Varieties of representations in $SL_{N+1}(\mathbb{C})$

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Aspects of representation theory in low-dimensional topology and 3-dimensional invariants

Carry-le-Rouet. November 9, 2012

- M^3 hyperbolic and orientable

  $hol : \pi_1(M^3) \to PSL_2(\mathbf{C})$

  $\mathrm{Sym}^N : SL_2(\mathbf{C}) \to SL_{N+1}(\mathbf{C})$ irreducible rep.

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Questions:

- Find a “natural domain” D where $\rho_N(\pi_1 M)$ acts properly
- Determine a neigborhood of ρ_N in $\hom(\pi_1 M, SL_{N+1}(\mathbf{C}))$ and $X(M, SL_{N+1}(\mathbf{C}))$.
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Joint with M. Heusener and P. Menal-Ferrer

Motivation

$hol : \pi_1(M^3) \to PSL_2(\mathbf{C})$ lifts to $\widetilde{hol} : \pi_1(M^3) \to SL_2(\mathbf{C})$.

- $\mathrm{Sym}^N : SL_2(\mathbf{C}) \to SL_{N+1}(\mathbf{C})$ irreducible
  $\mathbf{C}^{N+1} \cong \{p(x,y) \in \mathbf{C}[x,y]$ homogeneous & $\deg(p(x,y)) = N\}$
  If $\mathbf{C}^2 = \langle v_1, v_2 \rangle$, then $\mathrm{Sym}^N(\mathbf{C}^2) = \langle v_1^N, v_1^{N-1}v_2, \cdots, v_2^N \rangle$.
- $\rho_N := \mathrm{Sym}^N \circ \widetilde{hol} : \pi_1(M) \to SL_{N+1}(\mathbf{C})$

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- Extended to cusped manifolds (Menal-Ferrer- P.)
- Generalized to Seifert fibered spaces (Yamaguchi)
- Volume for $SL_N(\mathbf{C})$ reps (Garoufalidis-Thurston-Zickert)
- Analogue of Culler-Shalen theory for $SL_N(\mathbf{C})$ (Hara-Kitayama)
- Surface groups and $SL_N(\mathbf{R})$ (Labourie, Guichard-Wienhard,...)

Domains with proper action

- $\rho_N = \mathrm{Sym}^N \circ \widetilde{hol} : \pi_1 M \to SL_{N+1}(\mathbf{C}) \curvearrowright \mathbf{P}^N$

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- Veronese embedding

$$\begin{array}{rcl} \phi : \mathbf{P}^1 & \to & \mathbf{P}^N = \mathbf{P}(\{p \in \mathbf{C}[X,Y] \text{ homogeneous \& } \deg p = N\}) \\ [a:b] & \mapsto & [(a\,X + b\,Y)^N] \end{array}$$

Theorem (classical):

For $N > 2$, $\rho_N(\pi_1 M)$ acts properly on

$$D_N = \mathbf{P}^N \setminus Osc_{[(N+1)/2]}\phi(\mathbf{P}^1)$$

- Osc_k= k-th osculating manifold ($k=1$ is the tangent bundle, $k=2$ planes approaching second order, etc)

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- When M is compact, this action is cocompact for N odd and the quotient has a natural (but singular) compactification for N even.

Sketch of proof

$$\mathbf{P}^N \quad \longleftrightarrow \quad (\mathbf{P}^1)^N/\Sigma_N$$ ($SL_2(\mathbf{C})$-equivariant)

$$[p(X,Y)] \quad \longleftrightarrow \quad$$ roots of p ($p(Z,1)=0$, for $Z=X/Y \in \mathbf{P}^1$)

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$$\phi(\mathbf{P}^1) \quad\longleftrightarrow\quad \text{Diagonal } \Delta_N$$

$$Osc_k\phi(\mathbf{P}^1) \quad\longleftrightarrow\quad \Delta_{N-k} = \{ \text{ points with multiplicity} \geq N-k\}$$

$$\begin{aligned}\phi(X+\varepsilon Y) = (X+\varepsilon Y)^N &= X^N + O(\varepsilon)\\ &= X^{N-1}(X+\varepsilon NY) + O(\varepsilon^2)\\ &= X^{N-2}(X^2+\varepsilon NXY+\varepsilon^2\tfrac{N(N-1)}{2}Y^2)+O(\varepsilon^3)\end{aligned}$$

Claim: the action is proper on $(\mathbf{P}^1)^N \setminus \Delta_{[N/2]}$

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$$\begin{aligned} \mathbf{P}^N &\longleftrightarrow (\mathbf{P}^1)^N/\Sigma_N \text{ (}SL_2(\mathbf{C})\text{-equivariant)} \\ [p(X,Y)] &\longleftrightarrow \text{roots of } p \ \ (p(Z,1)=0, \text{ for } Z = X/Y \in \mathbf{P}^1) \\ \phi(\mathbf{P}^1) &\longleftrightarrow \text{Diagonal } \Delta_N \\ Osc_k\phi(\mathbf{P}^1) &\longleftrightarrow \Delta_{N-k} = \{ \text{ points with multiplicity} \geq N-k\} \end{aligned}$$

