THE COLORED JONES POLYNOMIAL AND AJ CONJECTURE

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Abstract. We present the basics of the colored Jones polynomials and discuss the AJ conjecture.

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1. Jones polynomial

1.1. **Knots and links in $\mathbb{R}^3 \subset S^3$.** Fix the standard $\mathbb{R}^3$. *An oriented link* L is a compact 1-dimensional oriented smooth submanifold of $\mathbb{R}^3 \subset S^3$. Denote by $\#L$ the number of connected components of L. A link of 1 component is called a *knot.* By convention, the empty set is also considered a link.

A *framed oriented link* L is a link equipped with a smooth normal vector field V, which is a function $V : L \to \mathbb{R}^3$, such that $V(x)$ is not in the tangent space $T_x L$ for every $x \in L$. One should consider $V(x)$ as an element in the tangent space $T_x\mathbb{R}^3$ of $\mathbb{R}^3$ at x.

Two oriented links L_1 and L_2 are *equivalent* if there is a smooth isotopy $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(L_1) = (L_2)$. Here h is a smooth isotopy if there is smooth map $H : \mathbb{R}^3 \times [0, 1] \to \mathbb{R}^3$ such that for every $t \in [0, 1]$, $h_t(x) : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $h_t(x) = H(x, t)$, is a diffeomorphism of $\mathbb{R}^3$, and $h_0 = \mathrm{id}, h_1 = h$.

Similarly, two framed oriented links (L_1, V_1) and (L_2, V_2) are equivalent if there is a smooth isotopy $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(L_1) = (L_2)$ and for every $x \in L_1$, $dh_x(V_1(x)) = V_2(h(x))$. Here dh_x is the derivative of h at x.

A (framed) link is *ordered* if there is an order on the set of its components. Then the equivalence relation is required to preserve the order.

The framing of a component can be specified by thickening the component to a ribbon in the direction of the framing vector.

Usually we don't distinguish between a link and its equivalence class.

Un-oriented links, un-oriented framed links and their equivalence classes are defined similarly.

A link invariant is a map

$$I : \{\text{equivalence classes of links}\} \to \mathcal{R},$$

where $\mathcal{R}$ is a set.

Example 1.1. For unoriented unframed links, the link group $\pi_1(\mathbb{R}^3 \setminus L)$ is a link invariant.

1.2. **Link diagram, blackboard framing.** One often studies an (oriented or unoriented) link L by its diagram on $\mathbb{R}^2$, which is the projection D of L onto $\mathbb{R}^2$ (in general position), together with the "over/under" information at each crossing point. An (oriented) link diagram defines an equivalence class of (oriented) links.

A link diagram comes with the *blackboard framing*, in which the framing vectors are in the plane $\mathbb{R}^2$. If the framed link determined by a link diagram D with the blackboard framing is isotopic the a framed link L, we say that D is a blackboard diagram of L.

It is known that two unoriented unframed link diagrams define the same class of links if and only if they are related by a sequence of Reidemeister moves RI, RII, and RIII and isotopies of the plane. The Reidemeister moves are listed in Figure 1 and 2. For framed unoriented link diagrams one replaces RI by RI_f. For oriented links one allows all possible orientations of the strands in the figures. For details, see e.g [BZ, Oh].

Thus, the map associating an unoriented unframed link diagram to its link class descends to an isomorphism of sets

$$\{\text{equiv. classes of links}\} \to \{\text{link diagrams}\}/(\text{RI,RII,RIII, isotopy of } \mathbb{R}^2).$$

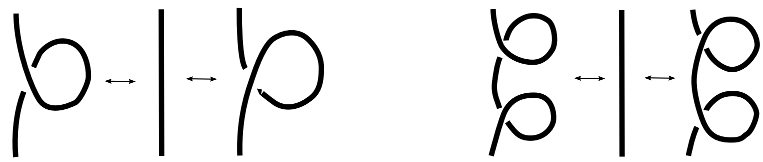


Figure 1. Reidemeister move RI on the left and RI_f on the right.

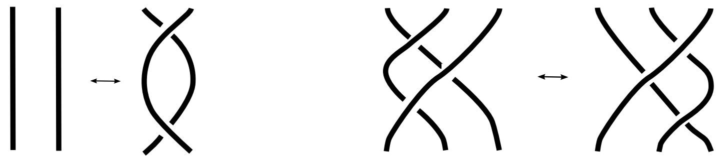


Figure 2. Reidemeister move RII on the left and RIII on the right.

The *mirror image* of link diagram D is the result of switching all the crossings of D from over to under and vice versa.

1.3. **Sign of a crossing, linking number, writhe.** Up to isotopy there are two types of crossings of oriented link diagrams, see Figure 3. The crossing on the left is called a positive

Figure 3. A positive crossing and a negative crossing

crossing, while the one on the right is called a negative crossing.

For a 2-component oriented link diagram $D = D_1 \cup D_2$, define

$$\mathrm{lk}(D) = \frac{1}{2} \sum_x \varepsilon(x),$$

where the sum is over all the crossings between D_1 and D_2, and $\varepsilon(x)$ is the sign of x. Then $\mathrm{lk}(D)$ does not changed under oriented Reidemeister moves and define un invariant of 2-components oriented links, known as the linking number.

Suppose D is the blackboard diagram of a framed oriented link. Define the writhe of L by

$$w(L) := \sum_{x \in C(D)} \varepsilon(x).$$

Exercise 1.2. (a) Show that $w(L)$ is an invariant of framed oriented links.

(b) Show that the writhe is an invariant of un-oriented knots.

(c) Suppose K is an unframed knot, and $fr(K)$ is the set of all framed knots whose underlying unframed knot is K. Show that the map $fr(K) \to \mathbb{Z}$, $K' \to w(K')$ is a bijection.

Suppose $K' \in fr(K)$. Let λ be a parallel of K',which is the result of pushing K off itself along the framed vector field of K'. Show that $w(K') = \mathrm{lk}(\lambda, K)$.

(d) Suppose $L = L_1 \cup L_2$ be a 2-component oriented link. Define the Gauss map

$$\gamma : L_1 \times L_2 \to S^2 = \{z \in \mathbb{R}^3 \mid ||z|| = 1\}, \quad \gamma(x, y) = \frac{x - y}{||x - y||}.$$

Show that up to sign, $\mathrm{lk}(L_1, L_2)$ is equal to the degree of γ.

The set of all framings of an unframed knot in $\mathbb{R}^3$ is naturally identified with $\mathbb{Z}$. For this reason, we also use integers to denote framing a knot.

1.4. **Alexander polynomial.** Suppose L is an m-component oriented link, and $X = S^3 \setminus L$. A small loop encircling the j-th component is called a meridian of the component, which is defined up to isotopy in the link complement. We choose the orientation of the meridians so that the linking number of the j-th component and its meridian is $+1$.

From Alexander duality, $H_1(X, \mathbb{Z}) = \mathbb{Z}^n$, with generators being the meridians of the links. The map $H_1(X, \mathbb{Z}) \to \mathbb{Z} = \langle t | \ \rangle$, mapping each meridian to t, gives rise to a surjective map $f : \pi_1(X) \to \mathbb{Z}$. The corresponding covering $\tilde{X} \to X$ has $\mathbb{Z}$ as the group of deck transformation. As a result, $H_1(\tilde{X}, \mathbb{Q})$ is a $\mathbb{Q}[\mathbb{Z}] \equiv \mathbb{Q}[t^{\pm 1}]$-module. Since $\mathbb{Q}[t^{\pm 1}]$ is a PID, and $H_1(\tilde{X}, \mathbb{Q})$ is finitely generated over $\mathbb{Q}[t^{\pm 1}]$ (prove this!), we have

$$H_1(\tilde{X}, \mathbb{Q}) \cong \bigoplus_{j=1}^{k} \mathbb{Q}[t^{\pm 1}]/(f_j),$$

where each $f_j \in \mathbb{Q}[t^{\pm 1}]$, $f_j | f_{j+1}$, and some of the f_j might be 0. The Alexander polynomial $\Delta_L(t) \in \mathbb{Q}[t^{\pm 1}]$ of L is defined to be $\prod_{j=1}^{k} f_k$.

The Alexander polynomial is defined up to a unit in $\mathbb{Q}[t^{\pm 1}]$. One can choose the unit normalization such that $\Delta_L(t) \in \mathbb{Z}[t^{\pm 1}]$.

If L is a knot, one can choose a unit normalization of Δ such that

$$\Delta_L(t^{-1}) = \Delta_L(t)$$

and $\Delta_L(1) = 1$. In particular, for any knot, $\Delta_L(t) \neq 0$.

1.5. **Kauffman bracket.** The Kauffman bracket was introduced by [Kau1]. For a good introduction see [Li].

There is a unique function

$$\{\text{unoriented link diagrams}\} \to \mathbb{Z}[t^{\pm 1}], \quad D \to \langle D \rangle$$

defined by

$$L = tL_+ + t^{-1}L_- \tag{1}$$

$$L \sqcup U = -(t^2 + t^{-2})L, \tag{2}$$

where in the first identity, L, L_+, L_- are identical except in a ball in which they look like in Figure 4, and in the second identity, the left hand side stands for the union of a link L and the trivial framed knot U in a ball disjoint from L. Here L might be the empty link diagram. In particular, if U is the unknot diagram, the

$$\langle U \rangle = \delta := -(t^2 + t^{-2}).$$

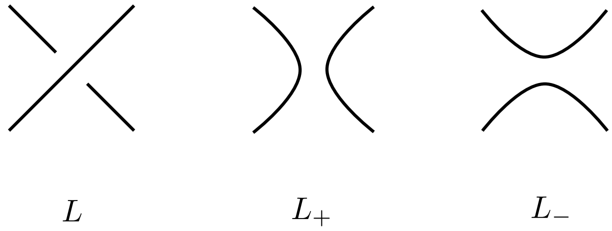
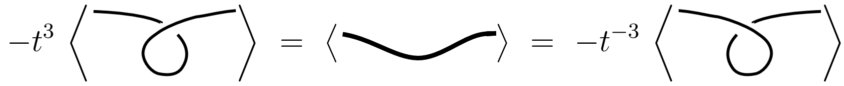
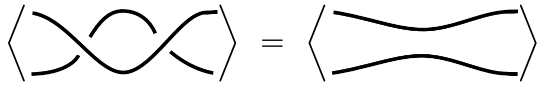
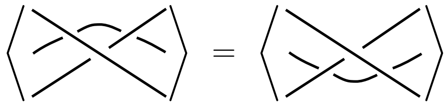


Figure 4. The links L, L_+, and L_-

Lemma 1.3. *One has*

(3) 

(4) 

(5) 

Exercise 1.4. Prove the lemma.

