Degeneration and dynamics

Greg McShane

November 19, 2012

Intro

- Σ = a surface of $\chi(\Sigma) < 0$, boundary $\partial\Sigma$
- Fricke space $\mathcal{F}(\Sigma)$ (Teichmueller space)
  := isotopy classes of marked hyperbolic structures on Σ
  + $\partial\Sigma$ geodesic.
- Fricke space $\hookrightarrow$ Character variety of $\pi_1 = \pi_1(\Sigma)$
- Mapping class group $\mathcal{MCG}$ = Diff/Isotopies

Definition G a Lie group then $\chi(\pi_1, G) =$ G-character variety
:= $\{$representations $\rho : \pi_1 \to G\}$ // G
"= $\{$representations $\rho : \pi_1 \to G\}$ up to conjugation. "

Representations and actions

$\rho \in \mathrm{Hom}(\pi_1, G)$
$[\rho] \in \mathrm{Hom}(\pi_1, G)//G = \chi(\pi_1, G)$

- Action on the representation variety

$$f \in \mathsf{Aut}(\pi_1),\ \rho \in \mathrm{Hom}(\pi_1, G),\ \ f^*\rho = \rho \circ f^{-1}$$

- Example: Action of interior automorphism $f : \gamma \mapsto \alpha\gamma\alpha^{-1}$

$$(f^*\rho)(\gamma) = \rho(\alpha^{-1}\gamma\alpha) = \rho(\alpha^{-1})(\gamma)\rho(\alpha).\ \Rightarrow [f\rho] = [\rho]$$

- Action of Out on $\mathrm{Hom}(\pi_1, G)//G$

$$[f], [\rho] \mapsto [f^*\rho]$$

Outer automorphisms $\mathrm{Out} := \mathsf{Aut}/\mathrm{Inn}$

The three big questions

1. What is the biggest subset on which $\mathcal{MCG}$ acts properly discontinuously?
2. On the complement of this set is the action ergodic?
3. Does the answer agree with *"geometric intuition"*?

On Teichmueller space the action is proper.

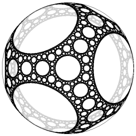
action proper $\leftrightarrow$ geometric structure associated to ρ.

When the action is proper one can take a quotient to form a "moduli space".

- How big is this moduli space?
- How does it degenerate?

What is the geography of the character variety relative to this action?

Decomposition of a group action



Γ torsion free, Kleinian group
acts on Riemann sphere $\hat{\mathbb{C}} = \Omega \sqcup \Lambda$

- Ω - discontinuity domain
  $\forall x, \exists U_x,\ g(U_x) \cap U_x = \emptyset, \forall g \neq 1$
- Λ- limit set
  - $\forall U, \exists g \neq 1,\ g(U) \cap U \neq \emptyset$
  - Topo transitive i.e. $\exists x,\ \Gamma.\{x\} \cap U \neq \emptyset$
    $(\Gamma.U = U) \Rightarrow U = \Lambda, \emptyset$
    $\Leftrightarrow$ Every invariant continuous function is constant.
  - Ergodic U meas.
    $(\Gamma.U = U) \Rightarrow m(U) \times m(U^c) = 0$
    $\Leftrightarrow$ Every invariant measurable function is constant.

Ergodic?

- Irrational rotation of the circle

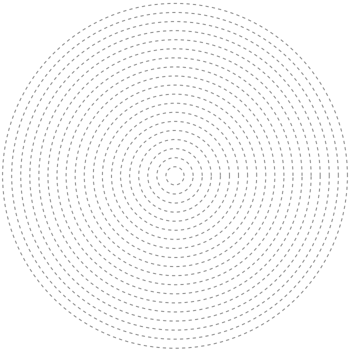
$$z \mapsto e^{i\theta} z$$

- Hyperbolic toral automorphism

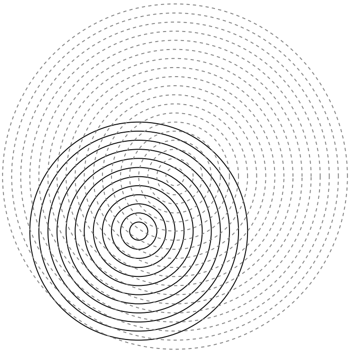
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{mod } 1$$

- $SL(2, \mathbb{Z})$ action on $\partial \mathbb{H} = P(\mathbb{R}^2)$
- $SL(2, \mathbb{Z})$ action on $\mathbb{R}^2$
- Semigroup generated by two irrational rotations with distinct fixed points.

