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The completed Goldman-Turaev Lie bialgebra and mapping class groups

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I. Goldman Lie algebra and its completion
II. Turaev cobracket
III. A tensorial description of the completed Goldman-Turaev Lie bialgebra

I. Goldman Lie algebra and its completion

Goldman Lie algebra

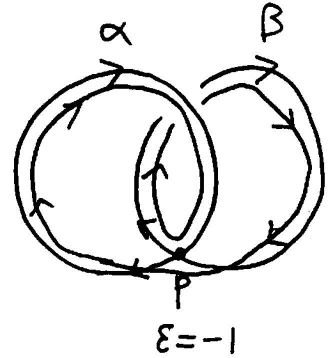
S : connected oriented surface

$\hat{\pi} = \hat{\pi}(S) := [S^1, S] = \pi_1(S)/_{\text{conj}}$ free loops on S

$|\ | : \pi_1(S) \to \hat{\pi}(S)$ forgetting the basepoint

$\alpha, \beta \in \hat{\pi}(S)$ in general position

$$[\alpha, \beta] \overset{\text{def}}{=} \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta) \, |\alpha_p \beta_p| \in \mathbb{Z}\hat{\pi} = \mathbb{Z}\hat{\pi}(S)$$



$\varepsilon(p; \alpha, \beta) \in \{\pm 1\}$ local intersection number

$\alpha_p, \beta_p \in \pi_1(S, p)$ based loop with basepoint p

Goldman

(1) $[\ ,\ ]$: well-defined

(2) $(\mathbb{Z}\hat{\pi}, [\ ,\ ])$: Lie algebra ⋯ Goldman Lie algebra of S

Action of a free loop on a path

$E \subset S$ closed subset, $E \setminus \partial S \overset{\text{closed}}{\subset} S$

$*_0, *_1 \in E$

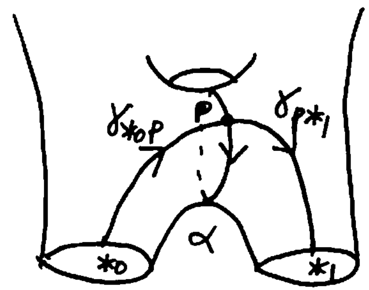
$\Pi S(*_0, *_1) = \pi_1(S, *_0, *_1) := [([0,1], 0, 1), (S, *_0, *_1)]$ fundamental groupoid

$\mathcal{C}(S,E)$ groupoid <u>object</u> $* \in E$, <u>morphism</u> $\gamma \in \Pi S(*_0, *_1)$

$S^* := S \setminus (E \setminus \partial S)$

$\alpha \in \hat{\pi}(S^*)$, $\gamma \in \Pi S(*_0, *_1)$ in general position

$$\sigma(\alpha)(\gamma) \overset{\text{def}}{=} \sum_{p \in \alpha \cap \gamma} \varepsilon(p; \alpha, \gamma)\, \gamma_{*_0 p}\, \alpha_p\, \gamma_{p *_1} \in \mathbb{Z}\Pi S(*_0, *_1)$$



<u>Kuno-K.</u>

(1) σ : well-defined

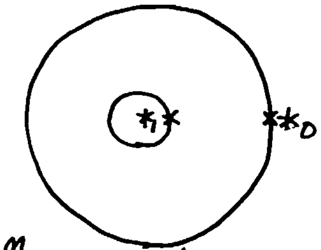
(2) $\sigma : \mathbb{Z}\hat{\pi}(S^*) \to \mathrm{Der}(\mathbb{Z}\mathcal{C}(S,E))$ Lie algebra homomorphism

(3) σ is <u>injective</u> if S : compact, $E \subset \partial S$ and $\pi_0(E) \underset{\text{surjective}}{\overset{\text{incl}_*}{\twoheadrightarrow}} \pi_0(\partial S)$

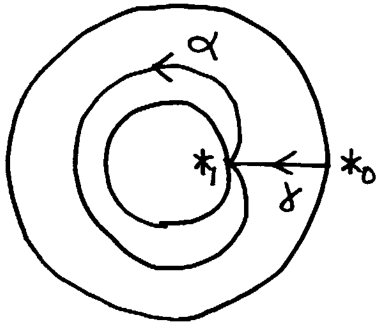
$$\mathcal{M}(S,E) \overset{\text{def}}{=} \{\varphi : S \to S : \text{ori. pres. diffeo} : \varphi|_{(\partial S) \cup E} = id_{(\partial S) \cup E}\} \Big/ \begin{array}{l}\text{isotopy fixing} \\ (\partial S) \cup E \\ \underline{\text{pointwise}}\end{array}$$