Corollary 1.5. *There exists a unique invariant*

$$\{\textit{oriented framed links}\} \to \mathbb{Z}[q^{\pm 1/4}], \quad L \to V_L \in \mathbb{Z}[q^{\pm 1/4}]$$

such that

$$q^{1/4} V_{L_+} - q^{-1/4} V_{L_-} = (q^{1/2} - q^{-1/2}) V_{L_0} \tag{6}$$

$$V_{L \sqcup U} = [2] V_L \tag{7}$$

$$V_{L^{+1}} = q^{3/4} V_L \tag{8}$$

Here in (6)*, the links* L_+, L_-, L_0 *are identical everywhere except for a small balls in which they look like in Figure 5. In* (7)*,* $L \sqcup U$ *is the union of* L *and a trivial 0-framed knot* U *which is far away from* L*. In* (8)*,* L^{+1} *is the same as* L*, with the framing of one of the components increased by* $+1$*.*

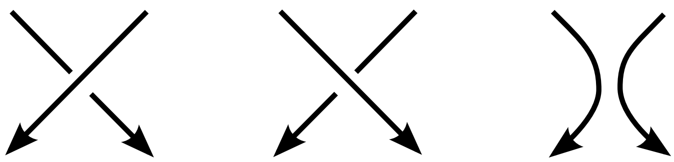


Figure 5. From left to right: the links L_+, L_- and L_0 in Equation (6)

Here we used the notation $[n]$ for the quantum integer

$$[n] := \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} = \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}}.$$

Proof. Suppose L is an oriented framed link with blackboard diagram D. Then

$$V_L(q) := (-1)^{\#L}\langle D\rangle\Big|_{t=-q^{1/4}}$$

is an invariant of L, satisfying the requirements of the corollary. □

If we define $\overset{\circ}{V}_L := q^{-(3/4)w(L)}V_L$, then $\overset{\circ}{V}$ is an invariant of oriented unframed links satisfying

$$q\overset{\circ}{V}_{L_+} - q\overset{\circ}{V}_{L_-} = (q^{1/2} - q^{-1/2})\overset{\circ}{V}_{L_0} \tag{9}$$

$$\overset{\circ}{V}_{L\sqcup U} = [2]\overset{\circ}{V}_L \tag{10}$$

Remark 1.6. The invariant $\overset{\circ}{V}_L$ is a version of the Jones polynomial [Jo]. Jones found his famous invariant using a special representation of the braid groups which he discovered in his study of subfactors. Soon after, there are many generalizations, the most comprehensive one is through the theory of ribbon category, see [RT].

1.6. **Examples.** The Hopf link

$$V_L = (t^4 + t^{-4})[2] = (q + q^{-1})[2].$$

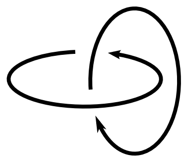


Figure 6. The Hopf link

Right hand trefoil with framing 3.

$$V_L = (-t^{-7} + t^{-3} + t^5)[2].$$

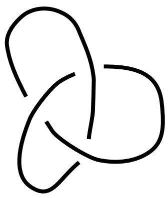


Figure 7. The right hand trefoil with framing 3

Exercise 1.7. (Kauffman) The Milnor link is given in Figure 8.

In his famous paper on Milnor's mu invariants, Milnor challenged us to find more invariants to distinguish links. He gave this example: at that time he did not know how to show that this link is not the trivial link.

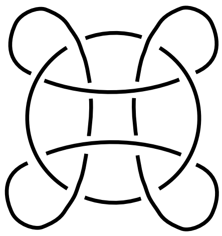


Figure 8. Milnor's link

Show that all the Milnor invariants are 0 (each component is contracted within the complement of the other component).

Calculate the Jones polynomial of the Milnor link and show that it is not the trivial link.

We see that the Jones polynomial captures very "fine" topology of knots and links which we don't fully understand yet.

1.7. **Some properties of the Jones polynomial.**

Proposition 1.8. *(0) One has*

$$V_L(q)\Big|_{q^{1/4}=1} = 2^{\#L}.$$

In particular, $V_L \neq 0$.

(1) Suppose $L!$ *is the mirror image of* L*, then*

$$V_{L!}(q) = V_L(q^{-1}).$$

(2) Suppose L *is a connected sum of* L_1 *and* L_2*. Then*

$$[2]\, V_L = V_{L_1} V_{L_2}.$$

(3) Suppose L' *is a Conway mutation of* L*, then*

$$V_L = V_{L'}.$$

(4) Suppose L *has* n *components. Then* $\overset{\circ}{V}_L(q) \in q^{n/2}\mathbb{Z}[q^{\pm 1}]$.

Exercise 1.9. Prove the proposition.

Thus, if L is an knot and $\overset{\circ}{V}_L(q) \neq \overset{\circ}{V}_L(q^{-1})$, then L is not amphichiral. For example, the right hand trefoil is not amphichiral.

1.8. **State sum of Kauffman bracket.** Let D be a c-crossing link diagram. Denote by C the set of crossings.

At a crossing x, the two strands of L divide a small neighborhood of x into four regions, two of them are marked $+$ and two are marked $-$ as in the middle part of Figure 9. The rule is: if one rotates the over-crossing strand counterclockwise slightly, the over-crossing strand will be in the two plus regions. There are two ways to resolve the singularity: the plus-resolution and the minus-resolution, see Figure 9. In the plus resolution, the two plus regions become connected (forget the dashed line). Similarly, in the minus resolution, the

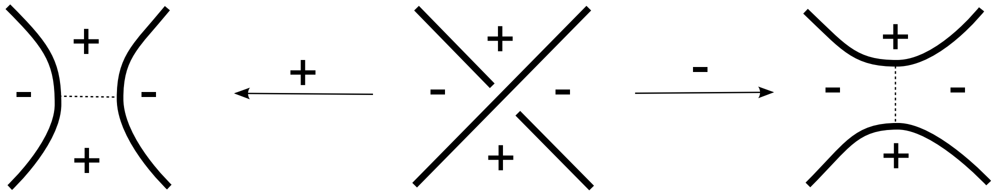


Figure 9. Positive resolution on the left and negative resolution on the right

two minus regions become connected (forget the dashed line). In each resolution, we use a dashed line to connect the two resulting (solid) arcs.

A *state* for D is a function $s : C \to \{1, -1\}$. There are in total 2^c states. For a state s let sD be the diagram constructed from D by doing $s(x)$-resolution at every crossing x (without dashed lines). Then sD consists of disjoint simple closed curves on $\mathbb{R}^2$. Let $|sD|$ be the number of connected components of sD, and $\sigma(s) = \sum_{c \in C} s(c)$.

Exercise 1.10. Show that one always has $\sigma(s) \equiv c := |C| \pmod 2$, for any state σ.

Let G_s denote the graph whose vertices are connected components of sD and whose edges are the dashed arcs constructed above. Thus, G_s has $|sD|$ vertices and $c = |C|$ edges.

For a state s define

$$\langle s \rangle = t^{\sigma(s)}(-t^2 - t^{-2})^{|sD|}.$$

Then clearly

$$\langle D \rangle = \sum_s \langle s \rangle.$$

1.9. **Maximal degree and minimal degree.** For a non-zero polynomial $f \in \mathbb{Z}[t^{\pm 1}]$ let $\deg_+(f)$ and $\deg_-(f)$ be respectively the maximal degree and the minimal degree non-zero monomials of f. The difference $\mathrm{br} := \deg_+ - \deg_-$ is called the *breadth* of the Laurent polynomial.

For non-zero $f, g \in \mathbb{Z}[t^{\pm 1}]$ one has

$$\deg_+(fg) = \deg_+(f) + \deg_+(g), \ \deg_-(fg) = \deg_-(f) + \deg_-(g), \tag{11}$$

$$\mathrm{br}(fg) = \mathrm{br}(f) + \mathrm{br}(g) \tag{12}$$

$$\deg_+(f+g) \le \max(\deg_+(f), \deg_+(g), \quad \text{if } f + g \neq 0. \tag{13}$$

Define a partial ordering on the set of states by: $s \ge s'$ if s' is obtained from s by switching some $+1$ to -1. We say that s' is *one step below* s if s' is obtained from s by switching exactly one $+1$ to -1.

If s' is one step below s, then $\sigma(s') = \sigma(s) - 2$, and $|s'D|$ is either $|sD| - 1$ or $|sD| + 1$.

Lemma 1.11. *(a) Suppose $s \ge s'$. Then* $\deg_+(s) \ge \deg_+(s')$.

(b) Suppose s' is one step below s and $|sD| > |s'D|$. Then $\deg_+(s) > \deg_+(s')$.

Exercise 1.12. Prove the lemma.

1.10. **Adequate diagrams and breadth of Jones polynomial.** One of the best known applications of the Jones polynomial is a proof (Kauffman, Murasugi, and Thistlethwaite) of the Tait conjecture on the crossing number of alternating links, based on an exact estimate of the crossing number using the *breadth* of the Jones polynomial. We will need a generalization of this estimate for the colored Jones polynomial.

Let $s_+ : C \to \{1, -1\}$ be the map $s_+(c) = 1$ for every $c \in C$. Similarly, $s_-(c) = -1$ for every $c \in C$. In other words, s_+ is the only s such that $\sigma(s) = n$, which is the largest possible. In our ordering of states, s_+ is the largest, and s_- is the smallest.

Definition 1. *A link diagram D is plus-adequate if $|s_+D| > |sD|$ for any state s one step below s_+. A link diagram D is minus-adequate if $|s_-D| > |sD|$ for any state s one step above s_-. If both conditions hold, then D is called adequate.*

A link is plus-adequate (minus-adequate, adequate) if it has a plus-adequate (minus-adequate, adequate) diagram.

Warning: There are knots which are both plus-adequate and minus-adequate, but not adequate.

Exercise 1.13. Suppose D is a link diagram. Show that the following are equivalent.
(a) D is plus-adequate
(b) The mirror image of D is minus adequate.
(c) At every crossing of D, the two arcs resulted in the positive resolution do not belong to the same connected component of s_+D.
(d) The graph G_{s_+} does not have one-loop edge.

Theorem 1.14. *Let D be a c-crossing link diagram. Then*
(i) $\deg_+(\langle D\rangle) \le c + 2|s_+D|$, *with equality if D is plus-adequate.*
(ii) $\deg_-(\langle D\rangle) \ge -c - 2|s_-D|$, *with equality if D is minus-adequate.*

Corollary 1.15. *Suppose L is a link with a c-crossing adequate diagram D. Then* $\mathrm{br}(V_L) = 2c + 2|s_+D| + 2|s_-D|$. *Consequently, $c + |s_+D| + |s_-D|$ is an invariant of L.*

Note that if L and L' differ by framings, then $\mathrm{br}(V_L) = \mathrm{br}(V_{L'})$.

A link diagram is called *alternating* if along any component, the over/under nature of crossings is alternate. A link diagram D is reduced if it has a *removable crossing*, i.e. a crossing c for which there is an embedded disk in $\mathbb{R}^2$ whose boundary intersects D at exactly 2 points, both are near the c and belong to different strands, see Figure 10.

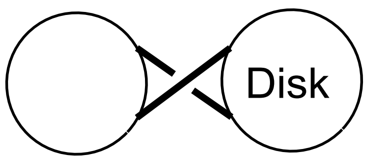


Figure 10. Removable crossing

Lemma 1.16. *(a) Suppose D is a connected link diagram with c crossings. Then $|s_+D| + |s_-D| \le c + 2$, with equality if D is alternating.*
(b) If an alternating link diagram D is reduced, then D is adequate.

Proof. (a) Induction on the number of crossings.

(b) Exercise. □

Corollary 1.17. *Suppose a link L has a connected, reduced, alternating diagram of c crossing, then it has no diagram of less than c crossings, and any alternating reduced diagram of L has c crossings.*

Thus, if L is a link possessing an alternating diagram, then any two reduced alternating diagrams of L have the same number c of crossings, and this number c is minimum among all crossing numbers of diagram of L. With a little more efforts one can also show that any non-alternating diagram of L has more than c crossings.

Exercise 1.18. Suppose D is a link diagram which does not have a trivial component, i.e. a component without crossing. Then the complement of D in S^2 consists of polygons. Each corner of every polygon is marked by $+$ or $-$, see Figure 9. Show that D is alternative if and only if the markings of all the corner of each region are the same.

