Splitting formulas for the LMO invariant

Gwénaël Massuyeau
(IRMA, Strasbourg)

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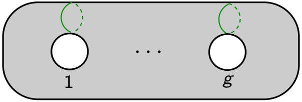

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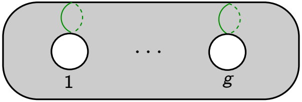

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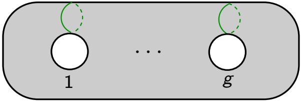

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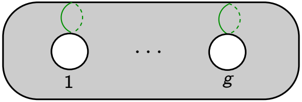

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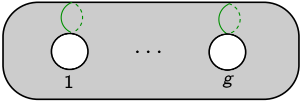

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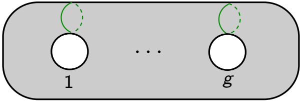

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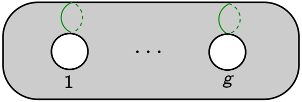

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Example: the genus 3 Heegaard splitting of the 3-torus

$T := S^1 \times S^1 \times S^1 = [0,1]^3/\sim$

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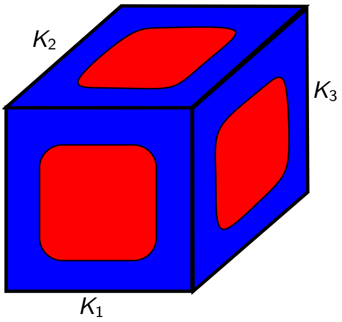

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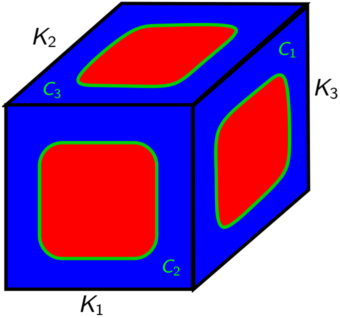

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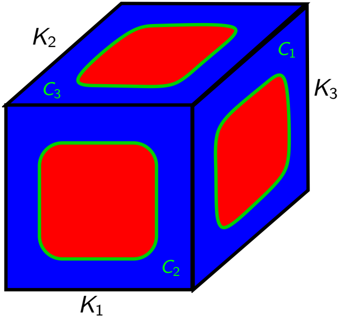

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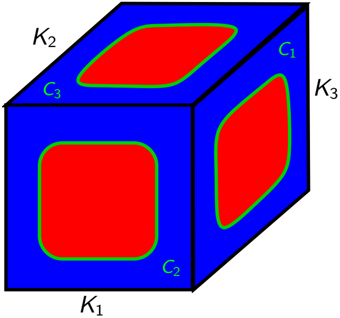

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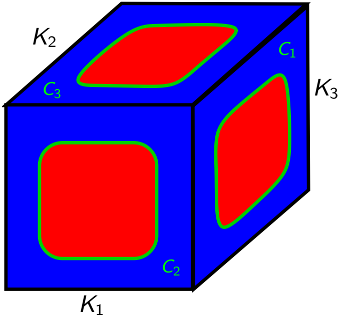

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Remark

Any genus 3 Heegaard splitting of the 3-torus is isotopic to this one (Frohman & Hass 1989).

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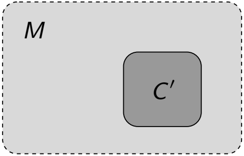
$\mathcal{C} = (C', C'')$: a $\mathbb{K}$-LP pair with $C' \subset M$



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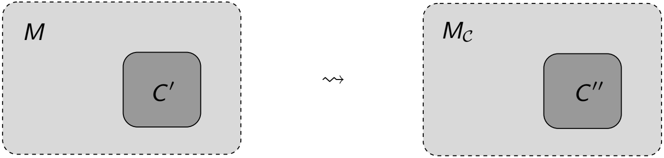

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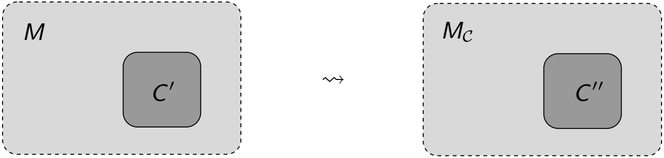

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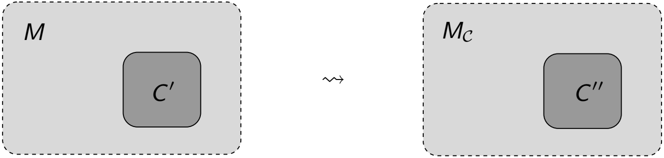

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The $\mathbb{Z}$-LP surgery $M \rightsquigarrow M_{\mathcal{T}}$ can be used to show that any trilinear alternate form is $\simeq$ to the $\mu(N)$ of a closed oriented 3-manifold N (Sullivan 1975).