- Semigroup generated by one irrational rotation on $\mathbb{R}^2$ NOT ergodic
- Semigroup generated by two irrational rotations with distinct fixed points.



- Semigroup generated by one irrational rotation on $\mathbb{R}^2$ NOT ergodic
- Semigroup generated by two irrational rotations with distinct fixed points.



Motivation

Motivation: Do,Norbury

Weil-Petersson volumes and cone surfaces, (2005)

- Mapping class group $\mathcal{MCG}$.
- ω_{WP} – $\mathcal{MCG}$-invar. symplectic form.
- $\Rightarrow$ Moduli space admits a symplectic vol. form

Symplectic volume of the moduli space of a surface

- $=$ a number for surface with marked points. Wolpert (1982), Penner, Harer-Zagier
- $=$ a polynomial for surface with boundary. Nakanishi-Naatanen (2001), Mirzakhani(2003).

$$\text{torus, one hole, } V_1(l_1) = \frac{1}{24}(4\pi^2 + l_1^2)$$

$$\text{torus, two hole, } V_1(l_1, l_2) = \frac{1}{192}(4\pi^2 + l_1^2 + l_2^2)(12\pi^2 + l_1^2 + l_2^2)$$

Motivation: Do,Norbury

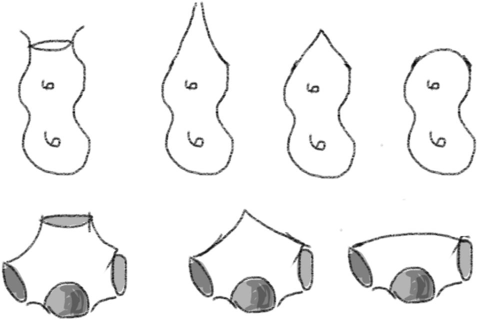
$$\begin{aligned} V_1(l_1) &= \frac{1}{24}(4\pi^2 + l_1^2) \\ V_1(l_1, l_2) &= \frac{1}{192}(4\pi^2 + l_1^2 + l_2^2)(12\pi^2 + l_1^2 + l_2^2) \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial l_2} V_1(l_1, l_2) &= \frac{1}{96} l_2(16\pi^2 + 2l_1^2 + 2l_2^2) \\ \left.\frac{\partial}{\partial l_2}\right|_{2\pi i} V_1(l_1, l_2) &= \frac{2\pi i}{96}(8\pi^2 + 2l_1^2) \\ &= \frac{2\pi i}{4.24}(4\pi^2 + l_1^2) = \frac{2\pi i}{4} V_1(l_1) \end{aligned}$$

Motivation: Interpolating the forgetful map

$$\frac{\partial}{\partial l_n} V_1(l_1, \ldots l_{n-1}, 2\pi i) = -2\pi i \times \chi(\Sigma_{g,n}) \times V_1(l_1, \ldots l_{n-1})$$

Cone point = geodesic boundary with complex length $i\theta$
Use cone surface with a cone point of angle $0 < \theta < 2\pi$.



$$\frac{\partial}{\partial l_n} V_1(l_1, \ldots l_{n-1}, 2\pi i) = -2\pi i \times \chi(\Sigma_{g,n}) \times V_1(l_1, \ldots l_{n-1})$$

Cone point = geodesic boundary with complex length $i\theta$
Use cone surface with a cone point of angle $0 < \theta < 2\pi$:

- to interpolate the forgetful map $(\Sigma_g, p) \to \Sigma_g$
- study degeneration of associated fibration(s)

$$\Sigma_g \to \mathcal{T}(\Sigma_{g,1})/\mathcal{MCG} \to \mathcal{T}(\Sigma_g)/\mathcal{MCG}$$

- Volume should go to zero (Schumacher-Trappani + some work)

$$V_g(\pm 2\pi i) = 0$$

- What happens to the topology of the moduli space.
- What happens to the dynamics of $\mathcal{MCG}$

as $\theta \to 2\pi$.