2. Colored Jones polynomial

2.1. **Chebyshev polynomials.** Define $T_n(z), S_n(z) \in \mathbb{Z}[z^{\pm 1}]$ inductively by

$$T_0 = 2, T_1(z) = z, T_n(z) = zT_{n-1}(z) - T_{n-2}(z) \tag{14}$$

$$S_0 = 1, S_1(z) = z, S_n(z) = zS_{n-1}(z) - S_{n-2}(z). \tag{15}$$

One can also extend the definition of S_n, T_n to $n \in \mathbb{Z}$, using the same recursion formula. Then

$$S_{-1-n} = -S_{-1+n}, \quad T_{-n} = T_n \tag{16}$$

$$T_n = S_n - S_{n-2}. \tag{17}$$

The T_n are known as Chebyshev's polynomials of type 1, and S_n are Chebyshev's polynomials of type 2.

Exercise 2.1. (a) If $z = \operatorname{tr} M$, where M is a 2×2 matrix. Then $\operatorname{tr}(M^n) = T_n(z)$.
(b) If $z = q^{1/2} + q^{-1/2} = [2]$. Then $S_{n-1}(z) = [n]$.

2.2. **Polynomials in framed links.** If L is an ordered k-component framed oriented link and $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$, then let $L^{\mathbf{n}}$ be the framed link obtained from L by replacing the j-th component of L with n_j of its parallels.

Let $\vec{\mathcal{L}}$ be the free $\mathbb{Z}[t^{\pm 1}]$-module with basis the set of all framed oriented links. For $P_1(z), \dots, P_k(z) \in \mathbb{Z}[t^{\pm 1}][z]$, let $\langle P_1 \otimes \dots \otimes P_k, L \rangle \in \mathcal{L}$ be the result of applying P_j to the j-th component of L, for all j. In other words, if

$$\prod P_j(z_j) = \sum a_{\mathbf{n}} z^{\mathbf{n}}, \text{ where } z^{\mathbf{n}} = z_1^{n_1} \dots z_k^{n_k} \text{ if } \mathbf{n} = (n_1, \dots, n_k),$$

then

$$\langle P_1 \otimes \dots \otimes P_k, L \rangle = \sum_{\mathbf{n}} a_{\mathbf{n}} L^{\mathbf{n}}.$$

We consider V_L as a function from the set of framed oriented links to $\mathbb{Z}[t^{\pm 1}]$. Linearly extend this function to a function from $\mathcal{L}$ to $\mathbb{Z}[t^{\pm 1}]$.

Suppose L is a k-component framed link, and $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$. Define the colored Jones polynomial of L by

$$J_L(n_1, \dots, n_k; q) = V_{\langle S_{n_1-1} \otimes \dots \otimes S_{n_k-1}, L \rangle}(q).$$

(Recall that $q^{1/4} = -t$.)

If all the colors are 2, then one recovers the Jones polynomial,

$$J_L(2, \dots, 2; q) = V_L(q).$$

One can remove any component with color 1 without affecting the value of the colored Jones polynomial. If a component has color 0, then the colored Jones polynomial of the link is 0.

Exercise 2.2. Show that

$$J_U(n) = [n],$$

$$J_H(n, m) = [nm],$$

where U is the unknot and H is the Hopf link.

2.3. **Properties.** Let K be a framed oriented knot. Define

$$J'_K(n) := \frac{J_K(n)}{[n]}.$$

Proposition 2.3. *Suppose K, K' are framed oriented knots.*

(1) If K has 0 framing, then $J'_K(n) \in \mathbb{Z}[q^{\pm 1}] = \mathbb{Z}[t^{\pm 4}]$.

(2) $J'_{K\#K'}(n;q) = J'_K(n)\, J_{K'}(n;q)$, where $K\#K'$ is the connected sum of K and K'.

(3) $J'_{K!}(n;q) = J'_K(n;q^{-1})$. Here $K!$ is the mirror image of K.

(4) $J_K(n;q) = J_{\overleftarrow{K}}(n;q)$, where $\overleftarrow{K}$ is the same knot K with reverse orientation.

(5) If K' is obtained from K by increasing the framing by 1,

$$J_{K'}(n) = q^{(n^2-1)/4} J_K(n).$$

Exercise 2.4. Prove parts (1), (3), and (4) of the proposition.

Remark 2.5. Property (2), showing that J'_K behaves well under connected sum, explains why cabling using the Chebyshev polynomials is interesting. This property can be proved using the Jones-Wenzl idempotent, see e.g. [Li]. This property, as well as property (5), is best understood in the frame work of links invariant coming from ribbon categories, as cabling by the Chebyshev polynomials corresponds to coloring by *simple* objects in the ribbon category, see [Tu3].

Recall that $w(K)$ is the writhe, or the integer value framing, of framed knots. Let

$$\mathring{J}_K(n;q) = q^{-w(K)(n^2-1)/4} J_K(n).$$

Then $\mathring{J}$ is an invariant of unframed un-oriented knots.

In general, if one changes the orientation of one component of a link, then the colored Jones polynomial change. If one reverse the orientation of all the components of a link, then the colored Jones polynomial does not change.

2.4. **Examples.** If K is the right handed trefoil with framing 0, then

$$J'_K(n) = q^{1-n} \sum_{k=0}^{\infty} q^{-kn} (q^{1-n};q)_k \tag{18}$$

$$= \sum_{k=0}^{\infty} q^{-k(k+3)/2} \prod_{j=1}^{k} (q^n + q^{-n} - q^j - q^{-j}). \tag{19}$$

Formula (18), see [HL], is valid when $n > 0$. The sum is actually finite, since if $k \ge n$, the summand is 0. Formula (19), see [Ha1], is valid for any $0 \ne n \in \mathbb{Z}$, and is a finite sum since the k-th summand is 0 whenever $k \ge |n|$. When $n < 0$, the right hand side of (18), while not being a finite sum, can be shown to be an element of the Habiro ring [Ha2], and in the Habiro ring, it is equal to the element defined by Formula (19). When $n = 0$, both formulas are infinite sums and equal in the Habiro ring, and give the Kashaev invariant of the knot.

If K is the figure 8 knot with 0 framing (see [Ha1])

$$J'_K(n) = \sum_{k=0}^{\infty} \prod_{j=1}^{k} (q^n + q^{-n} - q^j - q^{-j}). \tag{20}$$

See[Mas] for a proof of these formulas using skein approach.

2.5. **Breadth of colored Jones polynomial of adequate links.** The following is easy to prove.

Lemma 2.6. *Suppose D is a plus-adequate (minus-adequate, adequate) link diagram. Then for any $n \geq 0$, D^n is plus-adequate (minus-adequate, adequate).*

As a corollary, we have the following.

Proposition 2.7. *a) Suppose K is a framed oriented knot with a blackboard diagram D having c crossings. Then*

$$d_+(J_K(n)) \leq c(n-1)^2 + 2(n-1)s_+(D),$$
$$d_-(J_K(n)) \geq -c(n-1)^2 - 2(n-1)s_-(D).$$

Equalities hold if D is adequate.

(Hence the breadth of $J_K(n)$ grows at most as a quadratic function in n.)

b) If K is a non-trivial alternating knot with c crossings. Then the breadth of $J_K(n) \in \mathbb{Z}[t^{\pm 1}]$ is a quadratic polynomial in n. More precisely,

$$\mathrm{br}(J_K(n)) = 2c(n-1)^2 + 2(n-1)(c+2).$$

Proof. a) The n-parallel D^n of D will have cn^2 double points. In addition, it is easy to see that $s_\pm(D^n) = ns_\pm(D)$. Hence, Theorem 1.14 says

$$d_+\langle D^n\rangle \leq f(n) := cn^2 + 2ns_+.$$

Note that $f(n)$ is a strictly increasing function, $f(n+1) > f(n)$. Recall that $S_n(K) = D^n +$ terms of lower degrees in D. Hence one has

$$d_+\langle S_{n-1}(D)\rangle \leq f(n-1) = c(n-1)^2 + 2(n-1)s_+.$$

The proof for d_- is similar.

b) Choose a reduced alternating diagram D of K. Let K' be the framed oriented knot defined by D with blackboard framing. Then K' is the same as K with possibly different framing. Hence, their colored Jones polynomials have the same breadth.

For an alternating diagram one has $s_+ + s_- = c+2$. By Lemma 2.6 and Theorem 1.14,

$$d_+\langle D^n\rangle = cn^2 + 2ns_+,$$
$$d_-\langle D^n\rangle = -cn^2 - 2ns_-.$$

It follows that $d_+(D^n) > d_+(D^{n-1})$ and $d_-(D^n) < d_-(D^{n-1})$. We have that

$$S_n(K) = D^n + \text{terms of lower degrees in } K,$$

hence $d_\pm(\langle S_n(K)\rangle) = d_\pm(\langle D^n\rangle)$, and

$$\operatorname{br} J_K(n) = \operatorname{br}(S_{n-1}(D)) = d_+\langle D^{n-1}\rangle - d_-\langle D^{n-1}\rangle = 2c(n-1)^2 + 2(n-1)(c+2).$$

□

Exercise 2.8. Suppose K is an alternating knot with a reduced alternating diagram D which has c_+ positive crossings. Show that c_+ is an invariant of K.

By [Tu2, Mur], $s_+ - c_+ = \sigma + 1$, where σ is the signature of the knot.

2.6. **Melvin-Morton conjecture and volume conjecture.** Fix a 0-framed knot K. We will look at the colored Jones polynomial J'_K as a function on two variables: q and n, where $n \in \mathbb{Z}$ is the color. Suppose $u \in \mathbb{C}$ is a complex number, with $e^u = z$, or $u = \log z$. For a fixed z, various values of u differ by a multiple of $2\pi i$.

We will consider u near 0 and u near $2\pi i$. In both case, z is near 1.

Let

$$f_n(u) = J'_K(n, q = \exp(u/n)),$$

which is an analytic function in $u \in \mathbb{C}$.

Here is the strong Melvin-Morton conjecture.

Theorem 2.9. [GL2] *For every knot K there is a open set $S_K \subset \mathbb{C}$ containing 0 such that*

$$\lim_{n\to\infty} J'_K(n; q = \exp(u/n)) = \frac{1}{\Delta_K(e^u)}$$

uniformly on any compact in S_K. Here $\Delta_K(z)$ is the Alexander polynomial of the knot, normalized so that $\Delta_K(z) = \Delta_K(z^{-1})$ and $\Delta_K(1) = 1$.

The original Melvin-Morton conjecture [MM] (proved by Bar-Natan and Garoufalidis [BG]) says the Maclaurin series of $J'_K(n, q = \exp(u/n))$ converges coefficient-wise to the Maclaurin seris of $\frac{1}{\Delta_K(e^u)}$.

The (already proved) Melvin-Morton conjecture provides the first connection between the colored Jones polynomial and the fundamental group.

For the volume conjecture, one looks at u near $2\pi i$.

Conjecture 1 (Volume Conjecture)**.** *For any knot K,*

$$\lim_{n\to\infty} \frac{\log |J'_K(n; q = \exp(2\pi i/n))|}{n} = \frac{\operatorname{Vol}(K)}{2\pi}.$$

Here $\operatorname{Vol}(K)$ is the hyperbolic volume of the knot complement.

For more on the volume conjecture, see [Kas, MuM, Muk].

2.7. **Quantum link invariants associated to a simple Lie algebra.** Suppose $\mathfrak{g}$ is a simple Lie $\mathbb{C}$-algebra, L is a framed, oriented link with k-ordered components. Let $V_1, \dots, V_k$ be finite-dimensional $\mathfrak{g}$-module. One can define a *quantum invariant*

$$J_L^{\mathfrak{g}}(V_1, \dots, V_k) \in \mathbb{Z}[q^{\pm 1/d}]$$

using the Drinfeld-Jimbo quantized universal enveloping algebra of $\mathfrak{g}$ and the theory of ribbon category of Reshetikhin-Turaev, see e.g. [RT, Oh, Tu3]. Here d is twice the determinant of the Cartan matrix of $\mathfrak{g}$, see [Le1].