mapping class group

Dehn twists on an annulus

$S =$ 

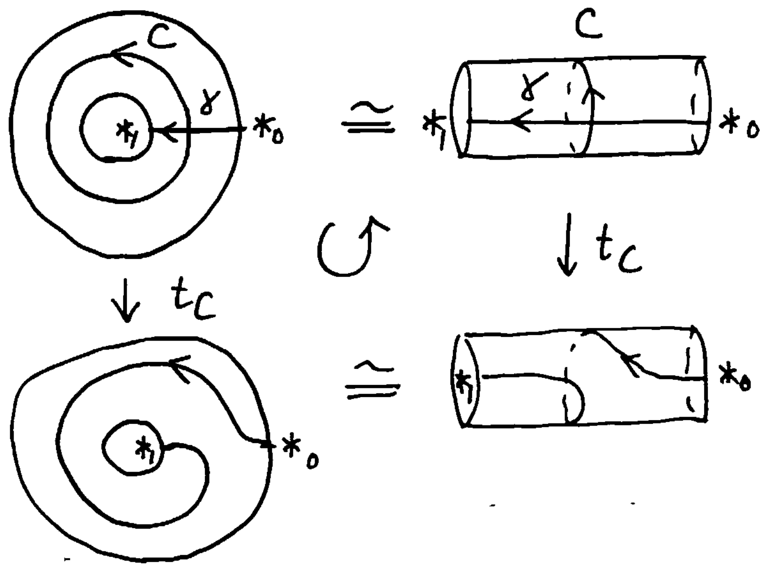
$E = \{*_0, *_1\} \subset \partial S$



$\alpha \in \pi_1(S, *_1)$

$\Pi S(*_0, *_1) = \{\gamma\alpha^n ;\ n \in \mathbb{Z}\}$

(right handed) Dehn twist $t_C \in \mathcal{M}(S, E)$, $C = |\alpha| \in \hat{\pi}(S)$



$$\begin{cases} t_C(\gamma) = \gamma\alpha \\ t_C(\alpha) = \alpha \end{cases}$$

$\widehat{\mathbb{Q}\Pi S}(*_0, *_1)$

$\widehat{\mathbb{Q}\Pi\mathcal{C}}(S, E)$

completion with respect to the augmentation ideal $I\pi_1(S)$

$\widehat{\mathbb{Q}\Pi\mathcal{C}}(S, E)$ "completed groupoid ring"

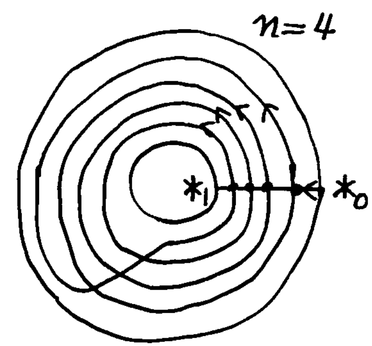
$$\gamma\alpha = \gamma e^{\log\alpha} \in \widehat{\mathbb{Q}\pi S}(*_0, *_1)$$

$\log(t_C) \in \mathrm{Der}(\widehat{\mathbb{Q}\mathcal{C}}(S,E))$ derivation.

$$(*1) \begin{cases} \log(t_C)(\gamma) \overset{\text{def}}{=} \gamma\log\alpha \\ \log(t_C)(\alpha) \overset{\text{def}}{=} 0 \end{cases}$$

$$\Rightarrow \quad e^{\log(t_C)} \left(= \sum_{n=0}^{\infty} \frac{1}{n!} (\log(t_C))^n \right) = t_C \text{ on } \widehat{\mathbb{Q}\mathcal{C}}(S,E)$$

<u>Goldman Lie algebra</u>



$\sigma(C^n)(\gamma) = n\gamma\alpha^n, \quad \sigma(C^n)(\alpha) = 0 \quad$ if $n \geq 0$

$\Big($ ($\because$) • n intersection points

• contribution of each point $= +\gamma\alpha^n$ $\Big)$

$f(x)$: polynomial in x

$$(*2) \begin{cases} \sigma(f(C))(\gamma) = \gamma\alpha f'(\alpha) \\ \sigma(f(C))(\alpha) = 0 \end{cases}$$

<u>Compare $(*1)$ and $(*2)$</u>

$$\alpha f'(\alpha) = \log\alpha$$

$$f(x) = \int_1 \frac{1}{x}\log x \, dx = \frac{1}{2}(\log x)^2$$

Hence

$$\log(t_C) = \frac{1}{2}(\log C)^2 \notin \mathbb{Q}\hat{\pi}$$

$$\underbrace{\in \widehat{\mathbb{Q}\hat{\pi}}}_{\text{completion}}$$

S : connected oriented surface , $E \overset{\text{closed}}{\subset} S$ as above

<u>$k \geqq 1$</u>

$$\mathbb{Q}\hat{\pi}(S)(k) := |\mathbb{Q}1 + (I\pi_1(S,*))^k|$$

where $* \in S$, $1 \in \pi_1(S,*)$ constant loop.