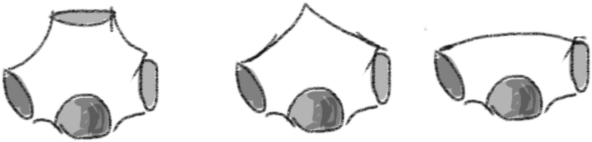
Four holed sphere

$$V_0(l_\alpha, l_\beta, l_\gamma, l_\delta) = \frac{1}{2}(4\pi^2 + l_\alpha^2 + l_\beta^2 + l_\gamma^2 + l_\delta^2)$$

What happens to

- the topology of the moduli space
- the dynamics of $\mathcal{MCG}$

as $l_\delta \to 2\pi i$?



Four holed sphere

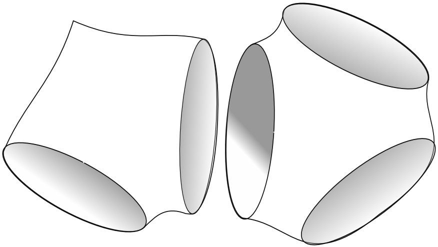
$$V_0(l_\alpha, l_\beta, l_\gamma, l_\delta) = \frac{1}{2}(4\pi^2 + l_\alpha^2 + l_\beta^2 + l_\gamma^2 + l_\delta^2)$$

What happens to

- the topology of the moduli space
- the dynamics of $\mathcal{MCG}$

as $l_\delta \to 2\pi i$?

Σ = Four-holed sphere = (pair of pants) $\sqcup$ (pair of pants) / $\sim$



Example: torus with a hole

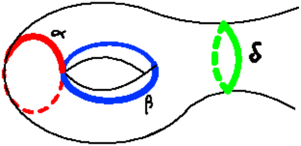
Σ = torus with a hole/cone point

Definition $SL(2,\mathbb{R})$-character variety

$:= \{$traces of representations $\rho : \pi_1 \to SL(2,\mathbb{R})\}$

"$= \{$representations $\rho : \pi_1 \to SL(2,\mathbb{R})\}$ up to conjugation. "

$= \{$representations $\rho : \pi_1 \to SL(2,\mathbb{R})\} \;//\; SL(2,\mathbb{R})$



$$\pi_1(\text{torus} \setminus \{pt\}) = \langle \alpha, \beta \rangle \simeq \mathbb{Z} * \mathbb{Z}$$

Theorem (Fricke/Vogt)

The $SL(2,\mathbb{R})$-character variety can be identified with a semi algebraic set (SAS) of $\mathbb{R}^3$ via

$$\chi : \rho \mapsto (\operatorname{tr}\rho(\alpha), \operatorname{tr}\rho(\beta), \operatorname{tr}\rho(\beta\alpha)).$$

Is $\chi : \mathrm{Hom}(\pi_1, SL(2,\mathbb{R})) \to \mathbb{R}^3$ surjective?

The character map is injective $\chi \to \mathbb{R}^3$
Is it surjective?Try to make a section:

$$\rho(\alpha) = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}, \ \rho(\alpha) = \begin{pmatrix} 0 & \zeta^{-1} \\ -\zeta & y \end{pmatrix}, \ \rho(\alpha\beta) = \begin{pmatrix} \zeta & x\zeta^{-1} + y \\ 0 & \zeta^{-1} \end{pmatrix}$$

$$z = \zeta + \zeta^{-1}$$

Conclusion

- $\exists SL(2,\mathbb{R})$ representation with traces (x, y, z) if (at least) one of $|x|, |y|, |z| \geq 2$.
- if $\zeta \notin \mathbb{R}$ then $\exists SU(2)$ representation with traces (x, y, z).

Slicing the character variety: Peripheral element

Lemma

$\delta = \alpha\beta\alpha^{-1}\beta^{-1}$, *$f \in Aut(\mathbb{Z} * \mathbb{Z})$ then $f(\delta)$ is conjugate to δ or δ^{-1}.*

Corollary

$\kappa : \rho \mapsto 2 + \mathrm{tr}\,(\rho(\delta))$ is an invariant function for $\mathcal{MCG}$ action (so action is not ergodic).