Suppose $\mathfrak{g} = sl_2(\mathbb{C})$, the simplest simple Lie algebra, which consists of 2×2 traceless matrices. For every $n \geq 1$, there is a unique, up to isomorphism, $\mathfrak{g}$-module V_n of dimension n.

Then

$$J_L(V_{n_1}, \dots, V_{n_k}) = J_L(n_1, \dots, n_k).$$

The invariants associated to sl_N and their fundamental representation can be used to define the HOMFLY polynomial. The invariants associated to so_N and their fundamental representation can be used to define the Kauffman polynomial [Kau2], see [Tu3].

2.8. **Habiro's expansion of the colored Jones polynomial.** Habiro [Ha2] showed that for every knot K and non-negative integer k, there exists $C_K(k) \in \mathbb{Z}[q^{\pm 1}]$ such that

$$\mathring{J'}_K(n) = \sum_{k=0}^{\infty} C_K(k) \prod_{j=1}^{k} (q^n + q^{-n} - q^j - q^{-j}). \tag{21}$$

Here

$$\mathring{J'}_K(n) = \frac{\mathring{J}_K(n)}{[n]}.$$

This expression has found applications in many works, see [Ha2, GL2].

2.9. **Colored Jones polynomial at roots of 1.** In the volume conjecture, we look at the value of $\mathring{J'}_K(n;q)$ at q a root of unity. Suppose ξ is a root of unity of order r. Then the colored Jones polynomial enjoy the following symmetry

Proposition 2.10. *For every knot K and every root ξ of unity of order r,*

$$\mathring{J'}_K(n) = \mathring{J'}_K(r-n) = \mathring{J'}_K(r+n),$$

when evaluated at $q = \xi$.

This type of symmetry was first discovered by Kirby and Melvin [KM] for sl_2 quantum invariants. For general Lie algebra see [Le1] and references therein.

Exercise 2.11. Use expression (21) to prove the above proposition.

In the volume conjecture, one cannot take $q = \exp(2\pi i/(n+1))$.

3. Recurrence relation: holonomicity

In this section $\mathcal{R} = \mathbb{Z}[t^{\pm 1}]$.

3.1. Recurrence relation with polynomial coefficients.

Here is an example of a recurrence relation with constant coefficients:

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2$$

We will need two initial values in order to determine F_n. For example, $F_1 = 1, F_2 = 1$.

And here is an example of a recurrence relation with polynomial coefficients.

$$(n-4)F_n = (n^3 - 4n^2 - 1)F_{n-1} + (n^2 - 3n - 2)F_{n-2}.$$

In general, two initial values are not enough, but 4 initial values $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = \sqrt{2}$, are enough to determine the whole sequence.

3.2. q-holonomicity, one variable.

We are looking at q-analog of recurrence relations with polynomial coefficients.

Suppose V is a $\mathbb{Z}[t^{\pm 1}]$-modude. The set $\mathrm{Map}(\mathbb{Z}, V)$ of all functions from $\mathbb{Z}$ to V is also a $\mathbb{Z}[t^{\pm 1}]$-module.

There are two operators L, M acting on $\mathrm{Map}(\mathbb{Z}, V)$:

$$(Lf)(n) := f(n+1)$$
$$(Mf)(n) := t^{2n} f(n).$$

Their inverses L^{-1}, M^{-1} exist. One has $LM = t^2 ML$.

The algebra

$$\mathcal{T} = \mathbb{Z}[t^{\pm 1}]\langle L^{\pm 1}, M^{\pm 1}\rangle / (LM = t^2 ML)$$

is called the *quantum torus*. Let $\mathcal{T}_+ \subset \mathcal{T}$ be the $\mathbb{Z}[t^{\pm 1}]$-submodule spanned by all monomials of the form $L^k M^p$ with $k, p \geq 0$. Then $\mathcal{T}_+$ is known as the quantum plane. It is easy to see that the set $\{M^a L^b \mid a, b \in \mathbb{Z}\}$ is a $\mathbb{Z}[t^{\pm 1}]$-basis of $\mathcal{T}$. Similarly, the set $\{M^a L^b \mid a, b \in \mathbb{N}\}$ is a $\mathbb{Z}[t^{\pm 1}]$-basis of $\mathcal{T}_+$.

With the above actions, $\mathrm{Map}(\mathbb{Z}, V)$ becomes a left module of $\mathcal{T}$.

A function $f \in \mathrm{Map}(\mathbb{Z}, V)$ is called *q-holonomic* if there is $0 \neq \alpha \in \mathcal{T}$ such that $\alpha(f) = 0$.

In general, the set $\mathcal{A}_f := \{\alpha \in \mathcal{T} \mid \alpha(f) = 0\}$ is called the annihilator ideal of f, which is a left ideal of $\mathcal{T}$. Thus, f is q-holonomic if and only its annihilator ideal is not 0.

An important characterization of q-holomorphic function: If f is q-holonomic, then there f is totally determined by a finite set of initial values: Suppose $0 \neq \alpha \in \mathcal{T}$. There exists $n, m \in \mathbb{Z}$, depending on α, such that if $\alpha(f) = \alpha(g) = 0$ and $f(j) = g(j)$ for $n \leq j \leq m$, then $f = g$.

The set of possible images of a fixed f under $\mathcal{T}$ is $\mathcal{T} \cdot f = \mathcal{T}/\mathcal{A}_f$. Hence, if f is not q-holomorphic, then $\mathcal{T} \cdot f \cong \mathcal{T}$ is much bigger than $\mathcal{T} \cdot g = \mathcal{T}/\mathcal{A}_g$ for some q-holonomic g.

Exercise 3.1. Show that each of functions $n \to t^{2n}$ and $n \to t^{4n^2}$ is q-holonomic. However, $n \to t^{8n^3}$ is not q-holonomic.

3.3. **q-holonomicity, many variables.** For a function $f : \mathbb{Z}^r \to V$, with $r \geq 2$ the definition of q-holonomicity is more complicated. The function must satisfy sufficiently many recurrence relations in order to be q-holonomic. But how many recurrence relations would be enough?

Let

$$\mathcal{T}_r = \mathbb{Z}[t^{\pm 1}]\langle L_1^{\pm 1}, \dots, L_r^{\pm 1}, L_1^{\pm 1}, \dots, L_r^{\pm 1}\rangle/(L_i M_i = t^2 M_i L_i, L_i M_j = M_j L_i \text{ for } i \neq j).$$

The algebra $\mathcal{T}_r$ acts on $\mathrm{Map}(\mathbb{Z}^r, V)$, where V is any $\mathbb{Z}[t^{\pm 1}]$-module, by

$$(L_i f)(n_1, \dots, n_i, \dots, n_r) = f(n_1, \dots, n_i + 1, \dots, n_r)$$
$$(M_i f)(n_1, \dots, n_r) = t^{2n_i} f(n_1, \dots, n_r).$$

Let $\mathcal{T}_{r,+}$ be the subalgebra of $\mathcal{T}_r$ generated by non-negative powers of M_j, L_j.

Suppose $f \neq 0$. From f, by actions of $\mathcal{T}_{r,+}$, we get other functions, $(\mathcal{T}_{r,+}) \cdot f$. Intuitively, the more recurrence relations f satisfies, the smaller $(\mathcal{T}_{r,+}) \cdot f$ is.

Informally, f is q-holonomic if the $(\mathcal{T}_{r,+}) \cdot f$ is as small as possible. A precise definition is the following. Berstein's inequality tells us that the *dimension* of $(\mathcal{T}_{r,+}) \cdot f$ is always $\geq r$, and one says f is *q-holonomic* if $f = 0$ or the dimension of $(\mathcal{T}_{r,+}) \cdot f$ is exactly r.

The dimension of $(\mathcal{T}_{r,+}) \cdot f$ can be defined as follows. Let $(\mathcal{T}_{r,+})^{\leq N}$ be the $\mathcal{R}$-span of all monomials in M_j, L_k with total degree $\leq N$. An analog of Hilbert's theorem for this non-commutative setting holds true: The $\mathbb{C}(t)$-dimension of $(\mathcal{T}_{r,+})^{\leq N} \cdot f$ is a polynomial in N, for big enough N. The degree of this polynomial is called the dimension of $\mathcal{T}_+ \cdot f$.

Another way to define the dimension: Suppose W is a non-zero $\mathcal{T}_r$-module. Its codimension and dimension are defined by

$$\mathrm{codim}(W) = \min\{j \in \mathbb{N} \mid \mathrm{Ext}^j_{\mathcal{T}_r}(W, \mathcal{T}_r) \neq 0\}, \quad \dim(N) = 2r - c(V).$$

Then Berstein inequality (proved by Sabbah [Sab] in the q-case) says that if $W \neq 0$ is finitely generated, then $\dim(N) \geq r$. *An $\mathcal{T}_r$-module W is q-holonomic* if either $W = 0$ or $\dim(W) = r$. A function $f \in \mathrm{Map}(\mathbb{Z}^r, V)$ is q-hononomic if the module $(\mathcal{T}_r) \cdot f$ is q-holonomic.

Exercise 3.2. Show that when $r = 1$, this definition of q-holonomicity is equivalent to the one given in Section 3.2.

3.4. **Examples of q–holonomic functions.** Here are a few examples of q–holonomic functions. In fact, we will encounter only sums, products, extensions, specializations, diagonals, and multisums of these functions. We use $v = t^2$, $q = t^4$.

Recall that for $n \in \mathbb{N}$,

$$(x; q)_n := \prod_{j=1}^{n} (1 - x q^{j-1}).$$

For $n, k \in \mathbb{Z}$, let

$$f(n,k) := \begin{cases} (q^n; q^{-1})_k, & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases}$$

$$g(n,k) := \begin{cases} \frac{(q^n;q^{-1})_k}{(q^k;q^{-1})_k} & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases}.$$

Then both f and g, as well as the delta function $\delta_{n,k}$, are q–holonomic. Note that $g(n,k)$ is the q-binomial and $f(n,k)$ is the q-combination number.

Exercise 3.3. Prove the above statement.

3.5. **Properties of q-holonomic functions.**

- Sums and products of q–holonomic functions are q–holonomic.
- Specializations and extensions of q–holonomic functions are q–holonomic. In other words, if $f(n_1,\dots,n_m)$ is q–holonomic, then so are the functions
$$g(n_2,\dots,n_m) := f(a,n_2,\dots,n_m) \quad \text{for fixed } a$$
$$\text{and} \qquad h(n_1,\dots,n_m,n_{m+1}) := f(n_1,\dots,n_m).$$
- Diagonals of q–holonomic functions are q–holonomic. In other words, if $f(n_1,\dots,n_m)$ is q–holonomic, then so is the function
$$g(n_2,\dots,n_m) := f(n_2,n_2,n_3,\dots,n_m).$$
- Linear substitution. If $f(n_1,\dots,n_m)$ is q–holonomic, then so is the function, $g(n'_1,\dots,n'_{m'})$, where each n'_j is a linear function of n_i.
- Multisums of q–holonomic functions are q–holonomic. In other words, if $f(n_1,\dots,n_m)$ is q–holonomic, the so are the functions g and h, defined by
$$g(a,b,n_2,\dots,n_m) := \sum_{n_1=a}^{b} f(n_1,n_2,\dots,n_m)$$
$$h(a,n_2,\dots,n_m) := \sum_{n_1=a}^{\infty} f(n_1,n_2,\dots,n_m)$$
(assuming that the latter sum is finite for each a).

