Subgroups of mapping class groups generated by Dehn twists around meridians on splitting surfaces

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Aspects of representation theory in low-dimensional topology
and
3-dimensional invariants

Splittings we consider

1. Heegaard splittings

A Heegaard splitting is a decomposition of a closed 3-manifold into two handlebodies glued along their boundaries.

For a Heegaard splitting $M=H_1\cup_S H_2$, we define Δ_j ($j=1,2$) to be the set of meridians (simple closed curves on S bounding compressing discs in H_j) in the curve complex $C(S)$ of S.

The Hempel (or Heegaard) distance of the splitting is defined to be the distance between Δ_1 and Δ_2 in the curve complex $C(S)$.

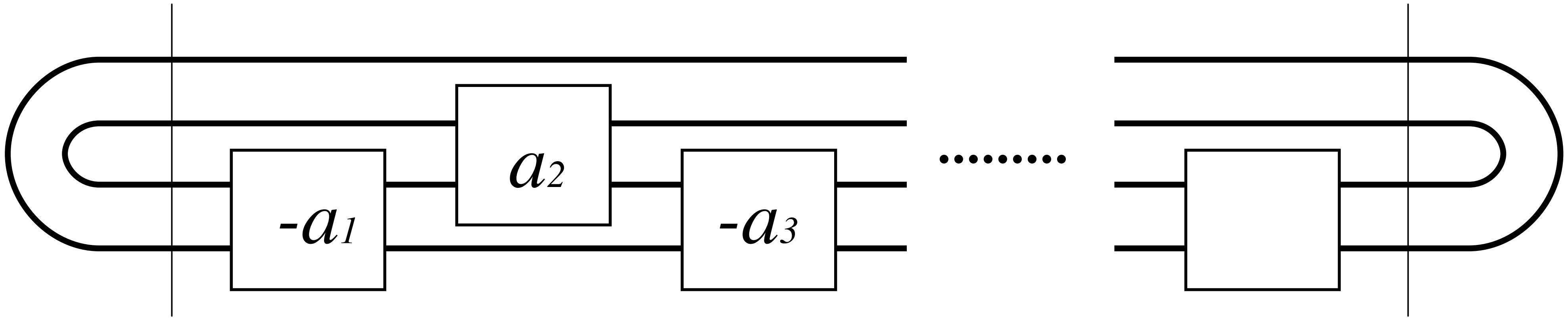
It is known that M is hyperbolic if the Hempel distance is greater than 2.

2. Bridge decomposition of knots/links

An n-bridge decomposition is a decomposition of a link in the 3-sphere into two trivial n-tangles (along $2n$-times punctured sphere).

More generally, we can consider a link L in a closed 3-manifold M and its Heegaard decomposition $M=H_1 \cup H_2$ such that both $H_1 \cap L$ and $H_2 \cap L$ are trivial tangles.
This can be regarded a relative version of a Heegaard splitting.

The most well-known are two-bridge decomposition of a link in the 3-sphere:



We can define the sets of meridians Δ_1,Δ_2 in the same way as Heegaard splittings, where meridians are assumed to be disjoint from the strands.

We define the Hempel distance to be the distance between Δ_1 and Δ_2 in the curve complex of the splitting punctured sphere for a link in the 3-sphere, or the splitting punctured surface in the general case.

Automorphism groups for splittings

Let $M=H_1\cup_S H_2$ be a Heegaard splitting or a bridge decomposition.

For $j=1,2$, we consider the inclusion $\iota_j : \pi_0\mathrm{Diff}^+(H_j)\to\pi_0\mathrm{Diff}^+(S)=\mathrm{Mod}(S)$.
The image of ι_j is denoted by Γ_j.

Let $\mathrm{Diff}^0(H_j)$ be the subgroup of $\mathrm{Diff}^+(H_j)$ consisting of diffeomorphisms **homotopic to the identity**.
We define G_j to be $\iota_j(\pi_0\mathrm{Diff}^0(H_j))$.
It is know that this group is generated by Dehn twist around meridians.

Minsky's questions:

1. Is $\Gamma_1\cap\Gamma_2$ finite if the Hempel distance is greater than 2?
2. Let $\langle\Gamma_1,\Gamma_2\rangle$ be the subgroup of $\mathrm{Mod}(S)$ generated by Γ_1 and Γ_2.
Does this group admit a decomposition $\langle\Gamma_1,\Gamma_2\rangle=\Gamma_1 *_{\Gamma_1\cap\Gamma_2}\Gamma_2$?