$I\pi_1(S,*) := \mathrm{Ker}(\mathbb{Q}\pi_1(S,*) \to \mathbb{Q})$ $\displaystyle\sum_{x \in \pi_1} a_x x \mapsto \sum a_x$, augmentation ideal

the RHS does <u>not</u> depend on the choice of $*$

$$[\mathbb{Q}\hat{\pi}(S)(k), \mathbb{Q}\hat{\pi}(S)(\ell)] \subset \mathbb{Q}\hat{\pi}(S)(k+\ell-2) \quad (\forall k, \forall \ell \geqq 1)$$

$\widehat{\mathbb{Q}\hat{\pi}}(S) \overset{\text{def}}{=} \varprojlim_{k \to \infty} \mathbb{Q}\hat{\pi}(S)/\mathbb{Q}\hat{\pi}(S)(k)$ the completed Goldman Lie algebra

$(\ni \frac{1}{2}(\log C)^2)$

$\widehat{\mathbb{Q}\hat{\pi}}(S)(k) := \mathrm{Ker}(\widehat{\mathbb{Q}\hat{\pi}}(S) \to \mathbb{Q}\hat{\pi}(S)/\mathbb{Q}\hat{\pi}(S)(k))$ Lie subalgebra.

$$\sigma : \widehat{\mathbb{Q}\hat{\pi}}(S^*) \to \mathrm{Der}(\widehat{\mathbb{Q}\mathcal{C}}(S,E))$$

well-defined Lie algebra homomorphism

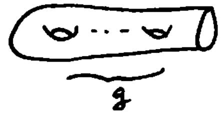
<u>injective</u> if S : compact, $E \subset \partial S$,

and $\pi_0(E) \underset{\mathrm{incl}_*}{\twoheadrightarrow} \pi_0(\partial S)$ surjective.

<u>Kuno-K.</u> $\phi \neq E \overset{\text{closed}}{\subset} S$, $E \setminus \partial S \overset{\text{closed}}{\subset} S$,

$C \subset S^* (= S \setminus (E \setminus \partial S))$ simple closed curve (SCC)

$\Longrightarrow \quad \exp\left(\sigma\left(\frac{1}{2}(\log C)^2\right)\right) = t_C$ on $\widehat{\mathbb{Q}\mathcal{C}}(S, E)$

(<u>Remark</u> · original formula was for $\Sigma_{g,1} =$  , $g \geq 1$ (Kuno-K.)

· <u>Massuyeau-Turaev</u> gave another generalization of the original formula

<u>Geometric Johnson homomorphism</u>

S : <u>compact</u> connected oriented surface with $\partial \neq \phi$

$E \subset \partial S$ finite subset $\pi_0(E) \xrightarrow[\text{incl}_*]{\cong} \pi_0(\partial S)$

$\Delta : \widehat{\mathbb{Q}\Pi S}(*_0, *_1) \longrightarrow \widehat{\mathbb{Q}\Pi S}(*_0, *_1) \hat{\otimes} \widehat{\mathbb{Q}\Pi S}(*_0, *_1)$ coproduct

$\gamma \in \Pi S(*_0, *_1) \longmapsto \gamma \hat{\otimes} \gamma$

$L^+(S, E) := \{ u \in \widehat{\mathbb{Q}\hat{\pi}}(S)(3) ; \ \Delta \sigma(u) = (\sigma(u) \hat{\otimes} \sigma(u)) \Delta \}$

$\subset \widehat{\mathbb{Q}\hat{\pi}}(S)$ Lie subalgebra

$\left(\begin{array}{l} \in \\ \frac{1}{2}(\log C)^2 \\ \text{if } [C] = 0 \in H_1(S; \mathbb{Z}) \end{array} \right)$

(<u>Remark</u> $L^+(\Sigma_{g,1}, \{*\}) \cong \widehat{\mathfrak{h}_{g,1}^+}$ the degree completion of Morita's $\mathfrak{h}_{g,1}^+$)

