On an analogue of Culler-Shalen theory for higher dimensional representations

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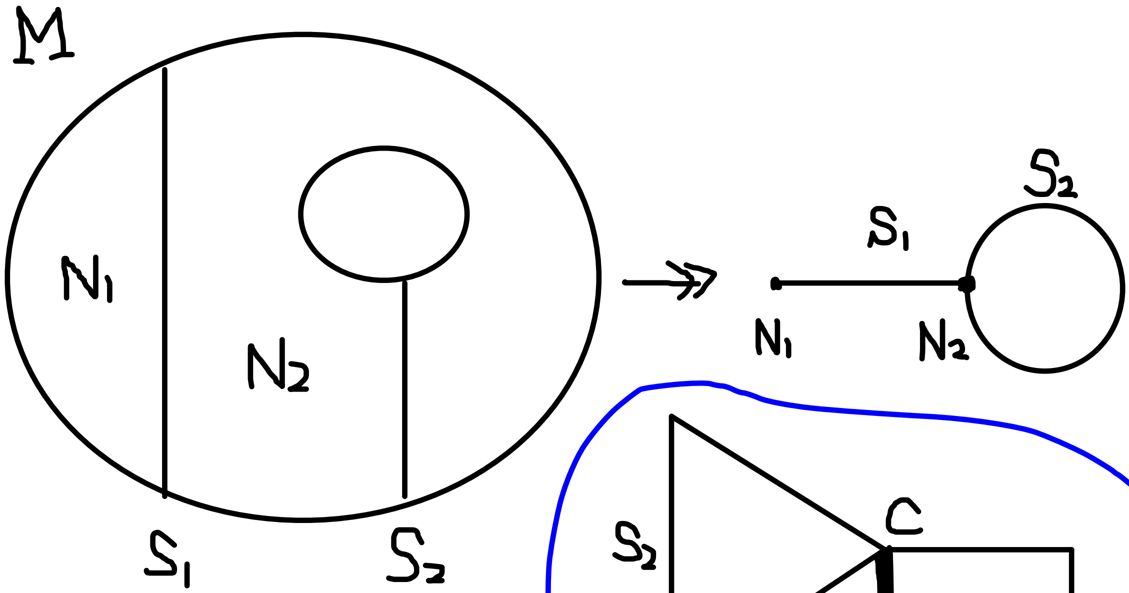
0. Introduction

M : compact irreducible oriented 3-manifold

[Culler-Shalen '83]
An ideal point ("∞-point") of $\mathrm{Hom}(\pi_1 M, SL_2(\mathbb{C}))$
gives essential surfaces in M.

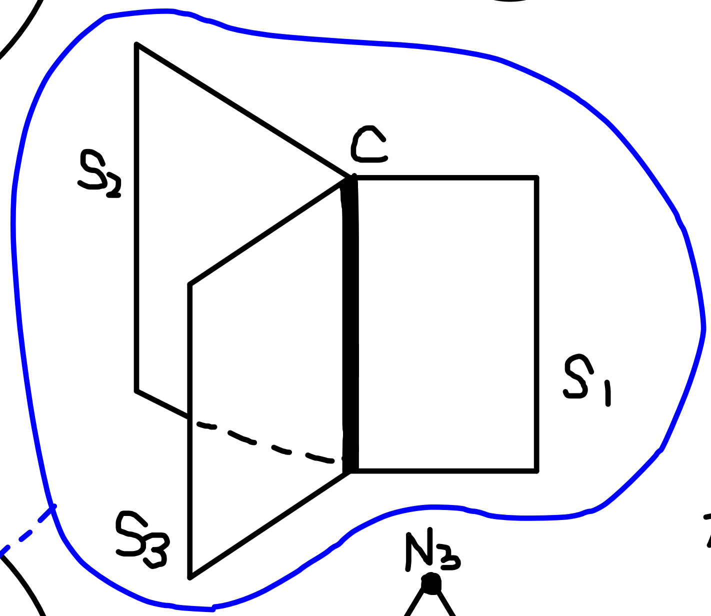
Today's Subject
An ideal point of $\mathrm{Hom}(\pi_1 M, SL_n(\mathbb{C}))$ $(n \geq 3)$
gives "essential" "Y-branched surfaces" in M.

