Extensions of mapping classes from the boundary of a handlebody

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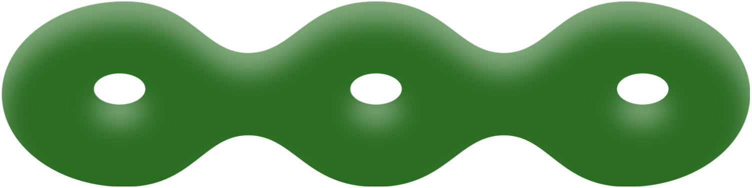
Extensions to handlebodies

Let S be a closed surface and H a handlebody of genus g (with $g \geq 2$) together with an identification $S \to \partial H$. Let $f : S \to S$ be a diffeomorphism.

Handlebody



Extensions to handlebodies

Let S be a closed surface and H a handlebody of genus g (with $g \geq 2$) together with an identification $S \to \partial H$. Let $f : S \to S$ be a diffeomorphism.

Does f extends to a diffeomorphism $F : H \to H$? I.e. Is there a diffeomorphism $F : H \to H$ whose restriction to ∂H is in the mapping class defined by f?

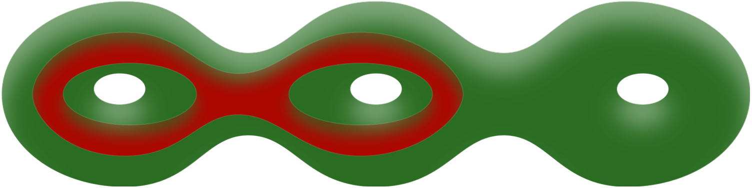
More reasonably, is there $n \in \mathbb{N}$ such that f^n extends to a diffeomorphism $F : H \to H$?

Partial extension

More generally, one can ask to what submanifold $W \subset H$ some power of f extends and what the "biggest" such submanifold is.

Let $W \subset H$ be a compression body such that $\partial H \subset \partial W$.

Compression body=hollow handlebody



Partial extension

More generally, one can ask to what submanifold $W \subset H$ some power of f extends and what the "biggest" such submanifold is.

Let $W \subset H$ be a compression body such that $\partial H \subset \partial W$.

We say that (some power of) f extends to W if there is $F : W \to W$ and $n \in \mathbb{N}$ such that $F_{|\partial H}$ is isotopic to f^n.

We say that (some power of) f partially extends if there is a non trivial compression body (i.e. W is not an I-bundle) $W \subset H$ to which f extends.

3-manifolds

Let M be a closed 3-manifold with a Heegaard splitting $M = H_1 \cup H_2$ with Heegaard surface $S = \partial H_1 = \partial H_2$ and let $f : S \to S$ be a diffeomorphism. Is f the restriction of a homeomorphism $F : M \to M$. More generally, to what submanifold of M does f extend.

Statements

Theorem (Biringer, Johnson, Minsky)

Let $f : S \to S$ be a pseudo-Anosov diffeomorphism. Then (some power of) f partially extends to H if and only if the (un)stable lamination of f is a projective limit of meridians.

Theorem (Biringer, L)

(Some power of) A non-annular diffeomorphsim $f : S \to S$ partially extends if and only if its invariant lamination can be finitely extended to contain a Hausdorff limit of meridians.

These statements can easily be extended to general 3-manifolds with compressible boundary.

Please look impressed

These statements are optimal in the following sense :

- $\exists f, n$ such that f does not extend and f^n extends.
- There are maps f that satisfy the conclusion and extend to H and others that only partially extend.

Notice also that $\exists f$ that extends to H but do not extend to any subcompression body.

Classification of diffeomorphsims

A diffeomorphsim $f : S \to S$ is :

- periodic if $\exists n \in \mathbb{N}$ such that f^n is homotopic to the identity map,
- pseudo-Anosov if the collection $f^n(c), n \in \mathbb{N}$ is infinite for any simple closed curve $c \subset S$,
- reducible otherwise.

Invariant laminations

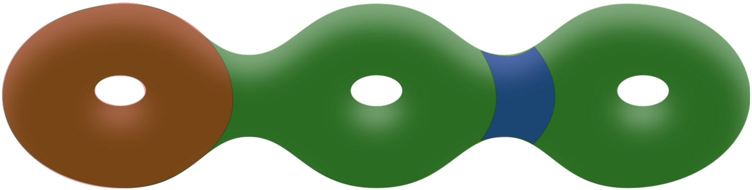
A pseudo-Anosov diffeomorphism has two invariant projective measured laminations $[\lambda^{\pm}]$, the stable one and the unstable one. They satisfy $f(\lambda^+) = r\lambda^+$ and $f(\lambda^-) = r^{-1}\lambda^-$ with $r > 1$.