$\exp\circ\sigma : L^+(S,E) \to \mathrm{Aut}(\widehat{\mathbb{Q}\mathcal{C}}(S,E))$ well-defined, <u>injective</u>

$$u \longmapsto \sum_{n=0}^{\infty} \frac{1}{n!}\sigma(u)^n$$

$\mathrm{Image}(\exp\circ\sigma) \subset \mathrm{Aut}(\widehat{\mathbb{Q}\mathcal{C}}(S,E))$ subgroup ($\because$ Baker-Campbell-Hausdorff formula)

$\mathcal{M}(S) := \mathcal{M}(S,E) = \pi_0 \mathrm{Diff}(S, \mathrm{id\ on\ }\partial S)$

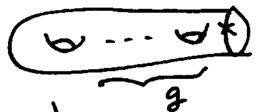
$\mathcal{I}(S) := \mathrm{Ker}(\mathcal{M}(S) \to \mathrm{Aut}(H_1(S,E;\mathbb{Z})))$

the "smallest" Torelli group in the sense of Putman

$$\begin{tikzcd} L^+(S,E) \arrow[rr, hook, "\exp\circ\sigma"] & \circlearrowleft & \mathrm{Aut}(\widehat{\mathbb{Q}\mathcal{C}}(S,E)) \\ & \mathcal{I}(S) \arrow[ul, dashed, "\exists!\tau"] \arrow[ur, hook] & \end{tikzcd}$$

injective (essentially due to Dehn-Nielsen)

($\because$ Putman's generators of $\mathcal{I}(S)$, Dehn twist formula (stated above))

If $(S,E) = (\Sigma_{g,1}, \{*\})$ $\quad \Sigma_{g,1} =$  $* \in \partial\Sigma_{g,1}$

$$\begin{tikzcd} \mathcal{I}_{g,1} = \mathcal{I}(\Sigma_{g,1}) \arrow[r, hook] \arrow[dr, "\tau^\theta"'] & L^+(\Sigma_{g,1}, \{*\}) \arrow[d, "\wr"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \widehat{\mathfrak{h}}^+_{g,1} \end{tikzcd}$$

$\leftarrow \theta$: symplectic expansion

Massuyeau's total Johnson map

$\mathrm{gr}(\tau) = \mathrm{gr}(\tau^\theta) =$ the (original) Johnson homomorphism :

$$\mathrm{gr}(\mathcal{I}_{g,1}) \to \mathfrak{h}^+_{g,1} = \mathrm{gr}(L^+(\Sigma_{g,1}, \{*\}))$$

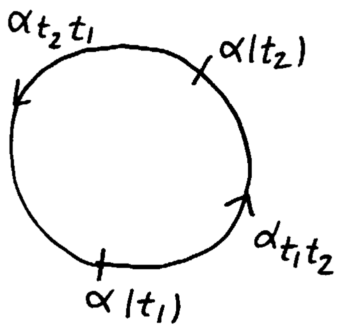
II. Turaev cobracket

S : connected oriented surface, $1 \in \hat{\pi}(S)$ constant loop

$\mathbb{Z}\hat{\pi}' = \mathbb{Z}\hat{\pi}'(S) \overset{\text{def}}{=} \mathbb{Z}\hat{\pi}/\mathbb{Z}1$ Lie algebra ($\because$ $1 \in \mathrm{Center}(\mathbb{Z}\hat{\pi})$)

$|\ |' : \mathbb{Z}\pi_1(S) \to \mathbb{Z}\hat{\pi}/\mathbb{Z}1 = \mathbb{Z}\hat{\pi}'$ quotient map

$\alpha \in \hat{\pi}$ in general position.



$D_\alpha := \{(t_1, t_2) \in S^1 \times S^1 \,;\, t_1 \neq t_2,\ \alpha(t_1) = \alpha(t_2)\}$

double points

$$\delta(\alpha) \overset{\text{def}}{=} \sum_{(t_1,t_2) \in D_\alpha} \varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2))\, |\alpha_{t_1 t_2}|' \otimes |\alpha_{t_2 t_1}|' \in \mathbb{Z}\hat{\pi}' \otimes \mathbb{Z}\hat{\pi}'$$

Turaev cobracket

$\varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2)) \in \{\pm 1\}$ local intersection number

<u>Turaev</u>

(1) $\delta : \mathbb{Z}\hat{\pi}' \to \mathbb{Z}\hat{\pi}' \otimes \mathbb{Z}\hat{\pi}'$ well-defined

(2) $(\mathbb{Z}\hat{\pi}', [\ ,\ ], \delta)$: Lie bialgebra --- <u>Chas</u> involutive

δ extends to $\widehat{\mathbb{Q}\hat{\pi}}$

$\widehat{\mathbb{Q}\hat{\pi}}$: the completed Goldman – Turaev Lie bialgebra

<u>Remark</u> $(\mathfrak{g}, [\ ,\ ], \delta)$: Lie bialgebra $\Rightarrow$ $\mathrm{Ker}\,\delta < \mathfrak{g}$: Lie subalgebra

($\because$) Compatibility Axiom for Lie bialgebra)

<u>Theorem 1</u> (Kuno-K.)