Namazi showed the answer to 1 is yes when the Hempel distance is large enough.
Johnson showed the same when the Hempel distance is greater than 2.

Questions posed by Sakuma (<u>our</u> problems):

1. Is $G_1 \cap G_2$ trivial if the Hempel distance is large enough?

2. Let $\langle G_1, G_2 \rangle$ be the subgroup of $\mathrm{Mod}(S)$ generated by G_1 and G_2.
   Does this group admit a free-product decomposition $\langle G_1, G_2 \rangle = G_1 * G_2$?

3. Characterise curves on S which are null-homotopic in M.
   Does the set of curves null-homotopic in M coincide with $\langle G_1, G_2 \rangle (\Delta_1 \cup \Delta_2)$ if the Hempel distance is large enough?

Let $\mathscr{PML}(S)$ denote the projective measured lamination space on S.

4. Is there an open set in $\mathscr{PML}(S)$ in which no curves are null-homotopic in M?

5. Does the action of $\langle G_1, G_2\rangle$ on $\mathscr{PML}(S)$ have non-empty domain of discontinuity if the Hempel distance is large enough?

6. Let Δ^* be the closure of $\langle G_1, G_2\rangle(\Delta_1 \cup \Delta_2)$ in $\mathscr{PML}(S)$.
Does Δ^* have measure 0 ?

D. Lee and Sakuma showed that all of these are true for two-bridge link complements.

Structure of automorphism groups: Answers to Sakuma's Questions 1 and 2

Theorem 1 (Bowditch-O-Sakuma).
There is a constant K depending only on the topological type of S such that for any Heegaard splitting or bridge decomposition with Hempel distance greater than K, the following hold.

(a) $G_1 \cap G_2$ is trivial.

(b) $\langle G_1, G_2 \rangle = G_1 * G_2$.

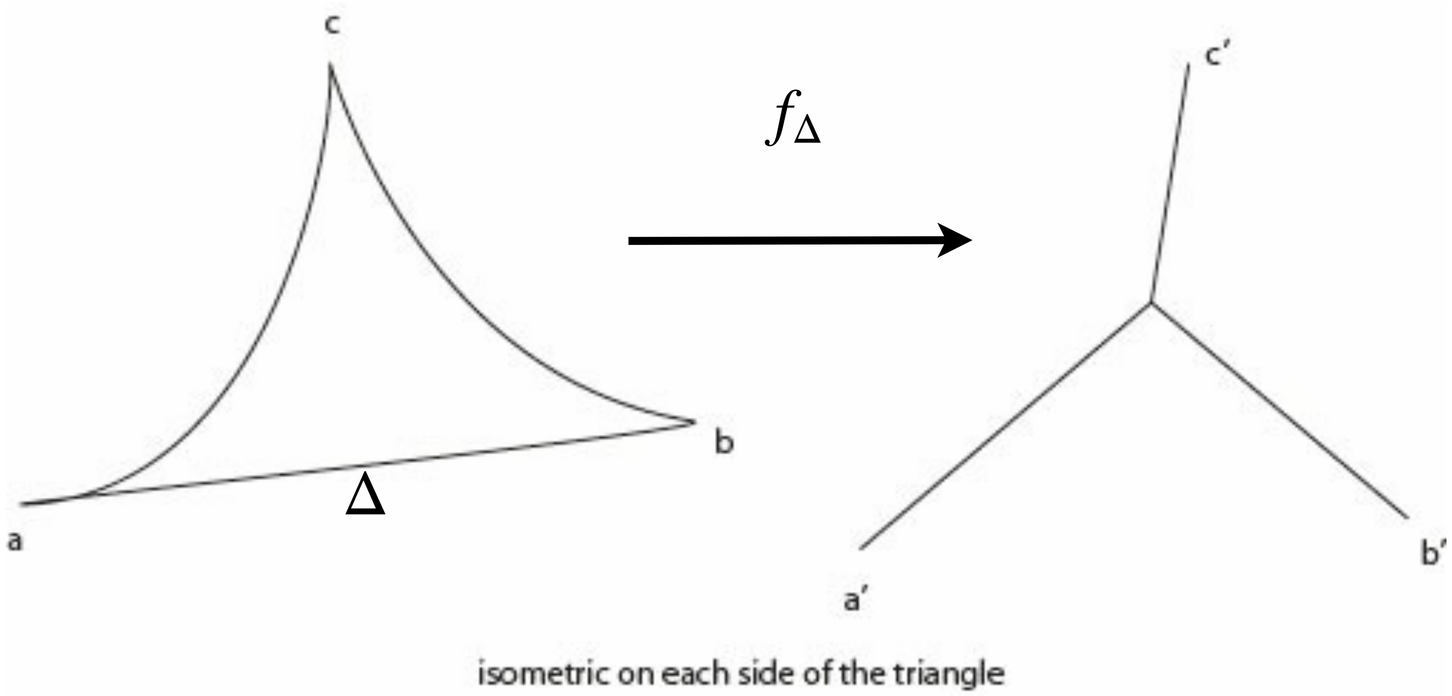
Remark: We have not yet succeeded in getting the same kind of answer to Minsky's Question 2.