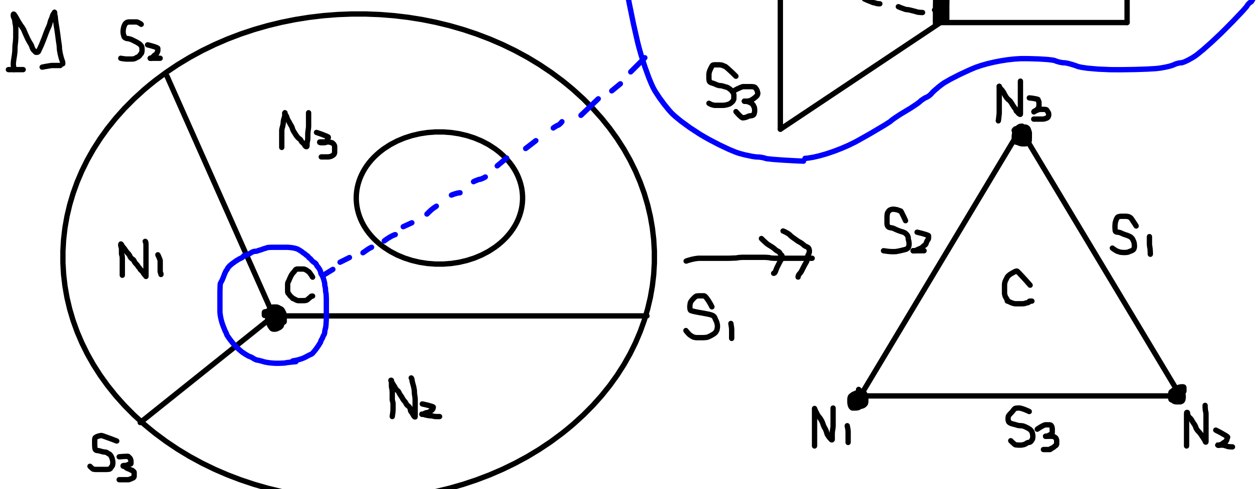


$\pi_1 M$

$$\pi_1 N_1 \hookleftarrow \pi_1 S_1 \hookrightarrow \pi_1 N_2 \leftleftarrows \pi_1 S_2$$

graph of groups





$\pi_1 M$

$$\begin{tikzcd}
 & & \pi_1 N_3 & & \\
 & \pi_1 S_2 \arrow[ur, hook] \arrow[ddl, hook] & & \pi_1 S_1 \arrow[ul, hook] \arrow[ddr, hook] & \\
 & & \pi_1 C \arrow[ul, hook] \arrow[ur, hook] \arrow[d, hook] & & \\
\pi_1 N_1 & & \pi_1 S_3 \arrow[ll, hook] \arrow[rr, hook] & & \pi_1 N_2
\end{tikzcd}$$

2-complex of groups

Overview of Culler-Shalen theory

① [Stallings]
A nontrivial action of $\pi_1 M$ on a tree T gives essential surfaces in M.

(nontrivial) $^\forall s \in T^{(0)}$, $(\pi_1 M)_s \neq \pi_1 M$

② [Bruhat-Tits, Serre]
canonical action of $SL_2(F)$ on a tree for a discrete valuation field (F, v)

e.g., lowest degree : $\mathbb{C}((t))^\times \to \mathbb{Z}$

③ (Heart of Culler-Shalen theory)
The action of $\pi_1 M$ on the tree associated to an ideal point of $\mathrm{Hom}(\pi_1 M, SL_2(\mathbb{C}))$ is nontrivial.