S : compact connected oriented surface with $\partial \neq \phi$

$E \subset \partial S$ finite subset s.t. $\pi_0(E) \underset{\mathrm{incl}_*}{\cong} \pi_0(\partial S)$)

$\Longrightarrow$

$$(\delta|_{L^+}) \circ \tau = 0 : \mathcal{I}(S) \xrightarrow{\tau} L^+(S,E) \xrightarrow{\delta|_{L^+}} \mathbb{Q}\hat{\pi}(S) \hat{\otimes} \mathbb{Q}\hat{\pi}(S)$$

$\because$) ◎ $\forall$mapping class $\in \mathcal{M}(S)$ preserves the self-intersections of any curves on S

- "Compatibility Axiom" for $\mathbb{Q}\hat{\pi}$ -"bimodule" $\mathbb{Q}\widehat{\Pi S}(*_0, *_1)$
- injectivity of $\sigma : \mathbb{Q}\hat{\pi} \to \mathrm{Der}(\mathbb{Q}\hat{\mathcal{C}}(S,E))$ //

III. A tensorial description of the completed Goldman-Turaev Lie bialgebra

Group-like expansion

π : free group of finite rank

$H := H_1(\pi; \mathbb{Q}) = \pi^{abel} \otimes_{\mathbb{Z}} \mathbb{Q} \ni [x] := (x \bmod [\pi, \pi]) \otimes_{\mathbb{Z}} 1. \quad (x \in \pi)$

$\widehat{T} = \widehat{T}(H) = \prod_{k=0}^{\infty} H^{\otimes k}$ completed tensor algebra

$\Delta : \widehat{T} \to \widehat{T} \hat{\otimes} \widehat{T}$ coproduct, $\quad X \in H \mapsto \Delta(X) = X \hat{\otimes} 1 + 1 \hat{\otimes} X$

$p \geq 1. \quad \widehat{T}_p := \prod_{k \geq p} H^{\otimes k} \subset \widehat{T}$ two-sided ideal

Definition (Massuyeau)

$\theta : \pi \to \widehat{T}$ group-like expansion

$\overset{\text{def}}{\iff}$ 1) $\forall x, \forall y \in \pi \quad \theta(xy) = \theta(x)\theta(y)$

2) $\forall x \in \pi \quad \theta(x) \equiv 1 + [x] \bmod \widehat{T}_2$

3) $\forall x \in \pi \ \Delta\theta(x) = \theta(x) \hat{\otimes} \theta(x)$ (group-like element)

$N : \widehat{T} \to \widehat{T}$ "cyclic symmetrizer" "cyclicizer"

$N|_{H^{\otimes 0}} := 0$

$N(Y_1 \cdots Y_k) \overset{\text{def}}{=} \sum_{i=1}^{k} Y_i \cdots Y_k Y_1 \cdots Y_{i-1} \quad (Y_j \in H)$

S : <u>compact</u> connected surface <u>with $\partial \neq \phi$</u>

$\Rightarrow \pi_1(S)$: free of finite rank

$H = H_1(S:\mathbb{Q})$

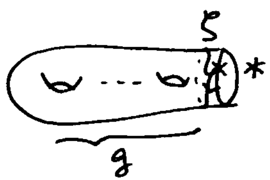
$\hat{T} = \hat{T}(H)$

<u>Kuno-K.</u>

$\theta : \pi_1(S) \to \hat{T}$ group-like expansion

$$\Rightarrow \quad \begin{array}{ccc} H_1(S : \coprod_{p \in S} \widehat{\mathbb{Q}\pi_1(S,p)}) & \underset{\cong}{\xrightarrow{\theta_*}} & H_1(\hat{T} ; \hat{T}^{\text{conjugate}}) \quad \left(\underset{/\!/}{} \mathrm{Tor}_1^{\hat{T}}(\hat{T}^{\text{conjugate}}, \mathbb{Q}) \right) \\ \| & \circlearrowleft & \| \\ \mathbb{Q}\hat{\pi}(S) & \underset{N\theta}{\xrightarrow{\cong}} & N(\hat{T}), \quad |\gamma| \mapsto N\theta(\gamma) \end{array}$$

isomorphism of filtered vector spaces.