3.6. **State sum formula for the colored Jones polynomial of a knot.** Suppose the knot K is represented as the closure of a braid β on k strands, see Figure 11. The diagram

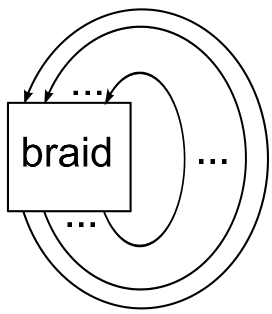
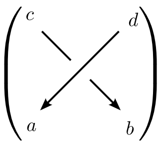


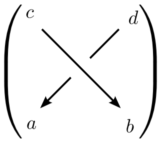
Figure 11. Oriented links as closures of braids

D of K is the closure of the diagram of the braid. Suppose β (or D) has c crossings. Then the underlying 4-valent graph of D has $2c$ edges. *A coloring of D* is a map
$$\text{col} : \{ \text{ edges of } D\} \to \mathbb{Z}.$$

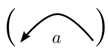
A local part of D is a crossing of D or a local maximal point.

For a non-negative integer n and a coloring of D, define the weight of a crossing by

w  $= q^{(n+nd+nb-ab-dc)/2}\, f(c, c-b) g(n-a, d-a) \delta_{a+b,c+d}$

w  $= (-1)^{b-c} q^{(-n-nb-nd+bd+ac-b+c)/2}\, f(a, a-d) g(n-c, b-c) \delta_{a+b,c+d}.$

and the weight of a maximal point by

w  $= q^{(2a-n)/2}.$

Here a, b, c, a are the colors of the edges.

The weight of a coloring

$$w(n, \mathrm{col}) = \prod_{x:\text{local parts}} w(x).$$

Then one has

$$\overset{\circ}{J}_K(n+1) = \sum_{\mathrm{col}} w(n, \mathrm{col}). \tag{22}$$

3.7. **The colored Jones polynomial is q-holonomic.** The colored Jones polynomial can be expressed as a multisum of terms, each is a q-holonomic in all of its variables.

Theorem 3.4. *For each framed oriented link L with n components, the function $J_L : \mathbb{Z}^n \to \mathbb{Z}[t^{\pm 1}]$ is q-holonomic.*

Let $\mathcal{A}_K$ be the annihilator ideal of K.

3.8. **Effect of Weyl symmetry.** Let $\sigma : \mathcal{T} \to \mathcal{T}$ be the $\mathcal{R}$-algebra involution defined by

$$\sigma(M^a L^b) = M^{-a} L^{-b}.$$

(Check that σ is a well-defined algebra involution.)

Proposition 3.5. *The annihilator ideal $\mathcal{A}_K$ is invariant under σ.*

Exercise 3.6. Prove the proposition, using the fact that $J_K(-n) = -J_K(n)$.

3.8.1. *An example.* For the right-handed trefoil, one has

$$\overset{\circ}{J}_K(n) = \frac{t^{2-2n}}{1-t^{-4}} \sum_{k=0}^{n-1} t^{-4nk} \prod_{i=0}^{k} (1 - t^{4i-4n}).$$

The function J_K satisfies $\alpha\, J_K = 0$, where

$$\alpha = (t^4 M^{10} - M^6) L^2 - (t^2 M^{10} + t^{-18} - t^{-10} M^6 - t^{-14} M^4) L + (t^{-16} - t^{-4} M^4). \tag{23}$$

Together with the initial conditions $J_K(0) = 0, J_K(1) = 1$, this recurrence relation determines $J_K(n)$ uniquely.

3.9. **Generator of the recurrence ideal.** The quantum torus $\mathcal{T}$ is not a principal ideal domain, and $\mathcal{A}_K$ might not be generated by a single element. Garoufalidis [Ga2] noticed that by adding to $\mathcal{T}$ all the inverses of polynomials in M one gets a principal ideal domain $\tilde{\mathcal{T}}$, and hence from the ideal $\mathcal{A}_K$ one can define a polynomial invariant. Formally one can proceed as follows. Let $\mathcal{R}(M)$ be the fractional field of the polynomial ring $\mathcal{R}[M]$. Let $\tilde{\mathcal{T}}$ be the set of all Laurent polynomials in the variable L with coefficients in $\mathcal{R}(M)$:

$$\tilde{\mathcal{T}} = \{\sum_{k\in\mathbb{Z}} a_k(M)L^k \mid \quad a_k(M) \in \mathcal{R}(M), \;\; a_k = 0 \quad \text{almost everywhere}\},$$

and define the product in $\tilde{\mathcal{T}}$ by $a(M)L^k \cdot b(M)L^l = a(M)\, b(t^{2k}M)L^{k+l}$.

Then it is known that every left ideal in $\tilde{\mathcal{T}}$ is principal, and $\mathcal{T}$ embeds as a subring of $\tilde{\mathcal{T}}$. The extension $\tilde{\mathcal{A}}_K := \tilde{\mathcal{T}}\mathcal{A}_K$ of $\mathcal{A}_K$ in $\tilde{\mathcal{T}}$ is then generated by a single polynomial

$$\alpha_K(t; M, L) = \sum_{i=0}^{n} \alpha_{K,i}(t; M)L^i \;\in\; \mathcal{T}_+,$$

where the degree in L is assumed to be minimal and all the coefficients $\alpha_{K,i}(t; M) \in \mathbb{Z}[t^{\pm1}, M]$ are assumed to be co-prime. That α_K can be chosen to have integer coefficients follows from the fact that $J_K(n) \in \mathbb{Z}[t^{\pm1}]$. It is clear that $\alpha_K(t; M, L)$ annihilates J_K, except for a finite number of values of the color. Note that $\alpha_K(t; M, L)$ is defined up to a factor $\pm t^a M^b, a, b \in \mathbb{Z}$. We will call α_K the *recurrence polynomial* of K. For example, the polynomial α in the previous subsection is the recurrence polynomial of the right-handed trefoil.

Remark 3.7. If P is a polynomial in t and M (no L), and $Pf = 0$ then $P = 0$. Hence adding all the inverses of polynomials in M does not affect the recurrence relations.

3.10. **Degree 1 recurrence relation.** It turns out that if J_K has a recurrence relation of degree 1, then the breadth of $J_K(n)$ can grow at most linearly with n.

Proposition 3.8. *Suppose the annihilator polynomial α_K has L-degree 1. Then there is a constant C such that for any $n \geq 1$,*

$$\mathrm{br}(J_K(n)) \leq Cn.$$

Consequently, if K is an alternating non-trivial knot, then α_K has L-degree ≥ 2.

Sketch of Proof. Assume $\alpha_K = P(t; M)\, L + P_0(t; M)$, where $P, P_0 \in \mathbb{Z}[t^{\pm1}, M^{\pm1}]$. Since $\sigma(\alpha_K) = P(t; M^{-1})\, L^{-1} + P_0(t; M^{-1})$ is also in the recurrence ideal, it is divisible by α_K.

One can then easily show that, (in the extension of $\mathcal{T}$ containing $\sqrt{M}$) after normalizing both P, P_0 by a same power of $M^{1/2}$, one has

$$P_0(t; M) = P(t; t^{-2}M^{-1}).$$

The equation $\alpha_K J_K = 0$ can now be rewritten as

$$J_K(n + 1) = -\frac{P(t; t^{-2-2n})}{P(t; t^{2n})} J_K(n).$$

Thus

$$\mathrm{br}(J_K(n+1)) = \mathrm{br}(J_K(n)) + \mathrm{br}(P(t;t^{-2-2n})) - \mathrm{br}(P(t;t^{2n})).$$

It is easy to see that for n big enough, the difference of the breadths $\mathrm{br}(P(t;t^{-2-2n})) - \mathrm{br}(P(t;t^{2n}))$ is a constant depending only on the polynomial $P(t;M)$, but not on n. From the above equation it follows that the breadth of $J_K(n)$, for n big enough, is a linear function on n. □

3.11. **Linear factor $L-1$.**

Proposition 3.9. *At $t=-1$, the recurrence polynomial α_K is divisible by $L-1$. In other words,* $\dfrac{\epsilon(\alpha_K)}{L-1} \in \mathbb{Z}[M,L]$.

Sketch of Proof. Suppose $\alpha_K = \sum_{j=0}^d a_j(t,M)L^j$. One has

$$\sum_{j=0}^d a_j(t,M) J_K(n+d) = 0$$

$$\sum_{j=0}^d a_j(t,M)\,(t^{2n+2j} - t^{-2n-2j})\,J'_K(n+d) = 0.$$

Setting $t = e^{u/4n}$ for small enough $|u|$ and taking the limit as $n \to \infty$, using Theorem 2.9, we have

$$\sum_{j=0}^d a_j(1, e^{u/2})\,(e^{u/2} - e^{-u/2}) \frac{1}{\Delta(e^{u/2})} = 0.$$

It follows that $\sum_{j=0}^d a_j(1,e^{u/2}) = 0$ for small $|u|$. Hence, $\alpha_K|_{t^2=1,M=z,L=1} = 0$, which is equivalent to the lemma. □

4. Kauffman bracket skein modules

We recall the definition and known facts about the colored Jones polynomial through the theory of Kauffman Bracket Skein Modules which were introduced by Przytycki [Pr] and Turaev [Tu1].

In this section $\mathcal{R} = \mathbb{C}[t^{\pm 1}]$.

4.1. **Skein modules.** A *framed link* in an oriented 3-manifold Y is a disjoint union of embedded circles, equipped with a non-zero normal vector field. Framed links are considered up to isotopy. In all figures we will draw framed links, or part of them, by lines as usual, with the convention that the framing is blackboard. Let $\mathcal{L}$ be the free $\mathcal{R}$-module with basis the set of isotopy classes of framed links in the manifold Y, including the empty link. Let Rel be the smallest submodule containing all expressions of the form ╳ $- t$ ≍ $- t^{-1}$)(and ○ $+ (t^2 + t^{-2})\emptyset$, where the links in each expression are identical except in a ball in which they look like depicted. The quotient $\mathcal{S}(Y) := \mathcal{L}/\text{Rel}$ is called the Kauffman bracket skein module, or just skein module, of Y.

When $Y = \Sigma \times [0, 1]$, the cylinder over the surface Σ, we also use the notation $\mathcal{S}(\Sigma)$ for $\mathcal{S}(Y)$. In this case $\mathcal{S}(\Sigma)$ has an algebra structure induced by the operation of gluing one cylinder on top of the other. The operation of gluing the cylinder over ∂Y to Y induces a $\mathcal{S}(\partial Y)$-left module structure on $\mathcal{S}(M)$.

Exercise 4.1. Suppose $f : Y_1 \hookrightarrow Y_2$ is an embedding. Show that $L \to f(L)$ gives rise to a well-define $\mathcal{R}$-module map $f_* : \mathcal{S}(Y_1) \to \mathcal{S}(Y_2)$.

4.2. **Example: $\mathbb{R}^3$, and the Jones polynomial.** When Y is the 3-space $\mathbb{R}^3$ or the 3-sphere S^3, the skein module $\mathcal{S}(Y)$ is free over $\mathcal{R}$ of rank one, and is spanned by the empty link. Thus if ℓ is a framed link in $\mathbb{R}^3$, then its value in the skein module $\mathcal{S}(\mathbb{R}^3)$ is $\langle \ell \rangle$ times the empty link, where $\langle \ell \rangle \in \mathcal{R}$ is the Kauffman bracket of ℓ.

One could think of $\mathcal{S}(Y)$ as the space of all Kauffman bracket type polynomial of framed links in Y.

4.3. **Example: The solid torus.** The solid torus ST is the cylinder over the annulus $\mathbb{A}$, and hence its skein module $\mathcal{S}(ST)$ has an algebra structure. The algebra $\mathcal{S}(ST)$ is the polynomial algebra $\mathcal{R}[z]$ in the variable z, which is a knot representing the core of the solid torus.

Instead of the $\mathcal{R}$-basis $\{1, z, z^2, \dots\}$, two other bases are often useful. Namely, each of $\{T_n(z) \mid n \in \mathbb{N}\}$ and $\{S_n(z) \mid n \in \mathbb{N}\}$ is a $\mathcal{R}$-basis of $\mathcal{S}(ST)$. Here T_n, S_n are the Chebyshev polynomials.