Outline

1. SL_n-character variety

2. Nontrivial actions on buildings

3. Construction of Y-branched surfaces

4. Further study

1. SL_n-character variety

M : compact irreducible oriented 3-manifold

$R_n := \mathrm{Hom}(\pi_1 M, SL_n(\mathbb{C}))$

: affine algebraic set / $\mathbb{C}$

$$\chi_\rho : \pi_1 M \to \mathbb{C}, \qquad \rho \in R_n$$
$$\gamma \mapsto \mathrm{tr}\,\rho(\gamma)$$

$X_n := \{\chi_\rho \,;\, \rho \in R_n\}$

: $SL_n(\mathbb{C})$-character variety

$$I_\gamma : X_n \to \mathbb{C}, \qquad \gamma \in \pi_1 M$$
$$\chi_\rho \mapsto \mathrm{tr}\,\rho(\gamma)$$

Theorem (Procesi).

$\{\gamma_1, \cdots, \gamma_m\}$: generator set of $\pi_1 M$

$\rightsquigarrow \{I_{\gamma_{i_1} \cdots \gamma_{i_k}}\}_{1 \leq i_1, \cdots, i_k \leq m}^{1 \leq k \leq 2^n - 1} : X_n \to \mathbb{C}^N$

gives affine coordinates.

$(X_n = R_n /\!/ SL_n(\mathbb{C}))$

Theorem (Menal-Ferrer - Porti).

M : hyperbolic 3-manifold with l torus cusps

$\rho_0 : \pi_1 M \to SL_2(\mathbb{C})$: lift of a holonomy representation

$\iota_n : SL_2(\mathbb{C}) \to SL_n(\mathbb{C})$: irreducible representation

$\rightsquigarrow$ (I) $\chi_{\iota_n \circ \rho_0}$ is a smooth point of X_n.

(II) irreducible component containing $\chi_{\iota_n \circ \rho_0}$ is $l(n-1)$-dimensional.

C : affine curve $/\mathbb{C}$

$$\rightsquigarrow \exists\ \begin{array}{ccc} \mathbb{C}^N & \to & \mathbb{CP}^N \\ \cup & & \cup \\ C & \to & \overline{C} \end{array} \quad \text{s.t.}\quad \overline{C}\setminus C : \text{smooth}$$

$\rightsquigarrow$ point of $\overline{C}\setminus C$ is called an **ideal point**

Suppose $\exists$ curve $C\subset X_n$

$$\begin{array}{ccccccc} \tilde{x} & \in & \overline{D} & \supset & D & \subset & R_n \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ x & \in & \overline{C} & \supset & C & \subset & X_n \end{array}$$

x : ideal point

$D\subset R_n$: lift of C

$\rightsquigarrow$ $\boxed{\begin{array}{l} P_n : \pi_1 M \to SL_n(\mathbb{C}(D)) : \text{tautological representation} \\ \qquad \gamma \mapsto (\rho\mapsto a_{ij})_{i,j}\,,\quad \rho(\gamma) = (a_{ij})_{i,j} \\ v_{\tilde{x}} : \mathbb{C}(D)^\times \to \mathbb{Z} : \text{discrete valuation at } \tilde{x} \end{array}}$

2. Nontrivial actions on buildings

[Bruhat-Tits, Iwahori-Matsumoto]

(F, v) : discrete valuation field

$\rightsquigarrow$ canonical action of $SL_n(F)$

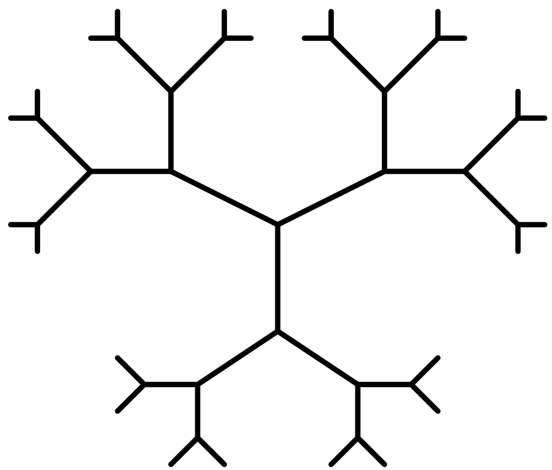
on a contractible $(n-1)$-building B_v s.t.