◎ $S = \Sigma_{g,1} =$ 

$* \in \partial S$, $\zeta \in \pi_1(S,*)$ ∂-loop

$g \geqq 1$.

Identify $H = H^*$, $X \mapsto (Y \mapsto X \cdot Y)$ Poincaré duality ($\cdot$: intersection number)

$\omega = \sum_{i=1}^{g} A_i B_i - B_i A_i \in H^{\otimes 2}$ symplectic form

independent of the choice of a symplectic basis $\{A_i, B_i\}_{i=1}^{g} \subset H$

$\mathrm{Der}_\omega(\hat{T}) := \{ D : \text{continuous derivation of } \hat{T} ;\ D\omega = 0 \}$

$\mathrm{Der}_\omega(\hat{T}) \cong N(\hat{T}) \,(\subset H \otimes \hat{T} \overset{\text{P.d.}}{=} H^* \otimes \hat{T})\ ,\ D \mapsto D|_H$

<u>Definition</u> (Massuyeau)

$\theta : \pi_1(\Sigma_{g.1}, *) \to \hat{T}$ symplectic expansion

$\overset{\text{def}}{\iff}$ 1) $\theta : \pi_1(\Sigma_{g.1}, *) \to \hat{T}$ group-like expansion

2) $\theta(\zeta) = \exp(\omega) \in \hat{T}$

<u>Kuno-K.</u> $\theta : \pi_1(\Sigma_{g.1}, *) \to \hat{T}$ symplectic expansion

(1) $-N\theta : \widehat{\mathbb{Q}\hat{\pi}} \xrightarrow{\cong} N(\hat{T}) = \mathrm{Der}_\omega(\hat{T})$ Lie algebra homomorphism

(2)

$$\begin{array}{ccc} \widehat{\mathbb{Q}\hat{\pi}} \otimes \widehat{\mathbb{Q}\pi_1} & \xrightarrow{\sigma} & \widehat{\mathbb{Q}\pi_1} \\ {\scriptstyle -N\theta\otimes\theta}\downarrow\wr & \circlearrowleft & {\scriptstyle\theta}\downarrow\wr \\ \mathrm{Der}_\omega(\hat{T}) \otimes \hat{T} & \xrightarrow[\text{derivation}]{} & \hat{T} \end{array}$$

(3) $(N\theta)(L^+(\Sigma_{g.1}, \{*\})) = \widehat{\mathfrak{h}^+_{g.1}} \subset \mathrm{Der}_\omega(\hat{T})$

<u>Remark</u> <u>Massuyeau-Turaev</u> : an alternative proof using the homotopy intersection form.

$\delta^\theta := ((-N\theta)\hat{\otimes}(-N\theta))\circ\delta\circ(-N\theta)^{-1} : \mathrm{Der}_\omega(\hat{T}) \to \mathrm{Der}_\omega(\hat{T})\hat{\otimes}\mathrm{Der}_\omega(\hat{T})$

Turaev cobracket

<u>Theorem 2</u> (Massuyeau-Turaev, Kuno-K., independently) $\forall X_1, \cdots, \forall X_k \in H$

$$\delta^\theta(N(X_1\cdots X_k)) = \underbrace{\delta^{alg}(N(X_1\cdots X_k))}_{\text{degree } k-2} + \underbrace{\text{higher terms}}_{\text{degree } \geq k+1}$$

where

$$\delta^{alg}(N(X_1\cdots X_k)) = \sum_{i<j}(X_i\cdot X_j)\left\{\begin{array}{l} N(X_{i+1}\cdots X_{j-1})\hat{\otimes}N(X_{j+1}\cdots X_k X_1\cdots X_{i-1}) \\ -N(X_{j+1}\cdots X_k X_1\cdots X_{i-1})\hat{\otimes}N(X_{i+1}\cdots X_{j-1}) \end{array}\right\}$$

Schedler's cobracket ($\Leftarrow$ quiver theory)

($\because$ <u>Massuyeau-Turaev</u>'s tensorial description of the homotopy intersection form)

<u>Theorem 3</u> (Kuno-K.)

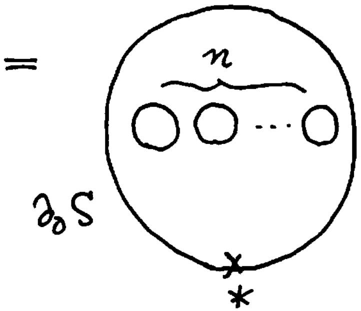
$\forall k \geq 2$ The k^{th} Morita trace factors through δ^{alg}.