A framed knot K in $\mathbb{R}^3$ gives rise to an embedding $f : \mathbb{A} \hookrightarrow \mathbb{R}^3$ which is defined up to isotopy. Then colored Jones polynomial is then

$$J_K(n) = (-1)^{n-1} f_*(S_{n-1}(z)).$$

4.4. **The skein module of the torus.** Let $\mathbb{T}^2$ be the torus with a fixed pair (μ, λ) of simple closed curves intersecting at exactly one point. For co-prime integers k and l, let $\lambda_{k,l}$ be a simple closed curve on the torus homologically equal to $k\mu + l\lambda$. It is not difficult to show that the skein algebra $\mathcal{S}(\mathbb{T}^2)$ of the torus is generated, as an $\mathcal{R}$-algebra, by all $\lambda_{k,l}$'s. In fact, Bullock and Przytycki [BP] showed that $\mathcal{S}(\mathbb{T}^2)$ is generated over $\mathcal{R}$ by 3 elements μ, λ and $\lambda_{1,1}$, subject to some explicit relations. If one adds the inverse of $(1-t^4)$ to the ground ring $\mathcal{R}$, then $\mathcal{S}(\mathbb{T}^2)$ is generated by just two elements μ, λ.

Recall that $\mathcal{T} = \mathcal{R}\langle M^{\pm 1}, L^{\pm 1}\rangle/(LM - t^2 ML)$ is the quantum torus. Let $\sigma : \mathcal{T} \to \mathcal{T}$ be the involution defined by $\sigma(M^k L^l) := M^{-k}L^{-l}$. Frohman and Gelca [FG] showed that there is an algebra isomorphism $\Upsilon : \mathcal{S}(\mathbb{T}^2) \to \mathcal{T}^\sigma$ given by

$$\Upsilon(\lambda_{k,l}) := (-1)^{k+l} t^{kl}(M^k L^l + M^{-k}L^{-l}).$$

The fact that $\mathcal{S}(\mathbb{T}^2)$ and $\mathcal{T}^\sigma$ are isomorphic algebras was also proved by Sallenave [Sal].

4.5. **Two-punctured disk and punctured torus.** Let $F_{0,3}$ be the disk with two points removed, and $F_{1,1}$ be the torus with one point removed.

Then $F_{0,3} \times [0,1] \cong F_{1,1} \times [0,1]$. Hence $\mathcal{S}(F_{0,3})$ and $\mathcal{S}(F_{1,1})$ are isomorphic as $\mathcal{R}$-modules. However, as $\mathcal{R}$-algebras, they are different. In fact, while $\mathcal{R}(F_{0,3})$ is commutative, $\mathcal{S}(F_{1,1})$ is not.

$\mathcal{S}(F_{0,3})$ is the polynomial algebra $\mathcal{R}[x, x', y]$, where x, x' are small loops surrounding the punctured points, and y is a loop parallel to the boundary of the disk. The skein algebra of the punctured torus is a quantization (non-commutative) algebra of the Lie algebra so_3, see [BP].

Many results and proofs in the theory reduce to calculations involving skein algebras of $F_{0,3}$ and $F_{1,1}$, see e.g. [BW, Le3].

4.6. **The orthogonal and peripheral ideals.** Let $N(K)$ be a tubular neighborhood of an oriented knot K in S^3, and X the closure of $S^3 \setminus N(K)$. Then $\partial(N(K)) = \partial(X) = \mathbb{T}^2$. There is a standard choice of a meridian μ and a longitude λ on $\mathbb{T}^2$ such that the linking number between the longitude and the knot is zero. We use this pair (μ, λ) and the map Υ in the previous subsection to identify $\mathcal{S}(\mathbb{T}^2)$ with $\mathcal{T}^\sigma$.

The torus $\mathbb{T}^2 = \partial(N(K))$ cut S^3 into two parts: $N(K)$ and X. We can consider $\mathcal{S}(X)$ as a left $\mathcal{S}(\mathbb{T}^2)$-module and $\mathcal{S}(N(K))$ as a right $\mathcal{S}(\mathbb{T}^2)$-module. There is a bilinear bracket

$$\langle \cdot, \cdot \rangle : \mathcal{S}(N(K)) \otimes_{\mathcal{S}(\mathbb{T}^2)} \mathcal{S}(X) \to \mathcal{S}(S^3) \equiv \mathcal{R}$$

given by $\langle \ell', \ell'' \rangle := \langle \ell' \cup \ell'' \rangle$, where ℓ' and ℓ'' are links in respectively $N(K)$ and X. Note that if $\ell \in \mathcal{S}(\mathbb{T}^2)$, then

$$\langle \ell' \cdot \ell, \ell'' \rangle = \langle \ell', \ell \cdot \ell'' \rangle.$$

In general $\mathcal{S}(X)$ does not have an algebra structure, but it has the identity element–the empty link. The map

$$\Theta : \mathcal{S}(\mathbb{T}^2) \to \mathcal{S}(X), \quad \Theta(\ell) := \ell \cdot \emptyset$$

is $\mathcal{S}(\mathbb{T}^2)$-linear. Its kernel $\mathcal{P} := \ker \Theta$ is called the *quantum peripheral ideal*, first introduced in [FGL]. In [FGL, Ge], it was proved that every element in $\mathcal{P}$ gives rise to a recurrence relation for the colored Jones polynomial. We will present a refinement of this result here.

The *orthogonal ideal* $\mathcal{O}$ in [FGL] is defined by

$$\mathcal{O} := \{\ell \in \mathcal{S}(\partial X) \quad | \quad \langle \ell', \Theta(\ell) \rangle = 0 \quad \text{for every } \ell' \in \mathcal{S}(N(K))\}.$$

It is clear that $\mathcal{O}$ is a left ideal of $\mathcal{S}(\partial X) \equiv \mathcal{T}^\sigma$ and $\mathcal{P} \subset \mathcal{O}$. In [FGL], $\mathcal{O}$ was called the formal ideal. From the definition, one has

$$\mathcal{P} \subset \mathcal{O}. \tag{24}$$

Conjecture 2. *For every knot, one has $\mathcal{P} = \mathcal{O}$.*

According to [Le2], if the conjecture holds, then the colored Jones polynomial distinguish the unknot from other knots.

4.7. **Recurrence relation from Kauffman bracket skein modules.** As mentioned above, the skein algebra of the torus $\mathcal{S}(\mathbb{T}^2)$ can be identified with $\mathcal{T}^\sigma$ via the $\mathcal{R}$-algebra isomorphism Υ sending μ, λ and $\lambda_{1,1}$ to respectively $-(M+M^{-1}), -(L+L^{-1})$ and $t(ML+M^{-1}L^{-1})$. We will use this identification.

Recall that $\mathcal{A} = \mathcal{A}_K$ is the recurrence ideal of the knot K.

Proposition 4.2. *For every knot one has*

$$\mathcal{P} \subset \mathcal{O} \subset \mathcal{A}.$$

Actually,

$$\mathcal{O} = \mathcal{A} \cap \mathcal{T}^\sigma. \tag{25}$$

To prove the above proposition, we first prove the following.

Proposition 4.3. *For any $\ell \in \mathcal{S}(\mathbb{T}^2)$. One has*

$$(-1)^{n-1}\langle S_{n-1}(\lambda), \Theta(\ell)\rangle = \ell \cdot J_K(n) \tag{26}$$

Here on the left hand side, ℓ is an element of $\mathcal{S}(\partial X)$, and on the right hand side ℓ is an element of $\mathcal{T}^\sigma \subset \mathcal{T}$, which acts on $\mathrm{Map}(\mathbb{Z}, \mathbb{Z}[t^{\pm 1}])$.

Proof. We know from the properties of the Jones-Wenzl idempotent (see e.g. [Oh]) that

$$\begin{aligned}
\langle S_{n-1}(\lambda)\cdot \mu,\, \emptyset\rangle &= (-t^{2n} - t^{-2n})\langle S_{n-1}(\lambda), \emptyset\rangle \\
\langle S_{n-1}(\lambda)\cdot \lambda,\, \emptyset\rangle &= \langle S_n(\lambda) + S_{n-2}(\lambda), \emptyset\rangle \\
\langle S_{n-1}(\lambda)\cdot \lambda_{1,1},\, \emptyset\rangle &= -\langle t^{2n+1}S_n(\lambda) + t^{-2n+1}S_{n-2}(\lambda), \emptyset\rangle.
\end{aligned}$$

By definition $J_K(n) = (-1)^{n-1}\langle S_{n-1}(\lambda), \emptyset\rangle$ and $(MJ_K)(n) = t^{2n}J_K(n)$, $(LJ_K)(n) = J_K(n+1)$. Hence the above equations can be rewritten as

$$\begin{aligned}
(-1)^{n-1}\langle S_{n-1}(\lambda),\, \Theta(\mu)\rangle &= -(M+M^{-1})J_K(n) = \mu\cdot J_K(n), \\
(-1)^{n-1}\langle S_{n-1}(\lambda),\, \Theta(\lambda)\rangle &= -(L+L^{-1})J_K(n) = \lambda\cdot J_K(n), \\
(-1)^{n-1}\langle S_{n-1}(\lambda),\, \Theta(\lambda_{1,1})\rangle &= t(ML+M^{-1}L^{-1})J_K(n) = \Upsilon(\lambda_{1,1})J(n).
\end{aligned}$$

Since $\mathcal{S}(\mathbb{T}^2)$ is generated by μ, λ and $\lambda_{1,1}$, we conclude that

$$(-1)^{n-1}\langle S_{n-1}(\lambda), \Theta(\ell)\rangle = \Upsilon(\ell)J_K(n)$$

for all $\ell \in \mathcal{S}(\mathbb{T}^2)$. □

Proof of Proposition 4.2. By the definition $\mathcal{P} \subset \mathcal{O}$. We now show $\mathcal{O} \subset \mathcal{A}$. Suppose $\ell \in \mathcal{O}$. Then by definition, the left hand side of 26 is 0. The right hand side of (26) is 0 means that $\ell \in cA$. Thus, $\mathcal{O} \subset \mathcal{A}$.

Next we show that $\mathcal{O} = \mathcal{A}_K \cap \mathcal{T}^\sigma$.

Since $\{S_n(\lambda)\}_n$ generates the skein module $\mathcal{S}(N(K))$, Proposition 4.3 implies that

$$\begin{aligned} \mathcal{O} &= \{\ell \in \mathcal{S}(\partial X) \mid \langle \ell', \Theta(\ell)\rangle = 0 \quad \text{for every } \ell' \in \mathcal{S}(N(K))\} \\ &= \{\ell \in \mathcal{S}(\partial X) \mid \langle S_n(\lambda), \Theta(\ell)\rangle = 0 \quad \text{for all integers } n\} \\ &= \{\ell \in \mathcal{S}(\partial X) = \mathcal{T}^\sigma \mid \ell \cdot J_K(n) = 0 \quad \text{for all integers } n\}. \end{aligned}$$

Hence $\mathcal{O} = \mathcal{A}_K \cap \mathcal{T}^\sigma$. □

Remark 4.4. Equation (25) was obtained in [Ga1] by another method. We present here a more geometric proof, using properties of the action of the longitude and the meridian on the Chebyshev polynomials.