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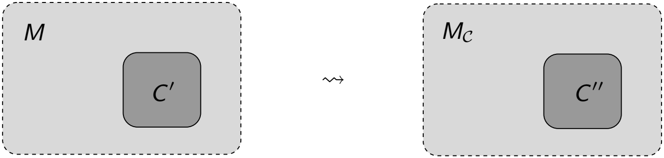

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This surgery is equivalent to Matveev's Borromean surgery and it is the main operation in the calculus of claspers by Goussarov and Habiro.

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Theorem (Matveev 1987)

The following statements are equivalent:

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for $\mathcal{M} = \mathbb{Q}\mathcal{HS}$, $A = \mathbb{Q}$ (Moussard 2012)

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An invariant $f : \mathcal{M} \longrightarrow A$ is of finite type of degree at most d if

$$\sum_{I \subset \{0,\ldots,d\}} (-1)^{|I|} \cdot f\left(M_{\mathcal{C}_I}\right) = 0 \in A$$

for any $M \in \mathcal{M}$, for any $\mathbb{K}$-LP pairs $\mathcal{C}_0, \ldots, \mathcal{C}_d$ with $C_0' \sqcup \cdots \sqcup C_d' \subset M$, where $M_{\mathcal{C}_I}$ results from the $\mathbb{K}$-LP surgeries $M \rightsquigarrow M_{\mathcal{C}_i}$ performed $\forall i \in I$.

$\mathbb{K} = \mathbb{Z}$: usual notion of finite-type inv. by Ohtsuki, Goussarov and Habiro

for $\mathcal{M} = \mathbb{Q}\mathcal{HS}$, $A = \mathbb{Q}$ (almost the same) (Moussard 2012)

$\mathbb{K} = \mathbb{Q}$: a stronger notion of finite-type invariant

Example: $\mathbb{K}\mathcal{HS} := \{\mathbb{K}\text{-homology spheres}\}$

- The Casson invariant $\lambda : \mathbb{Z}\mathcal{HS} \to \mathbb{Z}$ is a finite-type invariant of degree 2 in the usual sense (Morita 1991).
- Walker's extension $\lambda_W : \mathbb{Q}\mathcal{HS} \to \mathbb{Q}$ of λ is a finite-type invariant of deg. 2 in the strong sense (Lescop 1998).

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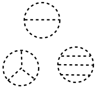

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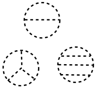

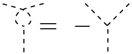

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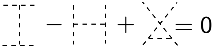

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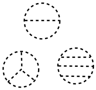

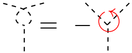

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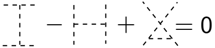

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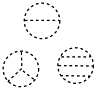

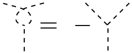

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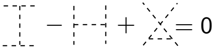

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Le, Murakami & Ohtsuki (1998) have constructed an inv. $\mathbb{Q}\mathcal{HS} \xrightarrow{Z} \mathcal{A}(\varnothing)$ such that

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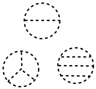

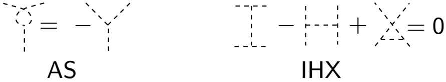

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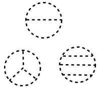

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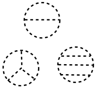

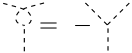

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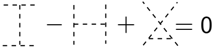

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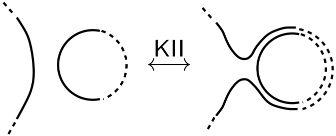

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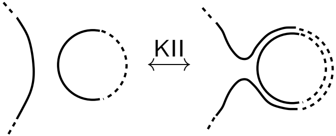

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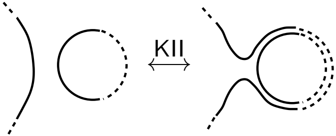

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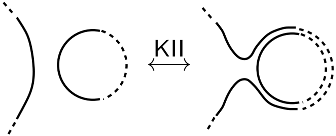

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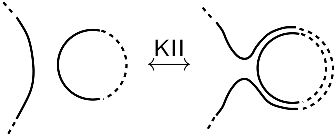

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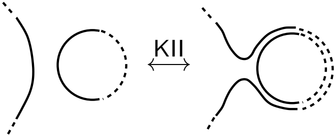

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(3) Divide by the values of the (± 1)-framed trivial knots ... and get KI.