Sketch of Proof

Ingredients:

1. The hyperbolicity of curve complexes. (Masur-Minsky)
2. The acylindricity of the action of the mapping class group on a curve complex. (Bowditch)
3. The quasi-convexity of Δ_1, Δ_2 (Minsky, Namazi).

1. The hyperbolicity: there is a constant δ depending only on the topological type of S such that every geodesic triangle in $C(S)$ is δ-thin.



Δ is said to be δ-thin when $d(x,y) \leq \delta$ for every x,y with $f_\Delta(x) = f_\Delta(y)$.

2. Acylindricity:

For any $K \geq 0$, there exist D and N depending only on the topological type of S with the following property.

For any points $x, y \in C(S)$ with $d(x, y) \geq D$, there are at most N elements of $g \in \mathrm{Mod}(S)$ such that $d(x, gx) \leq K$ and $d(y, gy) \leq K$.

3. Quasi-convexity:

There exists a constant L depending only on the topological type of S such that every geodesic segment connecting two points in Δ_j is contained in the L-neighbourhood of Δ_j.

(A) $G_1 \cap G_2$ is trivial.

The hyperbolicity and the acylindricity imply that $G_1 \cap G_2$ is finite if K is large.
(This part was shown by Namazi, we present a simplified version of his idea.)

Connect Δ_1 and Δ_2 by a shortest geodesic γ with endpoints $x \in \Delta_1$ and $y \in \Delta_2$.
For $g \in G_1 \cap G_2$, the distances $d(x,gx)$ and $d(y,gy)$ are uniformly bounded.
(Otherwise, we would have an arc shorter than γ connecting Δ_1 and Δ_2 by the hyperbolicity and the quasi-convexity of Δ_1 and Δ_2.)

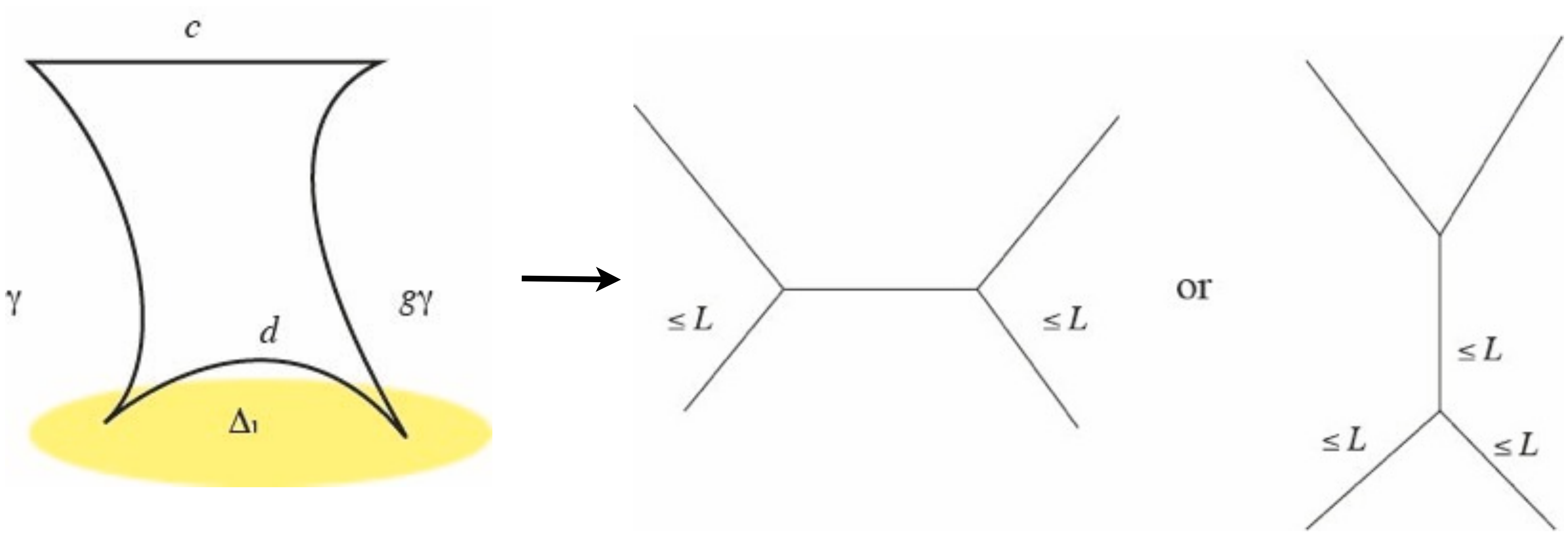
The acylindricity implies there are only finitely many elements in $G_1 \cap G_2$ provided that γ is sufficiently long.