4.8. **The character variety of a group.** The set of representations of a finitely presented group G into $SL_2(\mathbb{C})$ is an algebraic set defined over $\mathbb{C}$, on which $SL_2(\mathbb{C})$ acts by conjugation. The set-theoretic quotient of the representation space by that action does not have good topological properties, because two representations with the same character may belong to different orbits of that action. A better quotient, the algebro-geometric quotient denoted by $\chi(G)$ (see [CS, BH, LM]), has the structure of an algebraic set. There is a bijection between $\chi(G)$ and the set of all characters of representations of G into $SL_2(\mathbb{C})$, hence $\chi(G)$ is usually called the *character variety* of G. For a manifold Y we use $\chi(Y)$ also to denote $\chi(\pi_1(Y))$. The character variety has played an important role in geometric topology.

Suppose $G = \mathbb{Z}^2$, the free abelian group with two generators. Every pair of generators μ, λ will define an isomorphism between $\chi(G)$ and $(\mathbb{C}^*)^2/\tau$, where $(\mathbb{C}^*)^2$ is the set of non-zero complex pairs (L, M) and τ is the involution $\tau(M, L) := (M^{-1}, L^{-1})$, as follows: Every representation is conjugate to an upper diagonal one, with M and L being the upper left entry of μ and λ respectively. The isomorphism does not change if one replaces (μ, λ) with (μ^{-1}, λ^{-1}).

4.9. **Skein modules and character variety.** For a non-zero complex number ξ let $\mathcal{S}_\xi(Y)$ be the skein module of Y at $t = \xi$, i.e.

$$\mathcal{S}_\xi(Y) = \mathcal{S}(Y)/(t - \xi) = \mathcal{S}(Y) \otimes_{\mathcal{R}} \mathbb{C},$$

where $\mathbb{C}$ is considered as a $\mathcal{R}$-module by setting $t \to \xi$.

Then $\mathcal{S}_\xi(Y)$ is a vector space over $\mathbb{C}$. Note that $\mathcal{S}_{\pm 1}(Y)$ have a natural algebra structure where the product of two links in $\mathcal{S}_\pm(Y)$ is their disjoint union. It is easy to see that when $t = \pm 1$, this product is well-defined. Let $\sqrt{0}$ be the nil-radical of $\mathcal{S}_{-1}(Y)$. Then $\mathcal{S}_{-1}(Y)/\sqrt{0}$ is a reduced finitely generated $\mathbb{C}$-algebra. Hence, $\mathcal{S}_{-1}(Y)/\sqrt{0}$ is isomorphic to the ring of regular functions of some algebraic set.

An important result [Bul, PS] in the theory of skein modules is that $\mathcal{S}_{-1}(Y)/\sqrt{0}$ is isomorphic to the ring $\mathbb{C}[\chi(Y)]$ of regular functions of the character variety of $\pi_1(Y)$. The isomorphism between is given by $K(r) = -\operatorname{tr} r(K)$, where K is a knot in Y representing an element of $\pi_1(Y)$, and $r : \pi_1(Y) \to SL_2(\mathbb{C})$ is a representation of $\pi_1(Y)$.

The algebra $\mathcal{S}_{-1}(Y)$ is isomorphic to the *universal SL_2-character algebra* of $\pi_1(Y)$, see [PS, BH].

In many cases $\mathcal{S}_{-1}(Y)$ is reduced, i.e. its nilradical is zero, and hence $\mathcal{S}_{-1}(Y)$ is exactly the ring of regular functions on the SL_2-character variety of $\pi_1(Y)$. For example, this is the case when Y is a torus, or when Y is the complement of a two-bridge knot/link [Le2, PS, LT1], or when Y is the complement of the $(-2, 3, 2n+1)$-pretzel knot for any integer n (see [LT2]). We have the following conjecture.

Conjecture 3. *For every knot K the universal SL_2-character ring is reduced.*

4.10. **Skein modules at roots of 1.** Suppose $\xi \in \mathbb{C}$ such that ξ^2 is a root of unity of order exactly N for some odd integer $N > 0$. Define an action of $\mathcal{S}_{-1}(Y)$ on $\mathcal{S}_\xi(Y)$ as follows.

Suppose ℓ, ℓ' are disjoint framed links in Y. Let

$$\ell \cdot \ell' = T_N(\ell) \cup \ell',$$

where T_N is the N-th Chebyshev polynomial of type 1. On the left hand side ℓ is considered as an element of $\mathcal{S}_{-1}(Y)$, ℓ' is considered as an element of $\mathcal{S}_\xi(Y)$. On the right hand side both ℓ, ℓ' are considered as elements of $\mathcal{S}_\xi(Y)$. In [Le3], it was proved that this gives rise to an action of $\mathcal{S}_{-1}(Y)$ on $\mathcal{S}_\xi(Y)$, which is an extension of results of [BW] for skein algebras of surfaces.

5. AJ conjecture

5.1. **The A-polynomial.** Let X be the closure of S^3 minus a tubular neighborhood $N(K)$ of a knot K. The boundary of X is a torus whose fundamental group is free abelian of rank two. An orientation of K will define a unique pair of an oriented meridian and an oriented longitude such that the linking number between the longitude and the knot is zero, as in Subsection 4.6. The pair provides an identification of $\chi(\partial X)$ and $(\mathbb{C}^*)^2/\tau$ which actually does not depend on the orientation of K.

The inclusion $\partial X \hookrightarrow X$ induces the restriction map

$$\rho : \chi(X) \longmapsto \chi(\partial X) \equiv (\mathbb{C}^*)^2/\tau \tag{27}$$

Let Z be the image of ρ and $\hat{Z} \subset (\mathbb{C}^*)^2$ the lift of Z under the projection $(\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2/\tau$. The Zariski closure of $\hat{Z} \subset (\mathbb{C}^*)^2 \subset \mathbb{C}^2$ in $\mathbb{C}^2$ is an algebraic set consisting of components of dimension 0 or 1. The union of all the one-dimension components is defined by a single polynomial $A_K \in \mathbb{Z}[M, L]$, whose coefficients are co-prime. Note that A_K is defined up to ± 1. We call A_K the *A-polynomial* of K. By definition, A_K does not have repeated factors. It is known that A_K is always divisible by $L - 1$. The A-polynomial here is actually equal to $L - 1$ times the A-polynomial defined in [CCGLS].

The A-polynomial is an important geometric invariant. The slops of the Newton polygon of A_K are boundary slopes of the knot. The A-polynomial distinguishes the unknot from other knots, see [DG, BZ].

For a hyperbolic knot [Th], the character of a discrete faithful SL_2-representation is always a smooth point of the character variety, see e.g. [Po]. A component of the character variety containing the character of a discrete faithful representation is called a *geometric component*. By a result of Thurston, the complex dimension of each geometric component is 1. For knots in S^3 there are at most 4 geometric components, see e.g. [Du]. There is no known example of knots with more than one geometric components.

An important result of Dunfield [Du] that we will use is that the map ρ in (27), when restricted to a geometric component, is a birational equivalence onto its image.

5.2. **The B-polynomial.** It is also instructive to see the dual picture in the construction of the A-polynomial. For an algebraic set V (over $\mathbb{C}$) let $\mathbb{C}[V]$ denote the ring of regular functions on V. For example, $\mathbb{C}[(\mathbb{C}^*)^2/\tau] = \mathfrak{t}^\sigma$, the σ-invariant subspace of $\mathfrak{t} := \mathbb{C}[L^{\pm 1}, M^{\pm 1}]$, where $\sigma(M^k L^l) = M^{-k} L^{-l}$.

The map ρ in the previous subsection induces an algebra homomorphism

$$\theta : \mathbb{C}[\chi(\partial X)] \equiv \mathfrak{t}^\sigma \longrightarrow \mathbb{C}[\chi(X)].$$

We will call the kernel $\mathfrak{p}$ of θ the *classical peripheral ideal*; it is an ideal of $\mathfrak{t}^\sigma$. We have the exact sequence

$$0 \to \mathfrak{p} \to \mathfrak{t}^\sigma \xrightarrow{\theta} \mathbb{C}[\chi(X)]. \tag{28}$$

The ring $\mathfrak{t}^\sigma \subset \mathfrak{t} = \mathbb{C}[M^{\pm 1}, L^{\pm 1}]$ embeds naturally into the principal ideal domain $\widetilde{\mathfrak{t}} := \mathbb{C}(M)[L^{\pm 1}]$, where $\mathbb{C}(M)$ is the fractional field of $\mathbb{C}[M]$. The ideal extension $\widetilde{\mathfrak{p}} := \widetilde{\mathfrak{t}}\mathfrak{p}$ of $\mathfrak{p}$ in $\widetilde{\mathfrak{t}}$ is thus generated by a single polynomial $B_K \in \mathbb{Z}[M, L]$ which has co-prime coefficients and is defined up to a factor $\pm M^k$ with $k \in \mathbb{Z}$. Again B_K can be chosen to have integer coefficients because everything can be defined over $\mathbb{Z}$. We will call B_K the *B-polynomial* of K.

5.3. **Relation between the A-polynomial and B-polynomial.** From the definitions one has immediately that the polynomial B_K is M-essentially divisible by A_K. Moreover, their zero sets $\{B_K = 0\}$ and $\{A_K = 0\}$ are equal, up to some lines parallel to the L-axis in the LM-plane.

Lemma 5.1. *The field $\mathbb{C}(M)$ is a flat $\mathbb{C}[M^{\pm1}]^\sigma$-algebra, and $\widetilde{\mathfrak{t}} = \mathfrak{t}^\sigma \otimes_{\mathbb{C}[M^{\pm1}]^\sigma} \mathbb{C}(M)$.*

Proof. The extension from $\mathbb{C}[M^{\pm1}]^\sigma$ to $\mathbb{C}(M)$ can be done in two steps: The first one is from $\mathbb{C}[M^{\pm1}]^\sigma$ to $\mathbb{C}[M^{\pm1}]$ (note that $\mathbb{C}[M^{\pm1}]$ is free over $\mathbb{C}[M^{\pm1}]^\sigma$ since $\mathbb{C}[M^{\pm1}] = \mathbb{C}[M^{\pm1}]^\sigma \oplus M\mathbb{C}[M^{\pm1}]^\sigma)$; the second step is from $\mathbb{C}[M^{\pm1}]$ to its field of fractions $\mathbb{C}(M)$. Each step is a flat extension, hence $\mathbb{C}(M)$ is flat over $\mathbb{C}[M^{\pm1}]^\sigma$.

It follows that the extension $(\mathfrak{t}^\sigma \hookrightarrow \mathfrak{t}) \otimes \mathbb{C}(M)$ is still an injection, i.e.

$$\psi : \mathfrak{t}^\sigma \otimes_{\mathbb{C}[M^{\pm1}]^\sigma} \mathbb{C}(M) \to \mathfrak{t} \otimes_{\mathbb{C}[M^{\pm1}]} \mathbb{C}(M) = \widetilde{\mathfrak{t}}, \quad \psi(x \otimes y) = xy,$$

is injective. Let us show that ψ is surjective. For every $n \in \mathbb{Z}$,

$$L^n = \psi\left((ML^n + M^{-1}L^{-n}) \otimes \frac{1}{M - M^{-1}} - (L^n + L^{-n}) \otimes \frac{M^{-1}}{M - M^{-1}}\right).$$

Since $\{L^n \mid n \in \mathbb{Z}\}$ generates $\widetilde{\mathfrak{t}} = \mathbb{C}(M)[L^{\pm1}]$, ψ is surjective. Thus ψ is an isomorphism. □

Consider the exact sequence (28). The ring $\mathbb{C}[\chi(X)]$ has a $\mathfrak{t}^\sigma$-module structure via the algebra homomorphism $\theta : \mathbb{C}[\chi(\partial X)] \equiv \mathfrak{t}^\sigma \to \mathbb{C}[\chi(X)]$, hence a $\mathbb{C}[M^{\pm1}]^\sigma$-module structure since $\mathbb{C}[M^{\pm1}]^\sigma$ is a subring of $\mathfrak{t}^\sigma$. By Lemma 5.1, $\widetilde{\mathfrak{t}} = \mathfrak{t}^\sigma \otimes_{\mathbb{C}[M^{\pm1}]^\sigma} \mathbb{C}(M)$. It follows that $\widetilde{\mathfrak{p}} = \mathfrak{p} \otimes_{\mathbb{C}[M^{\pm1}]^\sigma} \mathbb{C}(M)$. Hence by taking the tensor product over $\mathbb{C}[M^{\pm1}]^\sigma$ of the exact sequence (28) with $\mathbb{C}(M)$, we get the exact sequence

$$0 \to \widetilde{\mathfrak{p}} \to \widetilde{\mathfrak{t}} \xrightarrow{\widetilde{\theta}} \widetilde{\mathbb{C}[\chi(X)]}, \tag{29}$$

where $\widetilde{\mathbb{C}[\chi(X)]} := \mathbb{C}[\chi(X)] \otimes_{\mathbb{C}[M^{\pm1}]^\sigma} \mathbb{C}(M)$.