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Splitting formulas for the LMO invariant

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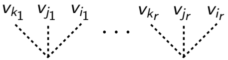

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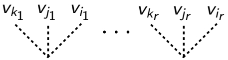

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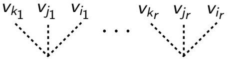

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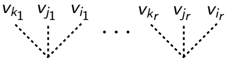

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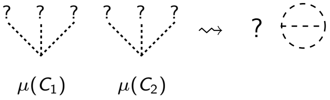

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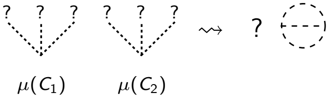

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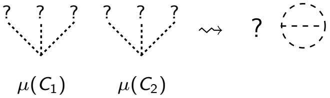

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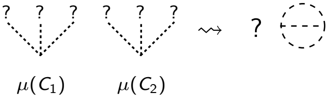

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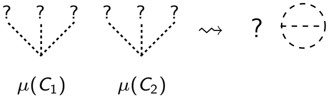

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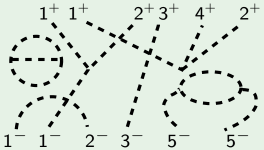

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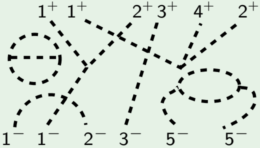

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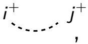
- morph. $g \to f$: elts of $\mathcal{A}\big(\{1^+, \ldots, g^+, 1^-, \ldots, f^-\}\big)$ without 

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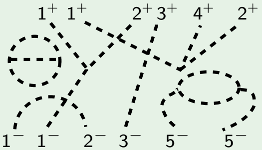

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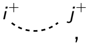
- morph. $g \to f$: elts of $\mathcal{A}\big(\{1^+, \dots, g^+, 1^-, \dots, f^-\}\big)$ without 

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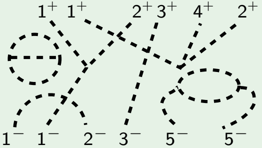

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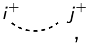
- morph. $g \to f$: elts of $\mathcal{A}\big(\{1^+, \dots, g^+, 1^-, \dots, f^-\}\big)$ without 

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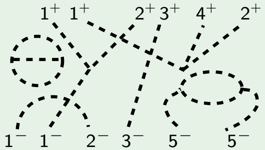

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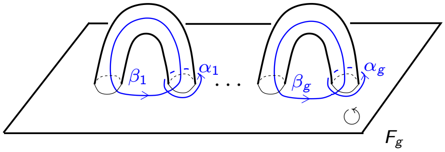

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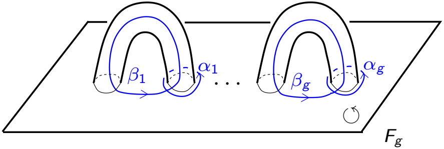

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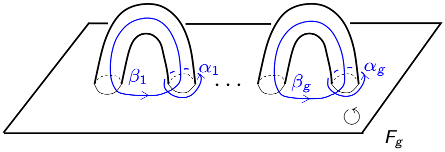

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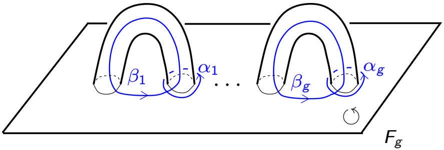

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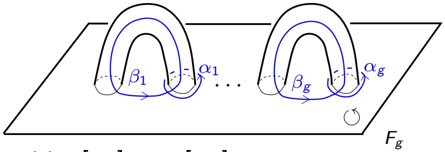

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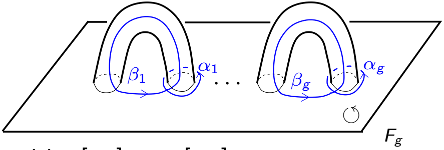

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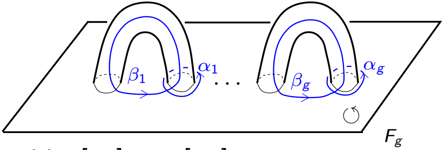

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The LMO functor

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There is a tensor-preserving functor $\widetilde{Z}$ extending the LMO invariant Z:

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Remark

Murakami & Ohtsuki (1997) and Cheptea & Le (2007) have constructed other functorial extensions of the LMO invariant.