Given a reducible diffeomorphism f, there is $k \in \mathbb{N}$ and a decomposition $S = S_{id} \cup S_1 \cup \ldots \cup S_r$ into essential subsurfaces preserved by f^k such that $f^k|S_{id} = id$ and there are projective measured laminations $\lambda_i \in \mathcal{PML}(S_i)$ such that if $f_i := f^k|S_i$ either

- S_i is an annulus and f_i is a power of a Dehn twist about λ_i,
- S_i is neither an annulus nor a pair of pants, f_i is pseudo-Anosov and λ_i is the stable lamination of f_i.

We say that $\lambda_f = \bigcup_i \lambda_i$ is the invariant geodesic lamination of f.

Invariant surfaces



Meridians and Masur domain

A merdian $m \subset \partial H$ is a simple closed curve that is homotopically trivial in H but not in ∂H.

Let $\Lambda(H) \subset \mathcal{PML}(\partial H)$ be the set of projective limits of meridians. Masur domain is the set $\mathcal{O}(H) = \{\lambda \in \mathcal{PML}(\partial H) | \, i(\lambda, \mu) > 0 \, \forall \mu \in \Lambda(H)\}$.

A geodesic lamination λ can be finitely extended to contain a Hausdorff limit of meridians if by adding finitely many leaves to λ we can get a lamination that contains a Hausdorff limit of meridians.
A filling laminations lies in Masur domain if and only it can be finitely extended to contain a Hausdorff limit of meridians.

Relative Masur domain

A geodesic lamination λ can be extended by finitely many leaves to contain a Hausdorff limit of meridians if and only if one of the following holds :

- λ is disjoint from a meridian in S,
- some sublamination $\lambda_1 \subset \lambda$ is the support of an element of $\Lambda(S(\lambda_1), M)$,
- there are sublaminations $\lambda_1, \lambda_2 \subset \lambda$ with $S(\lambda_1)$ and $S(\lambda_2)$ disjoint and incompressible, such that λ_1 and λ_2 are homotopic in M.

Non annular laminations

A homotopy of laminations is a continuous map $\lambda \times [0,1] \longrightarrow H$ where λ is a geodesic lamination on some hyperbolic surface, considered with the subspace topology.

A geodesic lamination $\mu \subset \partial H$ is non-annular if any homotopy of laminations $\lambda \times [0,1] \longrightarrow H$ with $\lambda \times \{0,1\} \subset \mu$ can be homotoped to a homotopy of laminations in the boundary $\lambda \times [0,1] \longrightarrow \partial H$.

Annular laminations

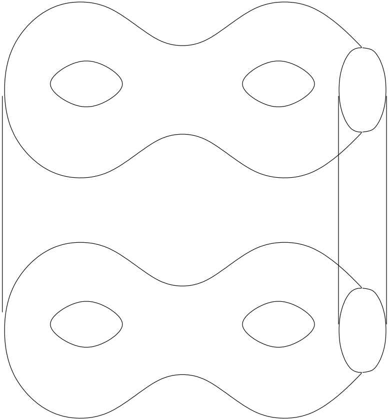
The boundary of an essential annulus is an annular lamination.

Suppose that $H = \Sigma \times [0,1]$, where Σ is a surface with boundary. Then if $f, g : \Sigma \to \Sigma$ are homeomorphisms that agree on $\partial\Sigma$, define

$$[f,g] : \partial H \longrightarrow \partial H, \quad [f,g](x,t) = \begin{cases} (f(x),1) & t = 1 \\ (g(x),0) & t = 0 \\ (f(x),t) & x \in \partial\Sigma \end{cases}$$

If $g = f^k$ for some $k \in \mathbb{N}$ then the invariant lamination of $[f,g]$ is annular.

Annular laminations



Issue with annular laminations

For any homeomorphism $f : \Sigma \to \Sigma$ the map $[f, f]$ extends to a homeomorphism of H.
However, if f is pseudo-Anosov the map $[f, f^2]$ will not have a power that extends even partially to H, even though it has the same associated projective measured lamination as $[f, f]$.

Restatements

Theorem (Biringer, Johnson, Minsky)

Let $f : S \to S$ be a pseudo-Anosov diffeomorphism. Then (some power of) f partially extends to H if and only if the (un)stable lamination of f lies in $\Lambda(H)$.

Theorem (Biringer, L)

Let $f : S \to S$ be a diffeomorphism with non-annular invariant lamination $\lambda_f = \bigcup_i \lambda_i$. Then if S_{id} is compressible or $\lambda_i \in \Lambda(S_i, H)$ for some i, (some power of) f partially extends to H. If f is non-annular, the converse holds as well.

Easy implication

If f partially extends then λ_f can be finitely extended to contain a Hausdorff limit of meridians.

Let $W \subset H$ be a compression body to which f^n extends and let $m \subset \partial H$ be a meridian (in W and hence in H). For any k, $f^{kn}(m)$ is a meridian. Let P be a pants decomposition of ∂H that contains m. Up to extracting a subsequence $f^{kn}(P)$ converges in the Hausdorff topology to a geodesic lamination that finitely extends λ_f and contains the Hausdorff limit of $f^{kn}(m)$.