$A \in SL_n(F)_\sigma \Rightarrow A$ fixes σ pointwise

($B_v^{(0)} = \{[\Lambda] ; \Lambda$: lattice in $F^n\}$, $\Lambda \sim \alpha\Lambda$, $\alpha \in F^\times$)

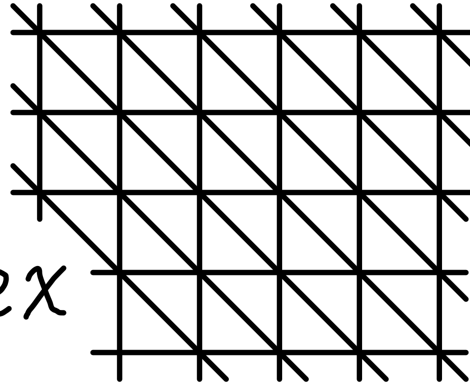
eg, $v_2 : \mathbb{Q}^\times \to \mathbb{Z}$

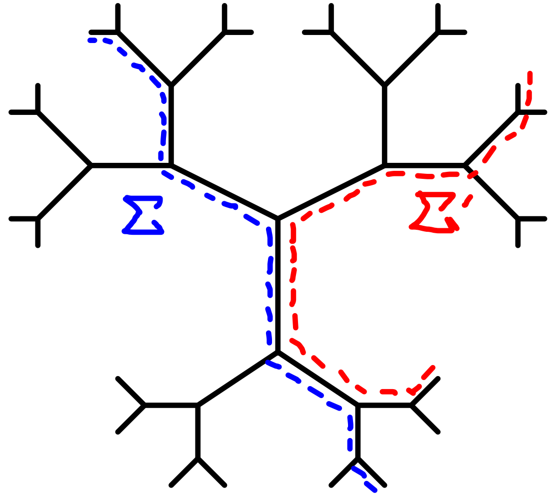
: 2-adic valuation

$SL_2(\mathbb{Q}) \curvearrowright$  $= B_{v_2}$

Axiomatic definition

$\mathbb{R}^2$





Δ : n-simplicial complex

Δ : (thin) chamber complex

$\overset{\text{def}}{\iff}$ (I) σ : maximal $\Rightarrow \dim \sigma = n$

(II) $\dim \sigma = n-1 \Rightarrow {}^{\exists}$distinct $\sigma_1, \sigma_2 \supsetneq \sigma$

(III) $\dim \sigma = \dim \sigma' = n \Rightarrow {}^{\exists}\sigma = \sigma_0 \sim \sigma_1 \sim \quad \sim \sigma_k = \sigma'$: adjacent

Δ : (thick) building

$\overset{\text{def}}{\iff} \Delta = \bigcup \Sigma$, Σ (thin) chamber complex s.t.

(I) $\dim \sigma = n-1 \Rightarrow {}^{\exists}$distinct $\sigma_1, \sigma_2, \sigma_3 \supsetneq \sigma$

(II) ${}^{\forall}\sigma, \sigma'$ ${}^{\exists}\Sigma \supset \sigma, \sigma'$

(III) $\sigma, \sigma' \subset \Sigma \cap \Sigma' \Rightarrow {}^{\exists}\Sigma \xrightarrow{\sim} \Sigma'$ fixing σ, σ' pointwise

x : ideal point of a curve in X_n

$\rightsquigarrow$ $\rho_n : \pi_1 M \to SL_n(\mathbb{C}(D)) \curvearrowright B_{v_{\tilde{x}}}$

$v_{\tilde{x}} : \mathbb{C}(D)^\times \to \mathbb{Z}$

$\rightsquigarrow$ $\pi_1 M \curvearrowright B_{v_{\tilde{x}}}$

Theorem A (Hara-K.).

x : ideal point of a curve in X_n

(I) ${}^\exists S \in B^{(0)}_{v_{\tilde{x}}}$ s.t. $\gamma \in (\pi_1 M)_S \Rightarrow I_\gamma(x) \in \mathbb{C}$