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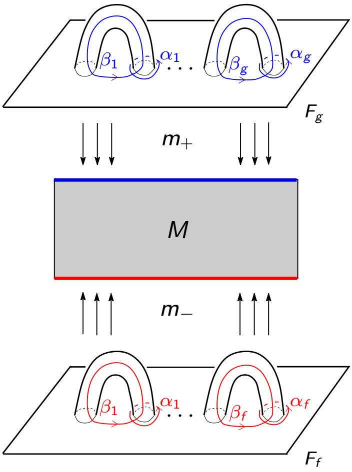

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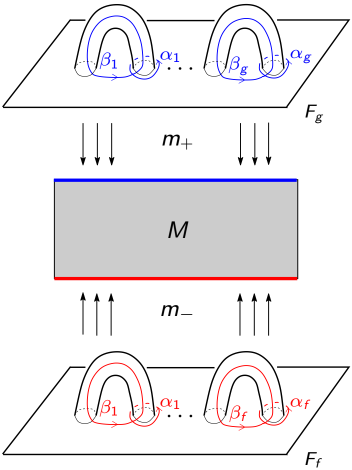

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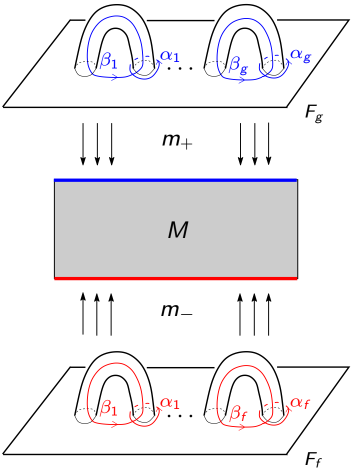

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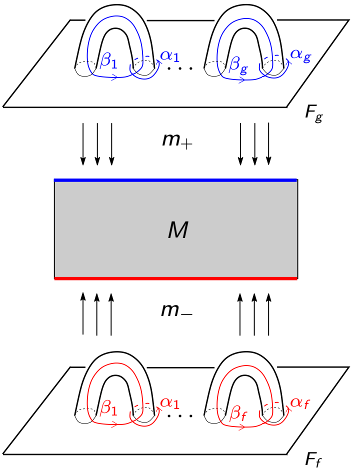

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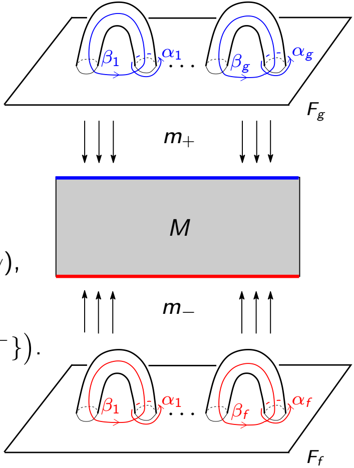
(3) Compute the Kontsevich–LMO invariant of (B, γ),

(4) and "symmetrize" this:

$\rightsquigarrow$ $Z(M) := \chi^{-1} Z(B, \gamma) \in \mathcal{A}\big(\{1^+, \ldots, g^+, 1^-, \ldots, f^-\}\big)$.



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(1) Glue 2-handles along $m_+(\beta_1),\dots,m_+(\beta_g)$,

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$\rightsquigarrow$ a $\mathbb{Q}$-homology cube B
... with a g-component "top" tangle γ^+
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(3) Compute the Kontsevich–LMO invariant of (B,γ),

(4) and "symmetrize" this:

$\rightsquigarrow Z(M) := \chi^{-1} Z(B,\gamma) \in \mathcal{A}\big(\{1^+,\dots,g^+,1^-,\dots,f^-\}\big)$.