Calculate ℓ_δ from the traces

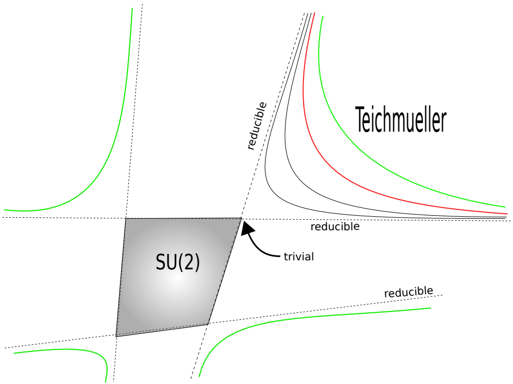
$$\begin{aligned}\kappa(\rho) &:= 2 + \mathrm{tr}\,\rho(\delta) \\ &= \mathrm{tr}\,^2\rho(\alpha) + \mathrm{tr}\,^2\rho(\beta) + \mathrm{tr}\,^2\rho(\gamma) - \mathrm{tr}\,\rho(\alpha).\mathrm{tr}\,\rho(\beta).\mathrm{tr}\,\rho(\gamma) \\ &= 2 - 2\cosh(\ell_\delta/2)\end{aligned}$$

The Fricke space of a holed torus is homeomorphic to (**a connected component)** of

$$(x, y, z) \in \mathbb{R}^3,\ \kappa(x, y, z) := x^2 + y^2 + z^2 - xyz \leq 0,$$

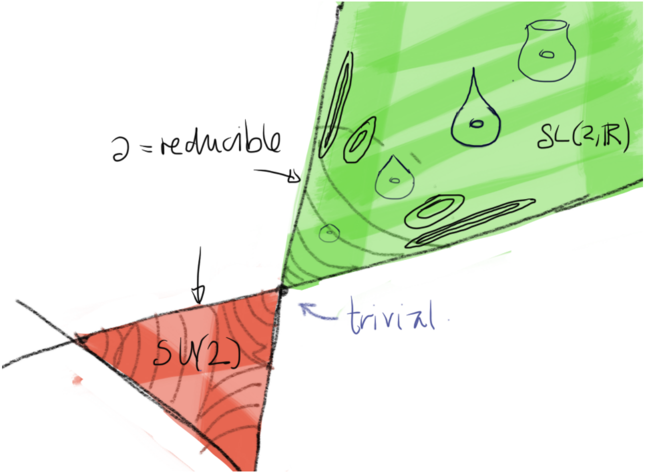
Geography of character variety

Loop round the hole $\delta = [\alpha, \beta] \in \pi_1$



Geography of character variety

Loop round the hole $\delta = [\alpha, \beta] \in \pi_1$



Topology of level sets of κ

Theorem (Fricke)

If $\rho \in Hom(\mathbb{Z} * \mathbb{Z}, SL(2,\mathbb{R}))$ is reducible then its image is in

$$\kappa^{-1}(4) \Leftrightarrow x^2 + y^2 + z^2 - xyz = 4.$$

.

1. Easy calculation $\Rightarrow \kappa^{-1}(t)$ singular iff $t = 4$
2. The singular points are permuted by the action.
3. The chararacter of the trivial representation is a singular point. It is a global fixed point.

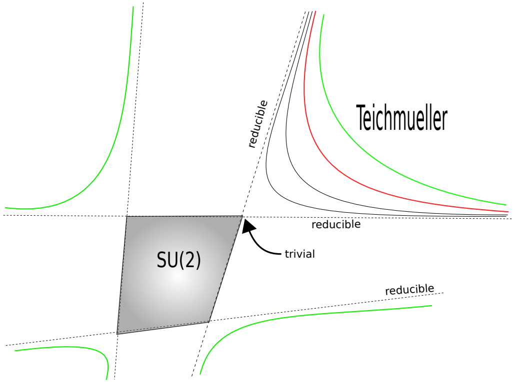
Topology of level sets of $\kappa^{-1}(t)$

$$\kappa(x, y, z) := x^2 + y^2 + z^2 - xyz$$

	$\sharp$ connected compents of $\kappa^{-1}(t)$	topology
$t = 4$	**single component**	complicated
$t < 0$	**4 components**	4 discs
$0 \leq t < 4$	**5 components**	4 discs a sphere
$4 < t$	**single component**	4 holed sphere

Remark: The sphere component $\subset \{|x|, |y|, |z| \leq 2\}$

Finally



Action of $\mathcal{MCG}$

t_α–Dehn twist round α.