Proposition 5.2. *The B-polynomial B_K does not have repeated factors.*

Corollary 5.3. *For every knot K one has*

$$B_K = \frac{A_K}{\text{its } M\text{-factor}}.$$

Here the M-factor of A_K is the maximal factor of A_K depending on M only; it is defined up to a non-zero complex number.

5.4. **AJ conjecture, example.** Garoufalidis [Ga2] formulated the following conjecture (see also [FGL, Ge]).

Conjecture 4. (AJ conjecture) *For every knot K, $\alpha_K|_{t=-1}$ is equal to the A-polynomial, up to a factor depending on M only.*

Some authors also call the recurrence polynomial α_K the quantum A-polynomial.

Example 5.4. For the right-handed trefoil, α_K is given by (23). One has

$$\alpha_K \mid_{t=-1} = (M^4-1)(L-1)(LM^6+1) = (M^4-1)\, A_K(L,M),$$

and the conjecture holds for the trefoil.

Exercise 5.5. Suppose the AJ conjecture holds for a framed knot K. Show that it holds for any knot K' differing from K by a framing.

The AJ conjecture gives a very deep relation between the colored Jones polynomial and the fundamental group.

The A-polynomial is difficult to calculate, the recurrence polynomial is even more difficult to calculate. There are only a few simple knots for which the AJ conjecture can be verified by direct calculation.

5.5. **Results.** Suppose K is a knot in $\mathbb{R}^3 \subset S^3$. Let $X = \overline{S^3 \setminus N(K)}$, where $N(K)$ is a tubular neighborhood of K. Then $\mathcal{S}(X)$ is a left $\mathcal{S}(\partial X)$-module. We already know that $\mathcal{S}(\partial X) = \mathcal{S}(\mathbb{T}^2) = \mathcal{T}^\sigma$. Let $\mathcal{M} = \mathcal{R}[M^{\pm 1}] \subset \mathcal{T}$. Then $\mathcal{M}^\sigma \subset \mathcal{T}^\sigma$. Since $\mathcal{S}(X)$ is a $\mathcal{T}^\sigma$-module, it is a module over $\mathcal{M}^\sigma$.

Theorem 1 (See [LT2])**.** *Suppose K is a knot satisfying all the following conditions:*

(i) K is hyperbolic,

(ii) the SL_2-character variety of $\pi_1(S^3 \setminus K)$ consists of two irreducible components (one abelian and one non-abelian),

(iii) the universal SL_2-character ring of $\pi_1(S^3 \setminus K)$ is reduced,

(iv) the skein module $\mathcal{S}(X)$ is finitely generated over $\mathcal{M}^\sigma$, and

(v) the recurrence polynomial of K has L-degree greater than 1.

Then the AJ conjecture holds true for K.

Note that if K is adequate, then (v) holds. If K is non-torus alternating, then (i) and (v) hold. On the other hand, if K is torus, then it is known that the AJ conjecture holds [Hi, Tr].

Condition (iv) can be relaxed, see below.

Theorem 2 (See [LT2])**.** *The following knots satisfy all the conditions (i)–(v) of Theorem 1 and hence the AJ conjecture holds true for them.*

(a) All pretzel knots of type $(-2, 3, 6n \pm 1)$, $n \in \mathbb{Z}$.

(b) All two-bridge knots for which the SL_2-character variety has exactly two irreducible components; these include

- *all double twist knots of the form $J(k,l)$ (see Figure 12) with $k \neq l$*
- *all two-bridge knots $\mathfrak{b}(p,m)$ with $m = 3$, and*
- *all two-bridge knots $\mathfrak{b}(p,m)$ with p prime and $\gcd(\frac{p-1}{2}, \frac{m-1}{2}) = 1$.*

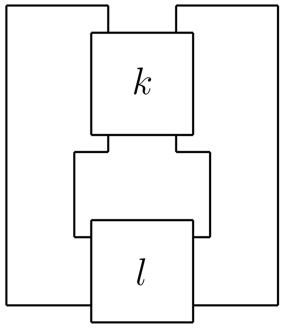


Figure 12. The double twist knot $J(k,l)$. Here k and l denote the numbers of half twists in the boxes. Positive (resp. negative) numbers correspond to right-handed (resp. left-handed) twists .

Here we use the notation $\mathfrak{b}(p,m)$ for two bridge knots from [BZ]. The fact that the character varieties of pretzel knots $(-2,3,6n\pm 1)$ and double twist knots have exactly 2 components was proved in [MPL] and in [Mat].

Actually, (b) can be strengthen as follows: if the non-abelian character variety of a two-bridge knot K is irreducible over $\mathbb{Z}$, then the AJ conjecture holds for K (joint work with X. Zhang).

Remark 5.6. Besides the infinitely many cases of two-bridge knots listed in Theorem 2, explicit calculations seem to confirm that "most two-bridge knots" satisfy the conditions of Theorem 1 and hence AJ conjecture holds for them. In fact, among 155 $\mathfrak{b}(p,m)$ with $p<45$, only 9 hyperbolic knots $\mathfrak{b}(15,11)$, $\mathfrak{b}(21,13)$, $\mathfrak{b}(27,5)$, $\mathfrak{b}(27,17)$, $\mathfrak{b}(27,19)$, $\mathfrak{b}(33,5)$, $\mathfrak{b}(33,13)$, $\mathfrak{b}(33,23)$, and $\mathfrak{b}(35,29)$ do not satisfy the condition (ii) of Theorem 1. Thus, the AJ conjecture holds for all two-bridge knots $\mathfrak{b}(p,m)$ with $p<45$ except for these 9 knots. Using explicit formula, Garoufalidis and Koutchan [GK] showed that the AJ conjecture holds for $\mathfrak{b}(15,11)$.

5.6. **Idea of proof.** We have the following commutative diagram

$$\begin{array}{ccc} \mathcal{T}^\sigma & \xrightarrow{\Theta} & \mathcal{S} \\ \varepsilon\big\downarrow & & \varepsilon\big\downarrow \\ \mathfrak{t}^\sigma & \xrightarrow{\theta} & \mathfrak{s} \end{array} \tag{30}$$

Here, for a $\mathbb{C}[t^{\pm 1}]$-module M, we denote ε the natural projection $M\to M/(t+1)$.

Let $\overline{\mathcal{M}}$ be the localization of $\mathcal{M}=\mathbb{C}[t^{\pm1},M^{\pm1}]$ at $(1+t)$, i.e.

$$\overline{\mathcal{M}}=\left\{\frac{f}{g}\ \mid\ f,g\in\mathbb{C}[t^{\pm1},M^{\pm1}], g\notin(1+t)\mathbb{C}[t^{\pm1},M^{\pm1}]\right\}.$$

Then $\overline{\mathcal{M}}$ is a local ring and a PID, and every ideal of $\overline{\mathcal{M}}$ is one of $((1+t)^k), k\in\mathbb{N}$. It is not difficult to show that $\overline{\mathcal{M}}$ is flat over $\mathcal{M}^\sigma$. Similarly, $\mathbb{C}(M)$ is flat over $\mathbb{C}[M^{\pm1}]^\sigma$.

Let $\overline{\mathcal{T}} = \mathcal{T}^\sigma \otimes_{\mathcal{M}^\sigma} \overline{\mathcal{M}}$ and $\overline{\mathcal{S}} = \mathcal{S} \otimes_{\mathcal{M}^\sigma} \overline{\mathcal{M}}$. Similarly, let $\bar{\mathfrak{t}} = \mathfrak{t}^\sigma \otimes_{\mathbb{C}[M^{\pm 1}]^\sigma} \mathbb{C}(M)$ and $\bar{\mathfrak{s}} = \mathfrak{s} \otimes_{\mathbb{C}[M^{\pm 1}]^\sigma} \mathbb{C}(M, L)$. Then one can show that

$$\overline{\mathcal{T}} = \overline{\mathcal{M}}[L^{\pm 1}]$$
$$\bar{\mathfrak{t}} = \mathbb{C}(M)[L^{\pm 1}] = \widetilde{\mathfrak{t}}.$$

From the Diagram (30), one has

$$\begin{array}{ccc} \overline{\mathcal{T}} & \xrightarrow{\overline{\Theta}} & \overline{\mathcal{S}} \\ \varepsilon\big\downarrow & & \varepsilon\big\downarrow \\ \bar{\mathfrak{t}} & \xrightarrow{\overline{\theta}} & \bar{\mathfrak{s}} \end{array}$$

According to condition (ii), the character variety of X has two components, one abelian and one non-abelian. Since K is hyperbolic, the non-abelian is the geometric component: it is the only irreducible component of the character variety containing the character of the discrete faithful SL_2 representation. By a result of Dunfield [Du], the map from the geometric component onto the character variety of the boundary torus is a birational map on its image. From here and the condition (iii) one can show that $\overline{\theta}$ is onto.

Now assume that $\overline{\mathcal{S}}$ is finitely generated over $\overline{\mathcal{M}}$. This condition is weaker than (iv). Then Nakayama's lemma shows that $\overline{\Theta}$ is surjective.

We have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{\mathcal{P}} & \xrightarrow{\iota} & \overline{\mathcal{T}} & \xrightarrow{\overline{\Theta}} & \overline{\mathcal{S}} & \longrightarrow & 0 \\ & & h\big\downarrow & & \varepsilon\big\downarrow & & \varepsilon\big\downarrow & & \\ 0 & \longrightarrow & \bar{\mathfrak{p}} & \longrightarrow & \bar{\mathfrak{t}} & \xrightarrow{\overline{\theta}} & \bar{\mathfrak{s}} & \longrightarrow & 0 \end{array}$$

Here $\overline{\mathcal{P}} = \ker(\overline{\Theta})$ and $\bar{\mathfrak{p}} = \ker(\overline{\theta})$, and h is the restriction of ε on $\overline{\mathcal{P}}$. One can show that h is surjective.

Since h is surjective, and B_K is the generator of $\bar{\mathfrak{p}}$, there is $\beta \in \overline{\mathcal{P}}$ such that $\beta|_{t=-1} = h(\beta) = B_K$. Since $\beta \in \overline{\mathcal{P}}$, $\alpha_K|\beta$. Then we have

$$(1 - L)|\varepsilon(\alpha_K)|\varepsilon(\beta) = B_K, \tag{31}$$

where the fact that $(1-L)|\varepsilon(\alpha_K)$ is Proposition 3.9. Since the character variety has exactly 2 component, $B_K = A_K = (1 - L)A'_K$, where A'_K is irreducible. It follows that either $\varepsilon(\alpha_K) = (1-L)$, or $\varepsilon(\alpha_K) = (1-L)A'_K = A_K$. The first possibility is excluded by condition (v). Hence, $\varepsilon(\alpha_K) = A_K$. This completes the proof of Theorem 1.

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