$\rightsquigarrow$ geometric interpretation of the Morita traces "self-intersections"

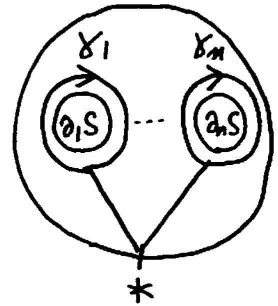
very recently

<u>Enomoto</u> The Enomoto-Satoh traces do <u>not</u> factor through δ^{alg}

<u>Problem</u> Do the Enomoto-Satoh traces factor through higher terms?

$S = \Sigma_{0,n+1} =$ 

$n \geqq 2$



$\partial S = \coprod_{i=0}^{n} \partial_i S$

$* \in \partial_0 S$

$\pi_1(S, *) = \langle \gamma_1, \gamma_2, \cdots, \gamma_n \rangle$ free of rank n

$x_i := [\gamma_i] \in H = H_1(S; \mathbb{Q})$, $1 \leq i \leq n$

$$\mathrm{sder}(\hat{T}) := \left\{ D : \hat{T} \to \hat{T} \text{ continuous derivation} \; ; \; \begin{array}{l} 1 \leq \forall i \leq n \;\; \exists u_i \in \hat{T} \;\; D(x_i) = [x_i, u_i] \\ D(x_1 + x_2 + \cdots + x_n) = 0 \end{array} \right\}$$

$\cup$

$$\mathrm{sder}(\hat{\mathcal{L}}) := \left\{ D \in \mathrm{sder}(\hat{T}) ; \; 1 \leq \forall i \leq n, \; \exists u_i \in \underbrace{\hat{\mathcal{L}}(H)}_{\text{completed free Lie algebra}} < \hat{T}(H) \;\; D(x_i) = [x_i, u_i] \right\}$$

$u = \sum_{i=1}^{n} x_i \otimes u_i \in H \otimes \hat{T} = \hat{T}_1$

$D_u \in \mathrm{Der}(\hat{T})$ defined by $D_u(x_i) := [x_i, u_i]$

- $D_u \in \mathrm{sder}(\hat{T}) \iff u \in N(\hat{T})$
- $u = \sum_{i=1}^{n} x_i \otimes u_i$, $v = \sum_{i=1}^{n} x_i \otimes v_i \in N(\hat{T})$

$[u, v] := N\left(\sum_{i=1}^{n} x_i \otimes (u_i v_i - v_i u_i)\right) \in N(\hat{T})$

$\Longrightarrow (N(\hat{T}), [\,,\,])$: Lie algebra

$\longrightarrow$ $\mathrm{sder}(\hat{T}) \cong N(\hat{T}) \Big/ \overline{\langle N(x_i^m) ; \; 1 \leq i \leq n, \; m \in \mathbb{N} \rangle}$

closure

Center $\cup$

Lie algebra isom.

Theorem 4 (Massuyeau-Turaev, Kuno-K., independently)

$S = \Sigma_{0,n+1}$, $n \geq 2$, $* \in \partial_0 S$.

$\theta : \pi_1(S, *) \to \widehat{T}$ group-like expansion

$1 \leq {}^\forall i \leq n$ $\ {}^\exists w_i \in \widehat{\mathcal{L}}(H)$ $\quad \theta(\gamma_i) = e^{w_i} e^{x_i} e^{-w_i}$

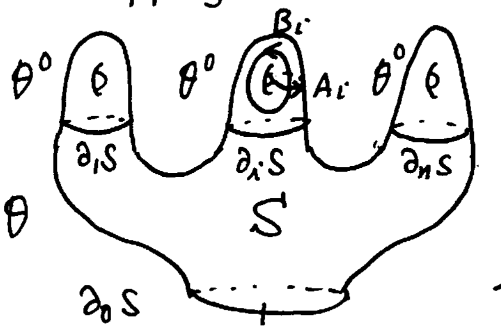
$\theta(\gamma_1 \gamma_2 \cdots \gamma_n) = e^{x_1 + x_2 + \cdots + x_n}$ (special / Artin / weak Kashiwara-Vergne)

$\Rightarrow -N\theta : \mathbb{Q}\widehat{\pi}(S) \xrightarrow{\cong} N(\widehat{T})$ Lie algebra homomorphism

((pf) · Kuno-K. : twisted homology on $(S, \partial S)$

· Massuyeau-Turaev : quiver theory ---- ${}^\forall g \geq 0$ ${}^\forall n \geq 0$ $\Sigma_{g,n+1}$)

capping trick



$\widetilde{S} \cong \Sigma_{n,1}$

θ : (special / Artin / weak KV) expansion for $\Sigma_{0,n+1}$

θ^0 : genus 1 symplectic expansion

$\Rightarrow \widetilde{\theta}$: symplectic expansion for $\widetilde{S} \cong \Sigma_{n,1}$