It is known that G_1 (or G_2) is torsion free (Otal).

→ contradiction

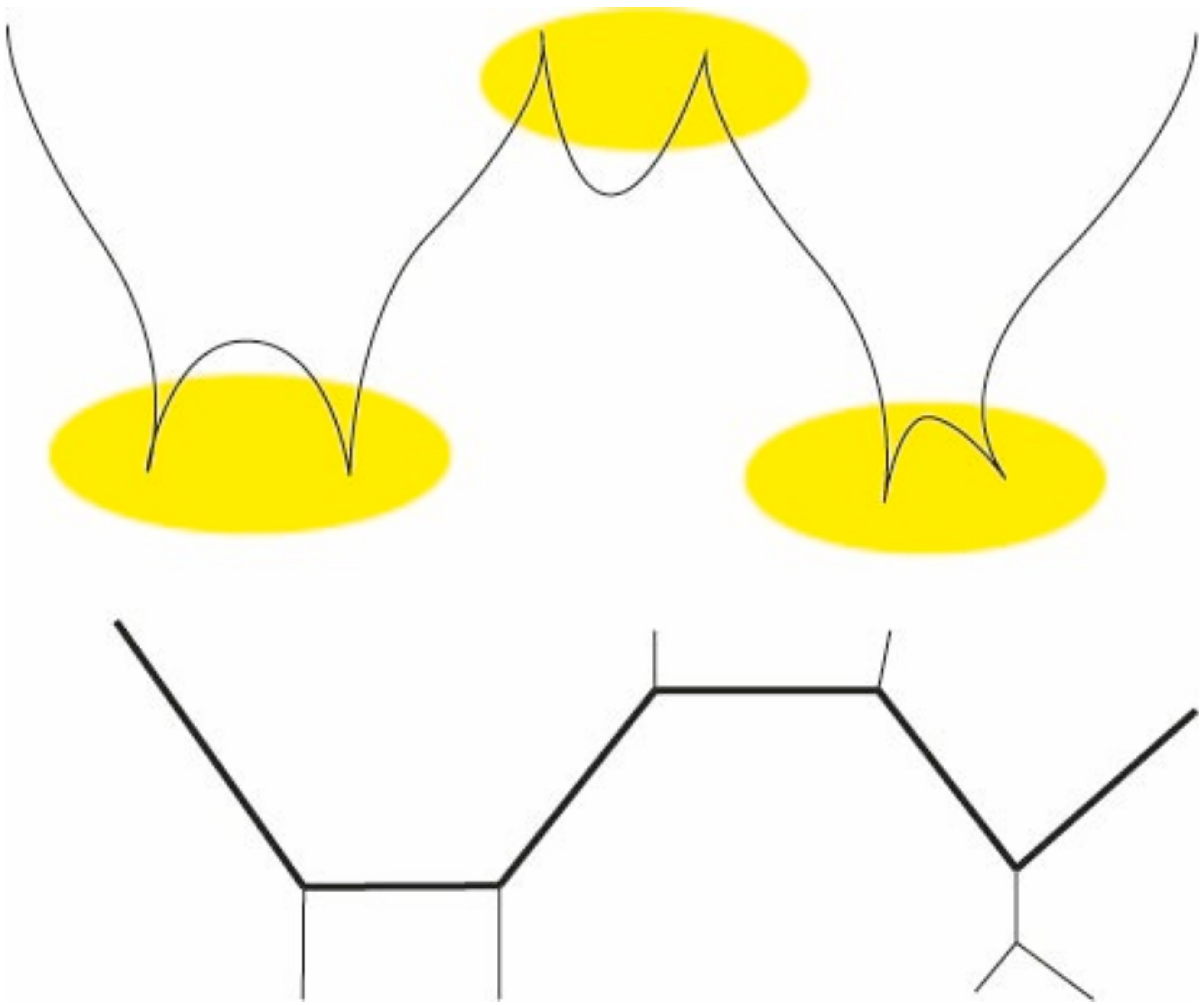
(B) $G = G_1 * G_2$.