(II) $\pi_1 M \curvearrowright B_{v_{\tilde{x}}}$: nontrivial,

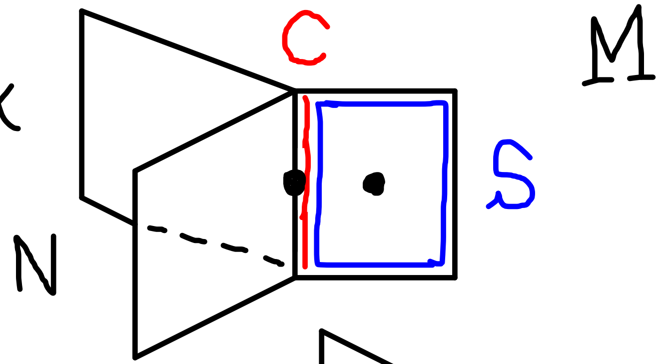
i.e., ${}^\forall S \in B^{(0)}_{v_{\tilde{x}}}$, $(\pi_1 M)_S \neq \pi_1 M$

(I) + Process $\Rightarrow$ (II)

$x \in \mathbb{C}P^N \setminus \mathbb{C}^N \quad \therefore\ {}^\exists \gamma \in \pi_1 M$ s.t. $I_\gamma(x) = \infty$.

3. Construction of Y-branched surfaces

$\Sigma \subset M$: connected 2-CW-complex

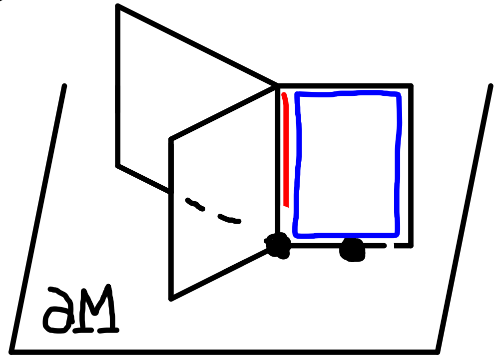


Σ : Y-branched surface

$\overset{def}{\iff}$ (I) locally, Σ looks like

(II) $\forall C$, $N(C) \cap \Sigma = S^1 \times Y$ or $[0,1] \times Y$

(III) $\forall S$: orientable



Σ : essential

$\overset{def}{\iff}$ (I) $\forall$component N of $M \setminus \mathrm{Int}\, N(\Sigma)$, $\pi_1 N \to \pi_1 M$: not surjective

(II) $\forall C, S, N$, $\pi_1 C \to \pi_1 S$, $\pi_1 S \to \pi_1 N$: injective

Remark.

An essential surface is an essential Y-branched surface (with no branch set).

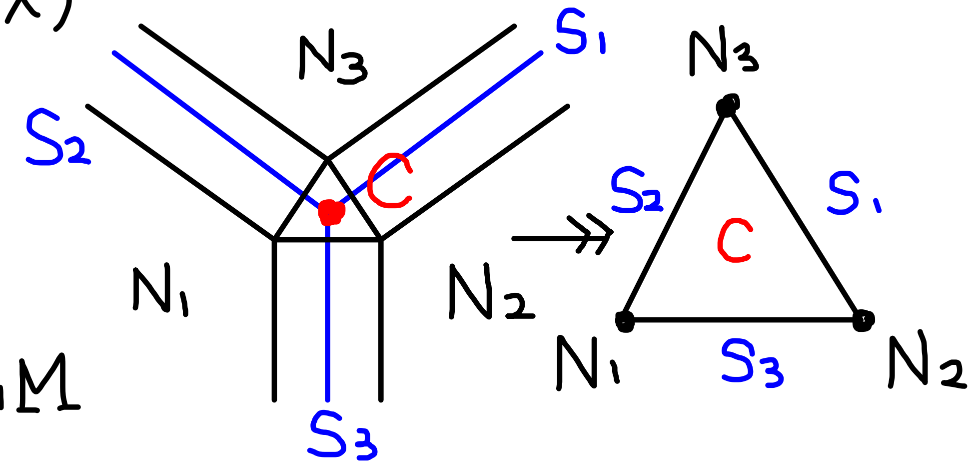
$\Sigma \subset M$: essential Y-branched surface

$\rightsquigarrow$ $M \twoheadrightarrow$ (2-simplicial complex)