- $(t_\alpha)_* : \pi_1 \to \pi_1, \ \ \alpha \mapsto \alpha, \beta \mapsto \beta.\alpha$
- Automorphism of $\mathbb{R}^3$

$$\begin{array}{ccc} \mathbb{R}^3 & \to & \mathbb{R}^3 \\ (x, y, z) & \mapsto & (x, xy - z, y) \end{array}$$

- Similarly Dehn twist round β

$$(x, y, z) \ \mapsto \ (xy - z, y, x)$$

Dehn twists generate $\mathcal{MCG}$.
$\Rightarrow$ The mapping class group acts on the whole of $\mathbb{R}^3$ by polynomial automorphisms.

Dynamical decomposition

Theorem (Goldman)

	$\kappa^{-1}(t)$ *topology*	*dynamics*
$t = 4$	*complicated*	*finite number of minimal sets*
$t < 0$	*4 discs*	*proper*
$0 \leq t < 4$	*4 discs* *a sphere*	*proper* *ergodic*
$4 < t \leq 20$	*4 holed sphere*	*ergodic*
$20 < t$	"	*wandering domains*

Definition an open set U is a *wandering domain* for the action of a group Γ iff

$$g(U) \cap U \neq \emptyset \Rightarrow g = 1_\Gamma$$

Faithful discrete $\Rightarrow$ action proper

1. $\kappa^{-1}(t < 0) = 4 \times$ Fricke space of a holed torus

length spectrum of a hyperbolic surface is discrete
$\Rightarrow$ action proper.

But if $0 \leq t < 4$ then 4 discs $\subset \kappa^{-1}(t)$ and action is still proper.
$\kappa(\rho) = 2 + \mathrm{tr}\,\rho(\delta) = 2 + 2\cosh(\ell_\delta/2) \geq 4$ if $\ell_\delta \in \mathbb{R}$
$\kappa(\rho) = 2 + \mathrm{tr}\,\rho(\delta) = 2 + 2\cosh(i\theta/2) \leq 4$

The discs are in the Fricke space of a torus with a cone point.

ρ not discrete, faithful for $0 < \theta < 2\pi$

If $\rho : \pi_1 \to SL(2, \mathbb{R})$ discrete, faithful then $\rho(\gamma)$ is hyperbolic $\forall \gamma (\neq 1) \in \pi_1$.

$\rho(\delta)$ is elliptic, but peripheral

Definition: Simple Hyperbolicity

$\forall \gamma \in \pi_1$ essential, simple loop the element $\rho(\gamma)$ is hyperbolic.

$\Rightarrow$ length function still well defined i.e.

$$\exists \ell_\gamma(\rho) > 0, |\mathrm{tr}\, \rho(\gamma)| = 2 \cosh(\ell_\gamma(\rho)/2)$$

Definition: Simple Spectrum

Simple spectrum $:= \{\ell_\gamma(\rho), \gamma \, \text{essential, simple loop}\}$.

want to show that the simple spectrum grows as quickly as for a discrete representation.

Theorem (Tan-Wong-Zhang)

For $\theta < 2\pi$: $\exists \epsilon(\rho) > 0$ such that $\forall \gamma$ essential simple loop

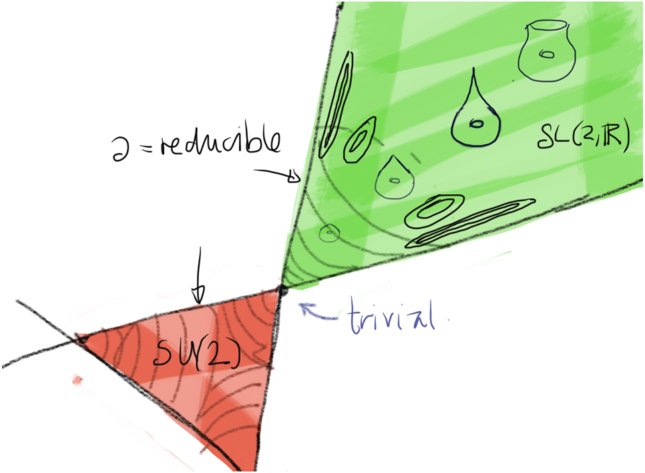
$$\ell_\gamma(\rho) \geq \epsilon \ell_\gamma(\rho_0).$$

where ρ_0 a fixed discrete faithful representation and $\mathbb{H}/\rho_0$ is a complete, hyperbolic one holed torus.