(5) Normalize $Z(M)$ to get functoriality:

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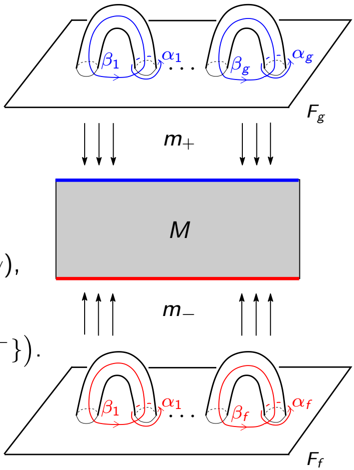

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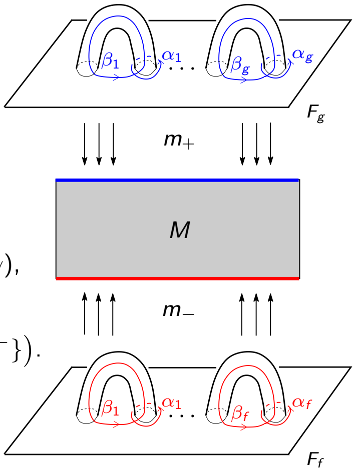

$$\widetilde{Z}(M) = \underbrace{\exp_\sqcup\left(\frac{1}{2}\mathrm{Lk}_B(\gamma^-)\ {}_{-}\!\frown\!{}_{-} + \mathrm{Lk}_B(\gamma^+,\gamma^-)\ \big(\begin{smallmatrix}+\\-\end{smallmatrix}\right)}_{\text{only struts}} \sqcup \underbrace{\left(\varnothing + \mathrm{Y} + \cdots\right)}_{\text{no strut at all}}$$

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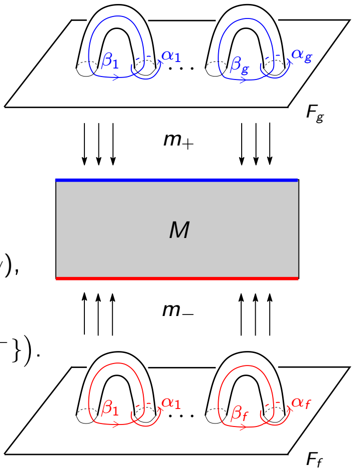

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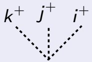

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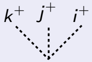

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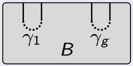

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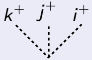

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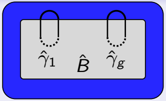

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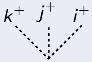

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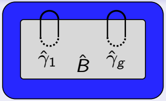

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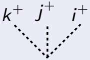

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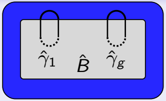

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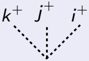

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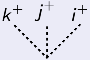

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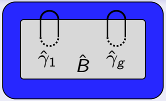

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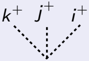

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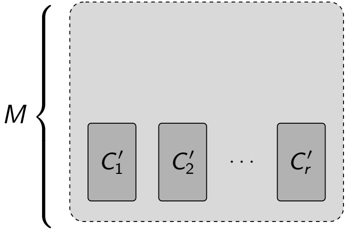

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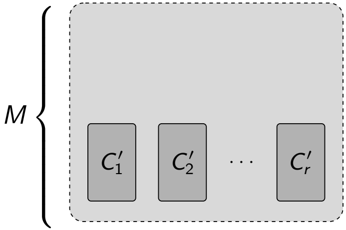

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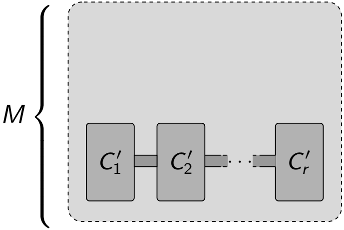

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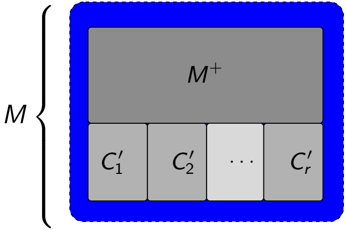

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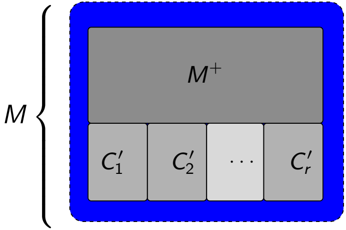

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$\rightsquigarrow$ a decomposition in the monoidal category $\mathbb{Q}\mathcal{LC}ob$:

