text{ weakly } F\text{-splits } f \implies h\ F\text{-splits } f.$$

Equivalently, either h F-spans f, or h F-splits f.

• In the exceptional cases, if h weakly F-splits f but does not F-split, then there exists a unique horizontal line $[-1,1] \times \{s\}$ which meets both closures $\overline{R_a^F}$ and $\overline{R_b^F}$ in a single point, which is a vertex of the graphic.

Λ F-spans Σ.

Proposition 0.8. *If Λ F-spans Σ then after isotoping Λ we obtain a surface whose intersection with Σ contains F.*

Sketch of Proof. There is a level surface Λ_s and values $t_-, t_+ \in (-1,1)$, $t_- < t_+$ such that $\Sigma_{t_-} \prec_F \Lambda_s$ and $\Sigma_{t_+} \succ_F \Lambda_s$.

Identify $f^{-1}((-1,1))$, with $\Sigma \times (-1,1)$. Set $J = [t_-, t_+]$ and consider $F \times J \subset M$.

$\Sigma_{t_-} \prec_F \Lambda_s \implies V_s^+ = h^{-1}([s,1])$ intersects $\Sigma \times \{t_-\}$ in a set that can be isotoped outside of $F \times \{t_-\}$; equivalently:

$F \times \{t_-\}$ can be isotoped within $\Sigma \times \{t_-\}$ so that it is contained in V_s^-.

Similarly $\Sigma_{t_+} \succ_F \Lambda_s \implies$ after isotopy the surface $F \times \{t_+\} \subset V_s^+$

Hence, after a level-preserving isotopy of $\Sigma \times (-1,1)$, we may assume that $F \times \{t_-\}$ and $F \times \{t_+\}$ are contained in V_s^- and V_s^+, respectively.

Since Λ_s separates V_s^- from V_s^+ the surface $S = \Lambda_s \cap F \times J$ separates $F \times \{t_-\}$ from $F \times \{t_+\}$ within $F \times J$, so $0 \neq [S] \in H_2(F \times J, \partial F \times J)$.

Compress S if necessary to S^*. Let S' be a connected component which is still nonzero in $H_2(F \times J, \partial F \times J)$.

Then S' separates $F \times \{t_-\}$ from $F \times \{t_+\}$ and hence (up to orientation) must be homologous to $F \times \{t_-\}$.

So the projection of S' to $F \times \{t_-\}$ is a proper, π_1-injective and of degree ± 1. Hence it must also be π_1-surjective, and thus S' is isotopic to a level surface $F \times \{t\}$.

Make the isotopy ambient and keeping track of the 1-handles corresponding to the compressions we obtain an isotopic copy of Λ which contains $F \times \{t\}$ minus attaching disks for the 1-handles. Now we can slide these disks outside of $F \times \{t\}$. Hence Λ itself is isotopic to a surface containing $F \times \{t\}$, as claimed.

$\square$

Λ F-splits (weakly splits) Σ. The intersections $\Lambda \cap \Sigma_t$ are the level sets of $f|_\Lambda$. Consider them as curves on Σ. The F-splitting property implies that they intersect F essentially. Now use the topological complexity of Λ to bound the diameter in $\mathcal{AC}(F)$ of the corresponding set.

First show:

Lemma 0.9. *If $d_F(\Sigma) > 3$, then there is some non-trivial interval $[u, v] \subset (-1, 1)$ such that for each regular $t \in [u, v]$, every loop of $\Sigma_t \cap \Lambda$ that is trivial in Λ is trivial in Σ_t.*

For $t \in (-1, 1)$ (a regular value) of $f|_\Lambda$, $\mathcal{L}_t$ denotes the set of nontrivial isotopy classes in Σ of the images of the curves of $(f|_\Lambda)^{-1}(t)$. For an interval $J \subset (-1, 1)$ let $\mathcal{L}_J$ denote the union of $\mathcal{L}_t$ over regular $t \in J$.

Lemma 0.10. *Let $[u, v]$ be an interval as above If $(u, v) \times \{s\}$ encounters no vertices of the graphic then*

$$\mathrm{diam}_F(\mathcal{L}_{[u,v]}) \leq 2g(\Lambda) - 2.$$

If $(u, v) \times \{s\}$ meets the weak splitting vertex then

$$\mathrm{diam}_F(\mathcal{L}_{[u,v]}) \leq 2g(\Lambda) - 1.$$

Proposition 0.11. *Let Σ and Λ be Heegaard surfaces for M, and let $F \subset \Sigma$ be a proper connected essential subsurface. If Λ F-splits Σ then*

$$d_F(\Sigma) \leq 2g(\Lambda).$$

If Λ weakly F-splits Σ then

$$d_F(\Sigma) \leq 2g(\Lambda) + 2.$$

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