Corollary

The action of $\mathcal{MCG}$ is proper on the discs.

Proper $\leftrightarrow$ geometric structure.


∂ = reducible
SL(2,ℝ)
SU(2)
trivial

Wolpert's Volume Calculation

Invariant area form on the level sets $\kappa^{-1}(t)$:

$$\frac{dx \wedge dy}{xy - 2z}$$

When the action is proper one can associate a volume and calculate it by finding a fundamental domain.

Nakashini/Nataanen generalised Wolpert's method to surfaces with boundary :
one holed torus $V_1(l) = \frac{1}{24}(4\pi^2 + l^2)$

ρ reducible $\Rightarrow \kappa = 4 = 2 - 2\cosh(i\ell_\delta/2) \Rightarrow i\ell_\delta = 2\pi i$
and $V_1(2\pi i) = 0$

Theorem (Goldman)

The $SL(2,\mathbb{Z}) < \mathcal{MCG}$ action on the $SL(2,\mathbb{R})$-reducibles is ergodic in fact it is conjugate to the usual linear action of $SL(2,\mathbb{Z})$ on $\mathbb{R}^2/v \to -v$.

Action on the $SU(2)$ representations

Theorem (Goldman)

The action of $\mathcal{MCG}$ on the $SU(2)$-reducibles is ergodic in fact it is conjugate to the usual linear action of $SL(2,\mathbb{Z})$ on $(S^1 \times S^1)/$inversion.

- Any $\phi \in \mathcal{MCG}$ pseudo anosov acts ergodically on reducibles.
- But any $\phi \in \mathcal{MCG}$ Dehn twist preserves non constant function.

Theorem (Goldman)

The action of $\mathcal{MCG}$ on $\kappa^{-1}(t) \subset SU(2)$-characters is ergodic.

Theorem (Brown)

There exists t and $\phi \in \mathcal{MCG}$ pseudo anosov such that ϕ does not act ergodically on $\kappa^{-1}(t)$.

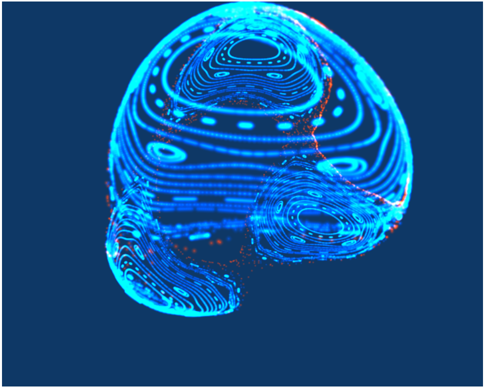
Action on the $SU(2)$ representations

Theorem (Goldman)

The action of $\mathcal{MCG}$ on $\kappa^{-1}(t) \subset SU(2)$-characters is ergodic.

Theorem (Brown)

There exists t and $\phi \in \mathcal{MCG}$ pseudo anosov such that ϕ does not act ergodically on $\kappa^{-1}(t)$.

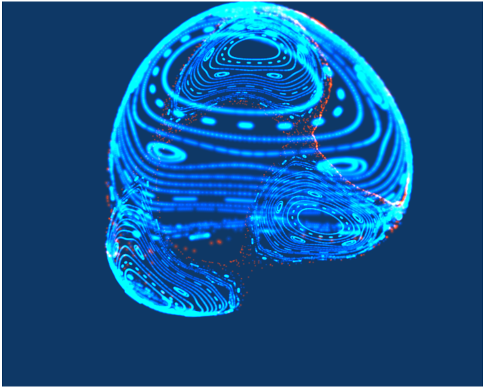


Action on the $SU(2)$ representations

Theorem (Goldman)

The action of $\mathcal{MCG}$ on $\kappa^{-1}(t) \subset SU(2)$-characters is ergodic.

Question (Funar): Does every non elementary subgroup of $\mathcal{MCG}$ act ergodically?