For $g \in G_j$, we connect the (non-fixed) endpoints of γ and $g\gamma$ by a geodesics c and d, and consider the quadrilateral $\gamma \cup d \cup g\gamma \cup c$.



Then the acylindricity of the action and the quasi-convexity of Δ_1 and Δ_2 give us a bound L for the lengths of the legs of the tree on which each pair of γ, d and $g\gamma$ is identified.

Express $g \in \langle G_1, G_2 \rangle$ as a reduced product $g_1 \ldots g_k$ of elements of G_1, G_2.
Consider the translates $\gamma, g_k\gamma, g_{k-1}g_k\gamma \cdots, g\gamma$.
Connect their endpoints by geodesics as d above.
These constitute a uniform quasi-geodesic (if the Hempel distance is large enough) and never comes back to the initial point.
Therefore g cannot be the identity.



Open sets containing no null-homotopic curves: An answer to Sakuma's problem 4

Theorem 2 (O-Sakuma). There is a non-empty open set U in $\mathscr{PML}(S)$ in which no curves are null-homotopic in M if the Hempel distance is large enough.

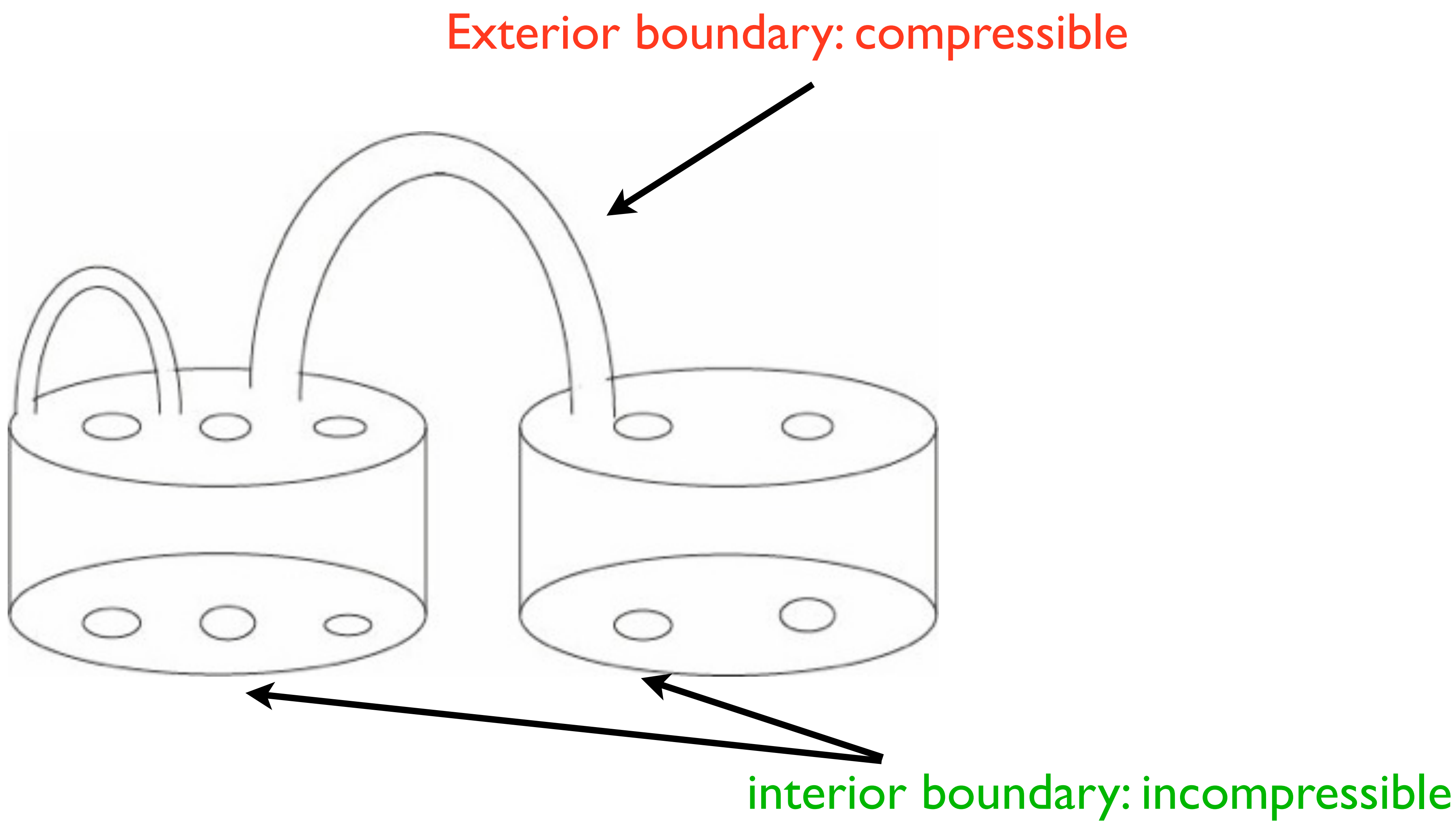
Basic tool:

We use model manifolds of Heegaard splittings/bridge decomposition by Namazi (partially collaborating with Brock, Minsky and Souto).

If the Hempel distance is larger than K, then there is an L-bilipschitz model manifold $\boldsymbol{N}$ of M with L depending on K and the genus of S.

The model manifold is constructed from hyperbolic compression bodies.

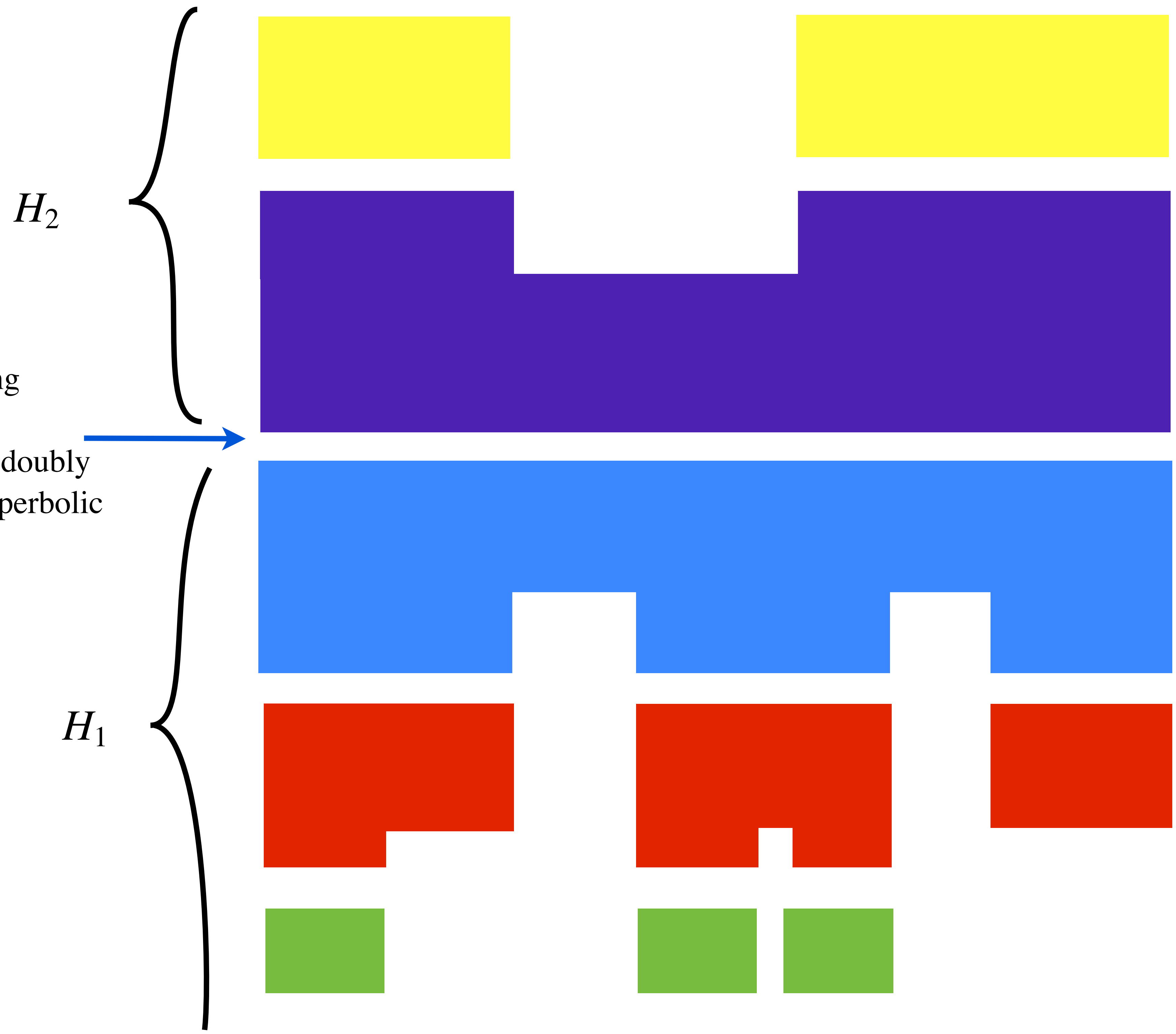
A compression body is a connected 3-manifold obtained from finitely many produce *I*-bundles by attaching finitely many 1-handles.