Here comes hyperbolic geometry

Fix a point $X \in \mathcal{T}(S)$ and consider the representation $\rho_n : \pi_1(S) \to PSL(2, \mathbb{C})$ defined as follows.

$\ker \rho_n = \ker(f^n \circ i)_*$ where $i_* : \pi_1(S) \to \pi_1(H)$ is the map induced by the inclusion.
Then $\mathbb{H}^3/\rho_n(\pi_1(S)$ is a handlebody and we assume that ρ_n is convex-cocompact and that its conformal structure at infinity corresponds to X.

Algebraic limit and extensions

Adapting classical results to the unfaithful setting, one proves that $\{\rho_n\}$ contains a subsequence that converges to a discrete representation $\rho_\infty : \pi_1(S) \to PSL(2,\mathbb{C})$.

Lemma (Biringer, Johnson, Minsky)

The map f does not partially extend if and only if any accumulation point of $\{\rho_n\}$ is faithful.

Such behavior may happen :

"If you are unfaithful in many different ways then you are eventually faithful" (An anonymous customer of Sofitel New York)

Ends and invariant laminations

The proof of the remaining implication ([λ_f can be extended to contain a limit of meridians] $\Rightarrow$ [f partially extends]) goes by contradiction.

Assuming that f does not partially extend, we want to show that λ_f does not satisfy the conditions of our Theorem.

If S_{id} is compressible then it is easy to see that f partially extends so we also assume that S_{id} is incompressible.

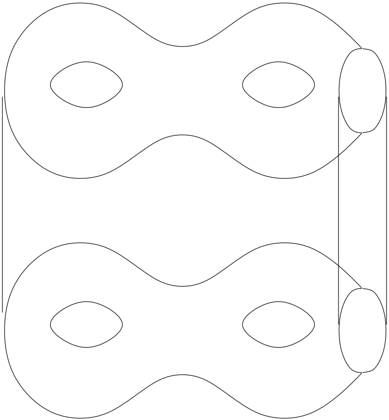
By the Lemma above, ρ_∞ is faithful. Furthermore by assumption, the top end of $M_\infty = \mathbb{H}^3/\rho_\infty(\pi_1(S)$ is geometrically finite without parabolic.

We show that each simple closed curve in λ_f must be a parabolic in M_∞ and that any stable lamination of f^k must be an ending lamination. Properties of parabolics, degenerate ends and disks sets give the expected contradicition.

Diffeomorphism with annular stable laminations

Given $f : S \to S$ with invariant lamination λ_f, there is $G : H \to H$ such that, if $g = G_{|\partial H}$, then $\lambda_{g \circ f} = \lambda_{f \circ g}$ is non-annular.

Stabilizing annular diffeomorphisms



Diffeomorphism with annular stable laminations

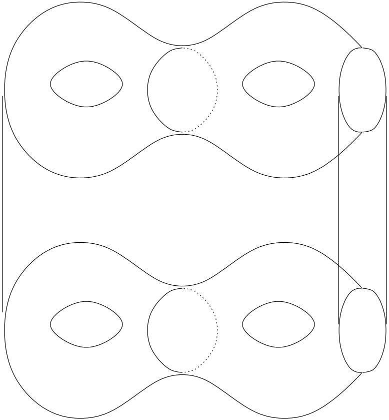
Given $f : S \to S$ with invariant lamination λ_f, there is $G : H \to H$ such that, if $g = G_{|\partial H}$, then $\lambda_{g \circ f} = \lambda_{f \circ g}$ is non-annular.

If $g \circ f$ partially extends, does f partially extends ?

One needs to check whether or not g and f extends to the same subcompression body.

Extending to incompressible submanifolds

How about partially extending to something that is not a compression body.



The other algebraic limit

We can also consider the representation $\sigma_n : \pi_1(H) \to PSL(2, \mathbb{C})$ whose conformal structure at infinity is $f_*^n(X)$.
The representations σ_n and ρ_n have the same image (up to conjugacy) but different markings.
If f does not partially extend (and λ_f is (strongly) non-annular), σ_n converges (by results of Kleineidam-Souto and Kim-L-Ohshika) to a representation $\sigma_\infty : \pi_1(H) \to PSL(2, \mathbb{C})$ that is discrete and faithful (by results of Chuckrow).

Geometric limits

Proposition

If f does not partially extend, λ_f is (strongly) non-annular and $S_{id} \neq \emptyset$ then the geometric limit of σ_n and ρ_n is obtained by gluing $\mathbb{H}^3/\sigma_\infty(\pi_1(H))$ to $\mathbb{H}^3/\rho_\infty(\pi_1(S))$ along its geometrically finite ends.

This provides a natural extension of Brock's examples (constructed as limit of quasi-Fuchsian groups) to manifolds with compressible boundary.

Geometric limits