Non orientable surfaces

Non orientable surfaces $PGL(2, \mathbb{R})$

- $C_{2,0}$ = One holed Moebius band, $\mathcal{MCG}$ finite
- $C_{1,1}$ = One holed Klein bottle, $\mathcal{MCG}$ infinite dihedral

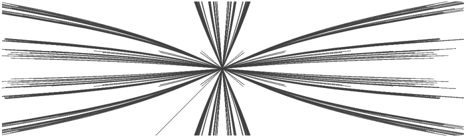
Want to use traces again to embed the character variety in $\mathbb{R}^3$.

Trick: the stabiliser of a plane $\mathbb{H} \subset \mathbb{H}^3$ in $PSL(2, \mathbb{C})$ is isomorphic to $PGL(2, \mathbb{R})$.

The orientation reversing reflections in $\mathbb{H}$ are restrictions of rotations of $\mathbb{H}^3$ through an angle π around an axis in $\mathbb{H}$.

$$z \mapsto -z \leftrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in SL(2, \mathbb{C})$$

Non orientable surfaces $PGL(2,\mathbb{R})$



Theorem

$\Gamma := PGL(2,\mathbb{Z})$

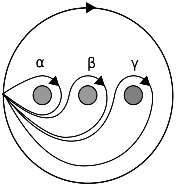
$\kappa(x,y,z) := -x^2 - y^2 + z^2 + xyz - 2$

1. *The action of* Γ *on* $\Gamma.\mathcal{F}(C_{2,0})$ *is proper.*
2. *For* $t < 2$ *the action on the complement of* $\kappa^{-1}(t) \setminus \Gamma.(C_{2,0})$ *is ergodic*
3. *The action of* Γ *on* $\Gamma.\mathcal{F}(C_{1,1})$ *is proper.*

The complement of $\Gamma.\mathcal{F}(C_{1,1})$ is a closed, nowhere dense subset. If the connected components of the complement of the dihedral representations are properly embedded copies of $\mathbb{R}$

Four holed spheres

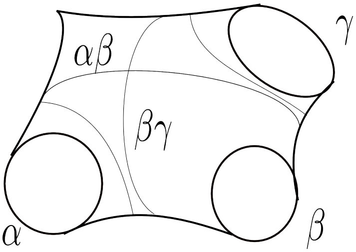
Four holed sphere: $\pi_1 = \langle \alpha, \beta, \gamma, \delta, \ \delta = \gamma\beta\alpha \rangle$.



- fix boundary lengths $l_\alpha, l_\beta, l_\gamma \geq 0$.
- vary the fourth $l_\delta = i\theta \Rightarrow |\mathrm{tr}\,\delta| = 2\cos(\theta/2))$

character variety	$\hookrightarrow$	(SAS. $\subset \mathbb{R}^7$)
ρ	$\mapsto$	traces of 4 peripheral elts
		+traces of 3 other elts
relative character variety	$\hookrightarrow$	(SAS. $\subset \mathbb{R}^3$)
ρ	$\mapsto$	traces of 3 other elts

Four holed sphere



Theorem (Fricke, Goldman, Gauglhofer-Semmler, Cantat-Loray)

The relative character variety defined by
$\mathrm{tr}\,\rho(\alpha) = 2\cosh(\ell_\alpha/2)$, $\mathrm{tr}\,\rho(\beta) = 2\cosh(\ell_\beta/2)$
$\mathrm{tr}\,\rho(\gamma) = 2\cosh(\ell_\gamma/2)$ *and* $\mathrm{tr}\,\rho(\delta) = 2\cos(\theta/2)$
is identified with a semi-algebraic subset of $\mathbb{R}^3$ *whose points satisfy*

$$x^2 + y^2 + z^2 + xyz = Ax + By + Cz + D$$

by $\rho \mapsto (\mathrm{tr}\,\rho(\alpha\beta), \mathrm{tr}\,\rho(\beta\gamma), \mathrm{tr}\,\rho(\gamma\alpha))$.

$\mathcal{MCG}$ dynamics on the relative character

$\exists$ three involutions

- induced by homeomorphisms of the four holed sphere
- that preserve a function $\rho \mapsto \mathrm{tr}\,\rho(\delta)$ (analogous to κ)

Because, though $x^2 + y^2 + z^2 + xyz = Ax + By + Cz + D$
is cubic, it is quadratic in x so there is an involution (on the SAS)
that swaps the two roots of

$$x^2 + x(yz - A) + y^2 + z^2 - By - Cz - D = 0$$

and this is induced by an orientation reversing homeomorphisms of
the sphere.