$N \rightarrow$ vertex

$N(S) \rightarrow$ edge

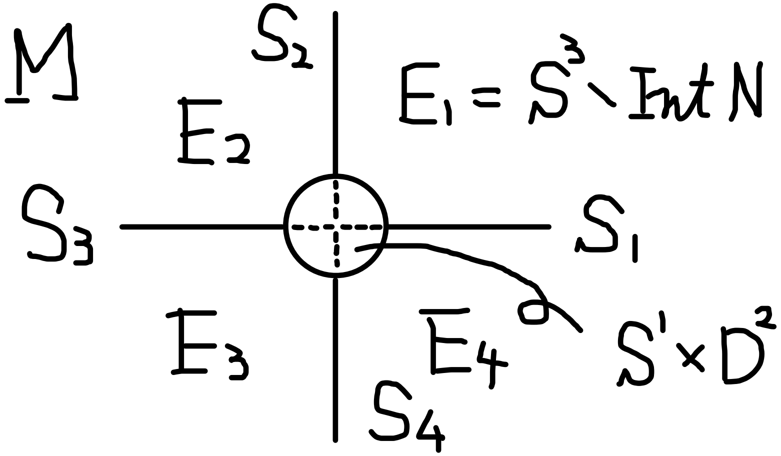
$N(C) \rightarrow$ triangle



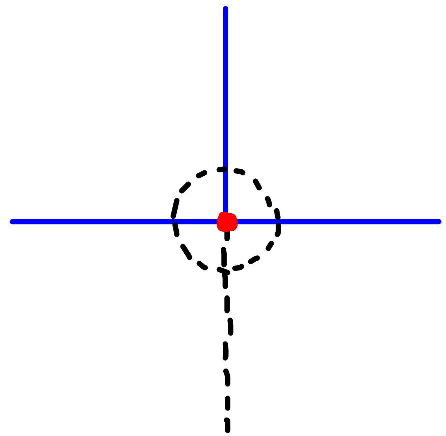
$\rightsquigarrow$ A nontrivial presentation of $\pi_1 M$ as a 2-complex of groups

Example. M = m-fold cyclic branched cover of S^3 along a nonfibered knot $(m \geq 3)$

$S \subset S^3$: minimal genus Seifert surface



$E_1 = S^3 \setminus \mathrm{Int}\, N(S)$

$\Sigma =$ 

Theorem B (Hara-K.).

$n = 3$ or ($n \geqslant 4$ and $\partial M \neq \phi$)

$\pi_1 M \curvearrowright B$: contractible $(n-1)$-building s.t.

(I) $\forall s \in B^{(0)}$, $(\pi_1 M)_s \neq \pi_1 M$

(II) $\gamma \in (\pi_1 M)_\sigma \Rightarrow \gamma$ fixes σ pointwise

$\rightsquigarrow$ essential Y-branched surfaces in M

Main Theorem (Theorem A + Theorem B).

$n = 3$ or ($n \geqslant 4$ and $\partial M \neq \phi$)

χ : ideal point of a curve in X_n

$\rightsquigarrow$ essential Y-branched surfaces in M

Idea of Proof of Theorem B

Take a triangulation of M.

① Construct a $\pi_1 M$-equivariant simplicial map

$\tilde{f} : \tilde{M} \to B^{(2)}$ ($\leftarrow$ spine of M, $\pi_1 B^{(2)} = 1$)