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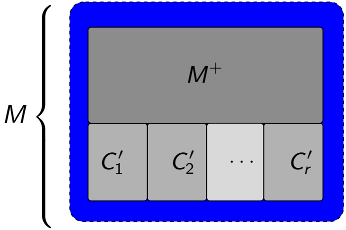

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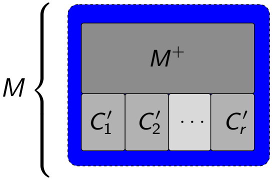

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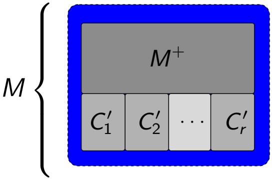

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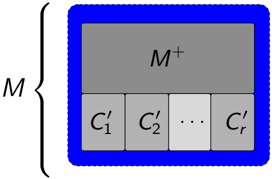

where $?$ $=$ $'$ or $''$

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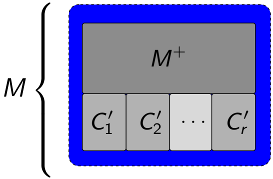

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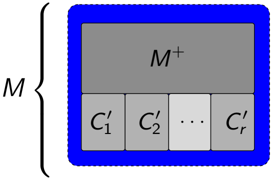

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$$= \sum_{I\subset\{1,\dots,r\}} (-1)^{|I|} \cdot \widetilde{Z}(\check{M}_{\mathcal{C}_I})$$

$$= \sum_{I\subset\{1,\dots,r\}} (-1)^{|I|} \cdot \widetilde{Z}\left((C_1^{?} \otimes \cdots \otimes C_r^{?}) \circ M^+\right)$$

$$= \sum_{I\subset\{1,\dots,r\}} (-1)^{|I|} \cdot \left(\widetilde{Z}(C_1^{?}) \otimes \cdots \otimes \widetilde{Z}(C_r^{?})\right) \circ \widetilde{Z}(M^+) \qquad \text{where } ? = \ ' \text{ or } ''$$

$$= \left(\left(\widetilde{Z}(C'_1) - \widetilde{Z}(C''_1)\right) \otimes \cdots \otimes \left(\widetilde{Z}(C'_r) - \widetilde{Z}(C''_r)\right)\right) \circ \widetilde{Z}(M^+)$$

Application of $\widetilde{Z}$ to the decomposition

$$\sum_{I\subset\{1,\dots,r\}} (-1)^{|I|}\cdot Z(M_{\mathcal{C}_I})$$

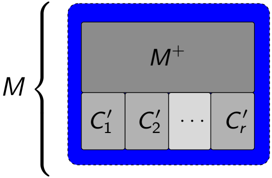
$$= \sum_{I\subset\{1,\dots,r\}} (-1)^{|I|}\cdot \widetilde{Z}(\check{M}_{\mathcal{C}_I})$$

$$= \sum_{I\subset\{1,\dots,r\}} (-1)^{|I|}\cdot \widetilde{Z}\left((C_1^{?}\otimes\cdots\otimes C_r^{?})\circ M^+\right)$$

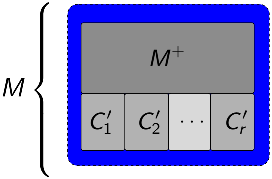
$$= \sum_{I\subset\{1,\dots,r\}} (-1)^{|I|}\cdot \left(\widetilde{Z}(C_1^{?})\otimes\cdots\otimes \widetilde{Z}(C_r^{?})\right)\circ \widetilde{Z}(M^+) \qquad \text{where } ? = \ ' \text{ or } ''$$

$$= \left(\left(\widetilde{Z}(C_1')-\widetilde{Z}(C_1'')\right)\otimes\cdots\otimes\left(\widetilde{Z}(C_r')-\widetilde{Z}(C_r'')\right)\right)\circ\widetilde{Z}(M^+)$$

$$= \left(\underbrace{\left(\widetilde{Z}_1(C_1')-\widetilde{Z}_1(C_1'')\right)}_{\mu(C_1)}\otimes\cdots\otimes\underbrace{\left(\widetilde{Z}_1(C_r')-\widetilde{Z}_1(C_r'')\right)}_{\mu(C_r)}\right)\circ\underbrace{\widetilde{Z}_0(M^+)}_{\exp_{\sqcup}(\ell_{\mathcal{C}}+\text{sthg else})} + (\text{i-deg} > r)$$