$\mathcal{MCG}$ dynamics when all the peripheral elements are parabolic

Observation

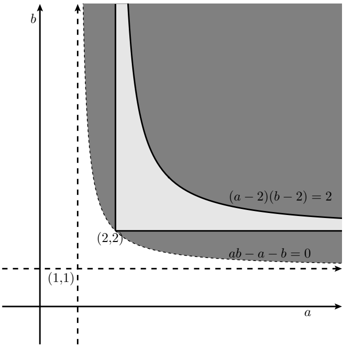
Involutions are good because the dynamics is determined by configuration of fixed point sets.

$\mathcal{MCG}$ dynamics: fixed point sets of involutions

When α, β, γ parabolic. $\exists$ a simpler model for the dynamics, (semi) conjugate to the action of 3 involutions on a subset of $\{(a, b)\} \subset \mathbb{R}^3$.

Fixed point sets of involutions

$\{a = 2\}$, $\{b = 2\}$
$\{c(a, b) = 2\} = \{(a - 2)(b - 2) = 2\cos(\frac{\theta}{2}) + 2\}$.



$\mathcal{MCG}$ dynamics: volume

Theorem (M)

When the peripheral elements ($\neq \delta$) are parabolic, $\theta < 2\pi$:

- *the three involutiuons generate a group isomorphic to* $\mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$
- *this group contains the pure braid group ($\mathcal{MCG}$) as a finite index subgroup.*
- *there is a fundamental domain which is a SAS.*

(Wolpert's) symplectic form on $\mathcal{T}(\Sigma)_\theta$

$$\omega_{WP} = \frac{da \wedge db}{ab - b - a}$$

Corollary

$Vol(\mathcal{T}(\Sigma)_\theta) = \frac{1}{2}(4\pi^2 - \theta^2)$

Where does the extra volume come from: Algebraic limits

Fix a sequence $\theta_n \to 2\pi$ consider limits of $\rho_n =$ holonomy of a 3 holed sphere with cone point of angle θ_n.

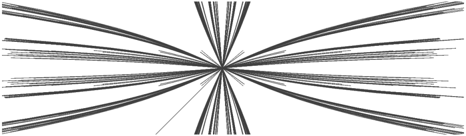
Theorem (M)

In the relative character variety of the 4 holed sphere.

- *space of possible limits of* $[\rho_n]$ *is a disc.*
- *this disc contains a single global fixed point for* $\mathcal{MCG}$*-action.*
- $\mathcal{MCG}$*-action is*
  - *wandering off a closed set on the disc if*

    $$l_\alpha^2 + l_\beta^2 + l_\gamma^2 > 0$$

  - *ergodic if* $l_\alpha^2 = l_\beta^2 = l_\gamma^2 = 0$



Algebraic limits

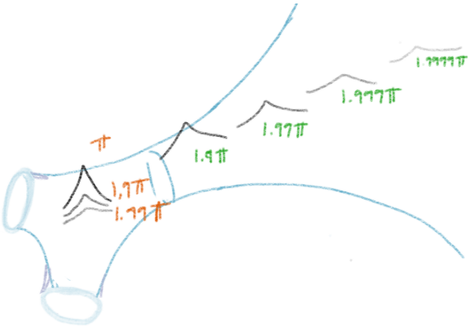
Extra volume comes from the quotient of the wandering part.

$$\begin{aligned} V_0(l_\alpha, l_\beta, l_\gamma, 2\pi i) &= \frac{1}{2}(4\pi^2 + l_\alpha^2 + l_\beta^2 + l_\gamma^2 - 4\pi^2) \\ &= \frac{1}{2}(l_\alpha^2 + l_\beta^2 + l_\gamma^2) \end{aligned}$$

$\rho_n(\delta) \to$ parabolic element. The fixed point is out one of the funnels on the 3-holed sphere.
From far away an elliptic element can look like a parabolic element:

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta)e^{2t} \\ -\sin(\theta)e^{-2t} & \cos(\theta) \end{pmatrix}$$

Algebraic limits



From far away an elliptic element can look like a parabolic element:

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta)e^{2t} \\ -\sin(\theta)e^{-2t} & \cos(\theta) \end{pmatrix}$$