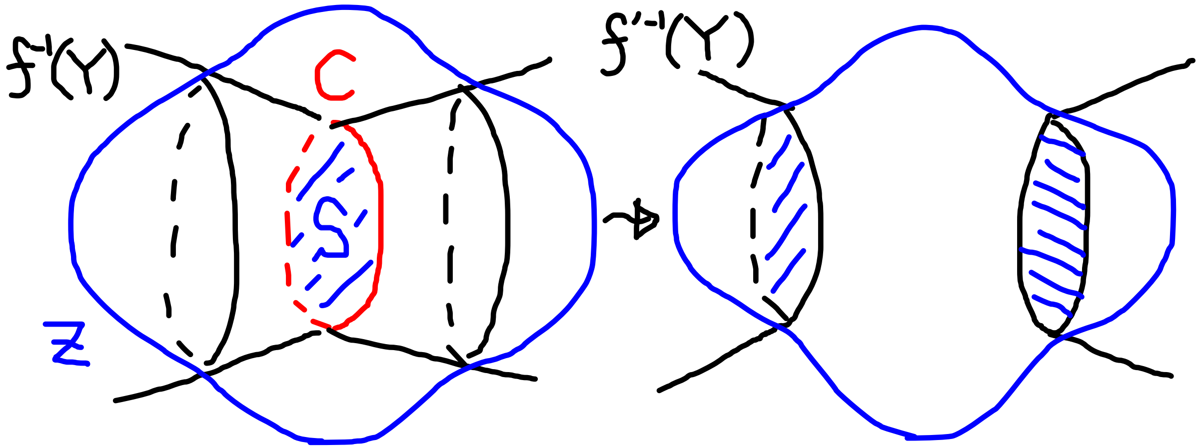
$\rightsquigarrow$ $f : M \to B^{(2)}/\pi_1 M$, $Y := \bigcup_{\Delta \subset B^{(2)}}$

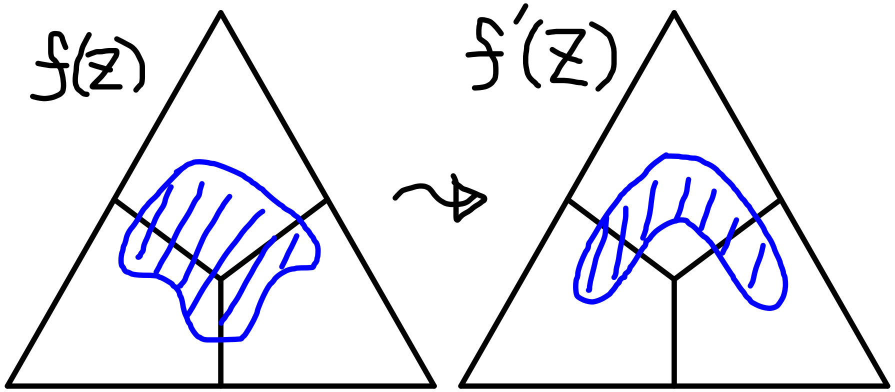
$\rightsquigarrow$ $f^{-1}(Y)$: Y-branched surfaces, $\neq \emptyset$

($\leftarrow$ $\pi_1 M \curvearrowright B$: nontrivial)

② Reduce $f^{-1}(Y)$ to be essential.

e.g., suppose $\ker(\pi_1 C \to \pi_1 S) \neq 1$ $\rightsquigarrow$ Replace $f|_Z$





4. Further study

$X_n^{ab} := \{\chi_\rho \,;\, \rho \in R_n : \text{abelian}\}$

Proposition (Hara-K.).
$K \subset S^3$: nontrivial knot, $M = S^3 \setminus \mathrm{Int}\, N(K)$
χ: ideal point of a curve in X_n^{ab}
$\rightsquigarrow$ A Seifert surface is obtained from χ

$(\iota_n)_* : X_2 \to X_n$, $\iota_n : SL_2(\mathbb{C}) \to SL_n(\mathbb{C})$: irreducible

Proposition (Hara-K.).
χ: ideal point of a curve $C \subset X_2$
χ': corresponding ideal point of $\overline{(\iota_n)_*(C)} \subset X_n$
$\rightsquigarrow$ Essential surfaces obtained from χ are also obtained from χ'.

Questions.

[Boyer-Zhang, Motegi, Schanuel-Zhang]
$\exists$ essential surfaces not obtained from X_2

① Does X_n $(n \geq 3)$ give (non-branched) essential surfaces not obtained from X_2?

[Cooper-Long-Thistlethwaite]
A holonomy representation $\rho_0 : \pi_1 M \to SO^+(3,1) \subset SL_4(\mathbb{C})$ of some closed hyperbolic manifold can be deformed in X_4.

② Find a non-Haken manifold s.t. $\dim X_3 \geq 1$.

③ Systematic way to construct examples.
([Fock-Goncharov, Garoufalidis-Goerner-Zickert])