We regard a handle body also as a compression body where the interior boundary is empty.

Namazi's model construction

Here is a pasting margin nearly isometric to a "doubly degenerate" hyperbolic 3-manifold.



The interior boundary of a bigger compression body is pasted to the exterior boundary of a smaller compression body.

This model is realised as a negatively curved manifold close to a hyperbolic manifold.

Sketch of Proof:

Consider a measured lamination λ which can be realised by a pleated surface homotopic to the inclusion of S in the pasting margin of along S.

We take λ to have rational depth 0, i.e. so that every complementary region of λ is an ideal triangle.

This is always possible since the set of projective laminations of rational depth 0 is an open dense set.

Then, there is an open neighbourhood U of $[\lambda]$ in $\mathscr{PML}(S)$ such that every measured lamination in U can also be realised by a pleated surface near the realisation of λ.

In particular, no simple closed curves in U are null-homotopic in M.
This is an open set as we wanted.

Non-empty domain of discontinuity: an answer to Sakuma's problem 2

Theorem 3. There is a non-empty domain of discontinuity for the action of $\langle G_1, G_2 \rangle$ on $\mathscr{PML}(S)$ if the Hempel distance if large enough. To be more precise, there is an open set U such that $\{g \in \langle G_1, G_2 \rangle \mid gU \cap U \neq \emptyset\} = \{id\}$.

Proof:

We take U to be an open set as in the proof of Theorem 2.
Suppose that $gU \cap U \neq \emptyset$.

Then for any simple closed curve $a \in gU \cap U$, there is $b \in U$ such that $a = g(b)$.
Since $g \in \langle G_1, G_2 \rangle$, this implies $a = b$ by the property *of* U.
Therefore g fixes all simple closed curves in $gU \cap U$.
Since simple closed curves are dense in $gU \cap U$, this shows g fixes all points in $gU \cap U$.
This is possible only when $g = id$.

Special case: gluing by iterations of a pseudo-Anosov map

Let $S^3=H_1\cup_\iota H_2$ be a standard Heegaard splitting of S^3 along S pasted by $\iota:\partial H_1\to\partial H_2$.

Alternatively, we consider an n-bridge decomposition of unknot $V=H_1\cup_\iota H_2$.

Let $\varphi:S\to S$ be a pseudo-Anosov map which does not extend any compression body in H_2.

μ_φ: a stable lamination of φ. (cf. Cyril Lecuire's talk)

Consider a Heegaard splitting $M_n=H_1\cup_{\iota\circ\varphi^n}H_2$.

G_1: the subgroup of the mapping class group of H_1 represented by homeomorphisms homotopic to the identity in H_1, regarded as a subgroup of the mapping class group of S.