$\widetilde{H} := H_1(\widetilde{S} : \mathbb{Q}) \supset \{A_i, B_i\}_{i=1}^n$ symplectic basis as in the left

$$\mathrm{Ker}(\mathrm{ind} : \mathbb{Q}\widehat{\pi}(S) \to \mathbb{Q}\widehat{\pi}(\widetilde{S})) = \langle |\log \gamma_i| : 1 \leq i \leq n \rangle = H$$

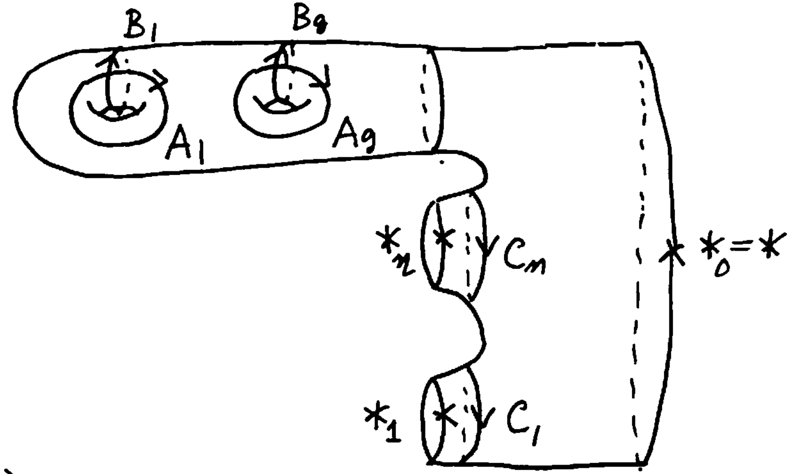
$\Rightarrow$ an alternative proof of

[a weak version of Theorem 4.

$-N\theta : \mathbb{Q}\widehat{\pi}(S)/H \xrightarrow{\cong} N(\widehat{T})/H$ Lie algebra homom.

(Satoh trace $\mapsto$ divergence cocycle

<u>general case</u> $S = \Sigma_{g,n+1}$, $g \geq 0$, $n \geq 1$



symplectic expansion | special / Artin / weak KV expansion

$\Downarrow$ gluing 2 expansions

$\theta : \pi_1(S, *) \longrightarrow \hat{T}$

a group expansion

$\{A_i, B_i\}_{i=1}^{g} \cup \{C_j\}_{j=1}^{n}$

a basis of $H_1(S;\mathbb{Q})$

$u, v \in N(\hat{T}) < \hat{T} \otimes H$

$$u = \sum_{i=1}^{g} u_i' \otimes A_i + u_i'' \otimes B_i + \sum_{j=1}^{n} u_j^0 \otimes C_j$$

$$v = \sum_{i=1}^{g} v_i' \otimes A_i + v_i'' \otimes B_i + \sum_{j=1}^{n} v_j^0 \otimes C_j$$

θ induces a Lie algebra structure on $N(\hat{T})$

<u>Goldman bracket</u>

$$[u, v] = +N\Big(\sum_{i=1}^{g} u_i' v_i'' - u_i'' v_i' + \sum_{j=1}^{n} C_j (u_j^0 v_j^0 - v_j^0 u_j^0)\Big)$$

θ induces a Lie algebra homomorphism

$$\sigma : N(\hat{T}) \to \mathrm{Der}(\widehat{\mathbb{Q}\mathcal{C}(\Sigma_{g,n+1}, E)})$$

Introduce a derivation D_u^0 on $\hat{T}$ by

$$D_u^0(A_i) = u_i''$$

$$D_u^0(B_i) = -u_i'$$

$$D_u^0(C_j) = C_j u_j^0 - u_j^0 C_j$$

Then, for $\forall w \in \widehat{\mathbb{Q}\Pi\Sigma_{g,n+1}}(*_a, *_b) \overset{\theta}{\cong} \hat{T}$, $0 \leq a, b \leq n$, we have

$$\sigma(u)(v) = \begin{cases} D_u^0(w) & \text{if } a = b = 0 \\ D_u^0(w) - u_a^0 w & \text{if } a \geqq 1,\ b = 0 \\ D_u^0(w) + w u_b^0 & \text{if } a = 0,\ b \geqq 1 \\ D_u^0(w) - u_a^0 w + w u_b^0 & \text{if } a, b \geqq 1. \end{cases}$$