Application of $\widetilde{Z}$ to the decomposition



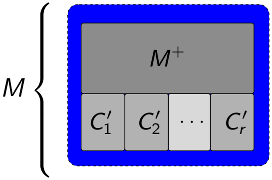
$$
\begin{aligned}
& \sum_{I\subset\{1,\ldots,r\}} (-1)^{|I|} \cdot Z(M_{\mathcal{C}_I}) \\
= & \sum_{I\subset\{1,\ldots,r\}} (-1)^{|I|} \cdot \widetilde{Z}(\check{M}_{\mathcal{C}_I}) \\
= & \sum_{I\subset\{1,\ldots,r\}} (-1)^{|I|} \cdot \widetilde{Z}\left((C_1^{?} \otimes \cdots \otimes C_r^{?}) \circ M^+\right) \\
= & \sum_{I\subset\{1,\ldots,r\}} (-1)^{|I|} \cdot \left(\widetilde{Z}(C_1^{?}) \otimes \cdots \otimes \widetilde{Z}(C_r^{?})\right) \circ \widetilde{Z}(M^+) \qquad \text{where } ? = {}' \text{ or } {}''
\end{aligned}
$$

$$
= \left(\left(\widetilde{Z}(C_1') - \widetilde{Z}(C_1'')\right) \otimes \cdots \otimes \left(\widetilde{Z}(C_r') - \widetilde{Z}(C_r'')\right)\right) \circ \widetilde{Z}(M^+)
$$

$$
= \Bigg(\underbrace{\left(\widetilde{Z}_1(C_1') - \widetilde{Z}_1(C_1'')\right)}_{\mu(C_1)} \otimes \cdots \otimes \underbrace{\left(\widetilde{Z}_1(C_r') - \widetilde{Z}_1(C_r'')\right)}_{\mu(C_r)} \Bigg) \circ \underbrace{\widetilde{Z}_0(M^+)}_{\exp_{\sqcup}(\ell_{\mathcal{C}} + \text{stgh else})} + (\text{i-deg} > r)
$$

$$
= \begin{pmatrix} \text{sum of all ways of identifying} \\ \text{pairwisely all legs of } \mu_{\mathcal{C}} \\ \text{by means of the pairing } \ell_{\mathcal{C}} \end{pmatrix} + (\text{i-deg} > r)
$$

Application of $\widetilde{Z}$ to the decomposition



$$\sum_{I\subset\{1,\dots,r\}} (-1)^{|I|}\cdot Z(M_{\mathcal{C}_I})$$

$$= \sum_{I\subset\{1,\dots,r\}} (-1)^{|I|}\cdot \widetilde{Z}(\check{M}_{\mathcal{C}_I})$$

$$= \sum_{I\subset\{1,\dots,r\}} (-1)^{|I|}\cdot \widetilde{Z}\left((C_1^{?}\otimes\cdots\otimes C_r^{?})\circ M^+\right)$$

$$= \sum_{I\subset\{1,\dots,r\}} (-1)^{|I|}\cdot \left(\widetilde{Z}(C_1^{?})\otimes\cdots\otimes\widetilde{Z}(C_r^{?})\right)\circ\widetilde{Z}(M^+) \qquad \text{where } ? = {}' \text{ or } {}''$$

$$= \left(\left(\widetilde{Z}(C_1')-\widetilde{Z}(C_1'')\right)\otimes\cdots\otimes\left(\widetilde{Z}(C_r')-\widetilde{Z}(C_r'')\right)\right)\circ\widetilde{Z}(M^+)$$

$$= \Bigg(\underbrace{\left(\widetilde{Z}_1(C_1')-\widetilde{Z}_1(C_1'')\right)}_{\mu(C_1)}\otimes\cdots\otimes\underbrace{\left(\widetilde{Z}_1(C_r')-\widetilde{Z}_1(C_r'')\right)}_{\mu(C_r)}\Bigg)\circ\underbrace{\widetilde{Z}_0(M^+)}_{\exp_{\sqcup}(\ell_{\mathcal{C}}+\text{stthg else})} + (\text{i-deg} > r)$$

$$= \begin{pmatrix}\text{sum of all ways of identifying}\\ \text{pairwisely all legs of } \mu_{\mathcal{C}}\\ \text{by means of the pairing } \ell_{\mathcal{C}}\end{pmatrix} + (\text{i-deg} > r) \qquad \square$$