$G^n{}_2$: the subgroup of the mapping class group of H_2 in M represented by homeomorphisms homotopic to the identity in H_2, regarded as a subgroup of the mapping class group of $S\subset M_n$.

$$\mathscr{PD}(H_1)=\{[\lambda]\in\mathscr{PML}(S)\,|\,\exists\eta>0 \text{ such that } i(\lambda,m)>\eta \text{ for any meridian } m \text{ of } H_1\}$$

Theorem 3. For any projective lamination $[\lambda]$ in $\mathscr{PD}(H_1)\backslash G_1\iota(\mu_\varphi)$ there exist an open neighbourhood U of $[\lambda]$ and n_0 such that $\{g\in\langle G_1,G^n{}_2\rangle \mid gU\cap U\neq\varnothing\} =\{\mathrm{id}\}$ if $n\geq n_0$.
In other words, $\mathscr{PD}(H_1)\backslash G_1\iota(\mu_\varphi)$ is covered by the domain of discontinuity of $\langle G_1,G^n{}_2\rangle$ as $n\to\infty$.

Sketch of Proof:

Namazi-Souto showed that if we put a basepoint in H_1, the manifold M_n (which is hyperbolic for large n) converges geometrically to a hyperbolic 3-manifold N such that $\pi_1(N)=\pi_1(H_1)$, and its ending lamination is $\iota(\mu_\varphi)$.

This implies that any lamination in $\mathscr{PD}(H_1)\backslash G_1\iota(\mu_\varphi)$ can be realised by a pleated surface in N.

Using the geometric convergence of $\{M_n\}$ to N, it follows that any $[\lambda]\in\mathscr{PD}(H_1)\backslash G_1\iota(\mu_\varphi)$ has a neighbourhood in which no simple closed curves are null-homotopic in M_n for $n\geq n_0$.

The same argument as in the proof of Theorem 2 shows that $[\lambda]$ has a neighbourhood U such that $\{g\in\langle G_1,G^n{}_2\rangle\}\mid gU\cap U\neq\varnothing\} =\{\mathrm{id}\}$ for $n\geq n_0$.

Related topics and prospects

1. A systematic construction of epimorphisms between n-bridge link groups.

For two-bridge links this was done by Ohtsuki-Riley-Sakuma.
Lee-Sakuma gave a necessary and sufficient condition for the existence of epimorphisms preserving meridians for two bridge link complements.

To get a similar result for n-bridge link complements, we need to refine Theorem 1.

For instance, if we can solve Sakuma's problem 1 affirmatively, and can give a lower bound of Hempel distances concretely, we are done.

Recall the problem 1:
Does the set of curves null-homotopic in M coincide with $\langle G_1, G_2 \rangle (\Delta_1 \cup \Delta_2)$ if the Hempel distance is large enough?

2. (Yet another) variation of McShane's identity

The original version of McShane's identity:

Fix a hyperbolic once-punctured torus S.
C is the set of essential simple closed curves on S.
l denotes the hyperbolic length on S.

$$\sum_{\gamma\in\mathcal{C}} \frac{1}{1+\exp(l(\gamma))} = \frac{1}{2}$$

There are generalisations and variations of this equality to various settings.
McShane, Bowditch, Akiyoshi-Miyachi-Sakuma, Tan-Wong-Zhang, Mirzakhani, etc.

Lee and Sakuma obtained a variation of McShane's identity for two bridge link complement.
In this case, $\mathscr{PML}(S)\cong S^1$ and it was shown that $S^1\setminus\langle G_1,G_2\rangle\{r,\infty\}$ is the domain of discontinuity of $\langle G_1,G_2\rangle$.
We can take two intervals I_1,I_2 such that $I_1\cup I_2$ is the fundamental domain of the action on the domain of discontinuity.

$$2\sum_{s\in \mathrm{Int}\, I_1} \frac{1}{1+\exp l(s)} + 2\sum_{s\in \mathrm{Int}\, I_2} \frac{1}{1+\exp l(s)} + \sum_{s\in\partial I_1\cup\partial I_2} \frac{1}{1+\exp l(s)} = -1$$

Can we generalise this to n-bridge link complements?

3. Points of discrete representations in the character variety of $\pi_1(S)$

If we put a basepoint in the "pasting margin" of two biggest compression bodies, M converges to a hyperbolic manifold homeomorphic to $S\times\mathbf{R}$ as the Hempel distance goes to ∞.

Can we realise this convergence as a continuous deformation of cone manifolds?

This can be done in the case of two bridge link complement.
(Akiyoshi-Sakuma-Wada-Yamashita + α)

This would lead to some understanding of the characteristic variety outside the closure of the quasi-Fuchsian space.