Claim: the action is proper on $(\mathbf{P}^1)^N \setminus \Delta_{[N/2]}$

- Either use Mumford's GIT

  or use the barycenter or center of mass (CM) in $\mathbf{H}^3$

  (CM exists on $(\mathbf{P}^1)^N \setminus \Delta_{[N/2]}$) and prove that

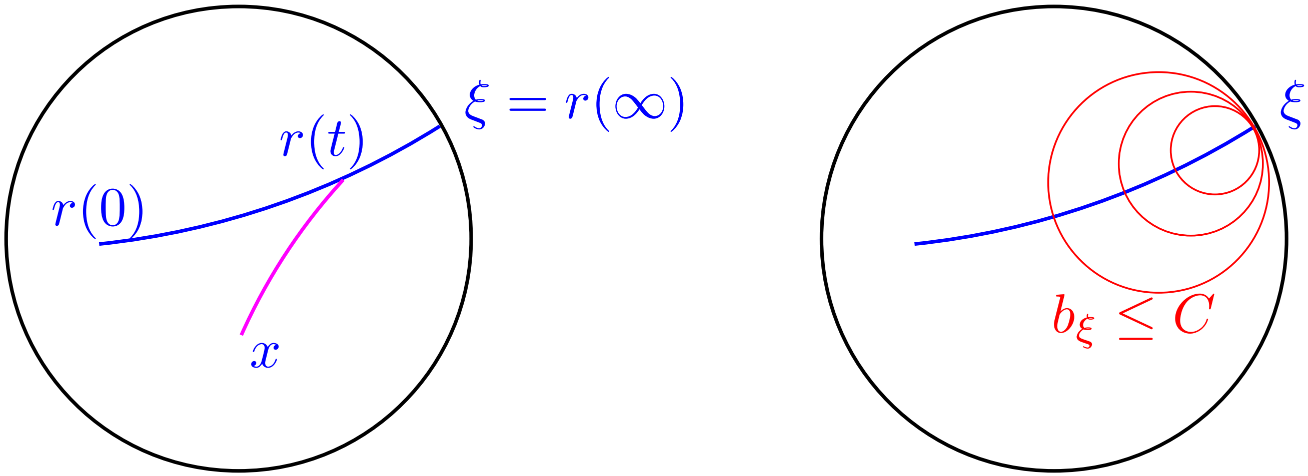
$$\mathbf{P}^N \setminus Osc_{[(N+1)/2]} \overset{roots}{\longleftrightarrow} (\mathbf{P}^1)^N/\Sigma_N \setminus \Delta_{[N/2]} \overset{CM}{\longrightarrow} \mathbf{H}^3$$

has compact fibre

For $\xi \in \partial_\infty \mathbf{H}^3 \cong \mathbf{P}^1$, if $r : [0, +\infty) \to \mathbf{H}^3$ ray ending at ξ:

$$b_\xi(x) = \lim_{t \to +\infty} d(x, r(t)) - t,$$

- b_ξ is well defined up to additive constant (choice of $r(0)$) and convex. $\{x \in \mathbf{H}^3 \mid b_\xi(x) \leq C\}$ = horosphere centered at ξ



Example: Halfspace model $\mathbf{H}^3 = \{(x, y, z) \in \mathbf{R}^3 \mid z > 0\}$

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}, \qquad \partial_\infty \mathbf{H}^3 = \mathbf{C} \cup \{\infty\}$$

$$b_\infty(x, y, z) = -\log(z)$$

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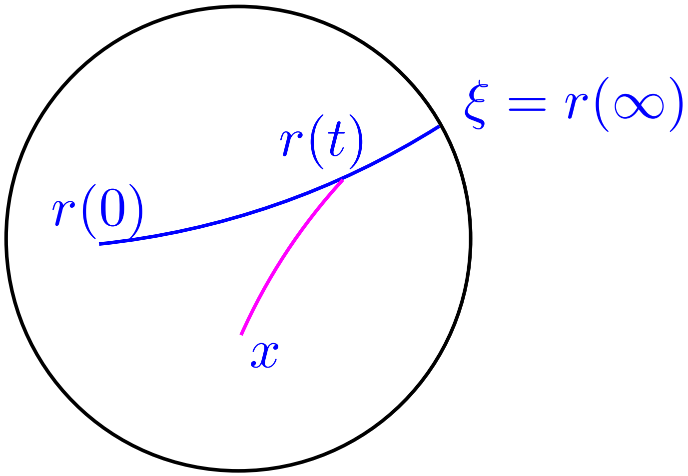

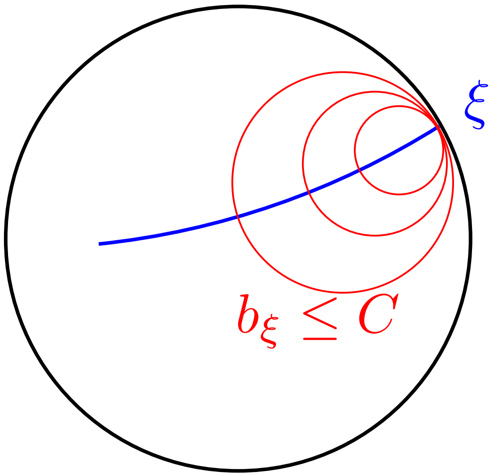

- $\sum b_{\xi_i}$ proper and strictly convex if $(\xi_1, \ldots, \xi_N) \in (\mathbf{P}^1)^N / \Sigma_N \setminus \Delta_{N/2}$.

- Define $CM(\xi_1, \ldots, \xi_N) \in \mathbf{H}^3$ to be *the minimizer of* $\sum b_{\xi_i}$

Busemann functions

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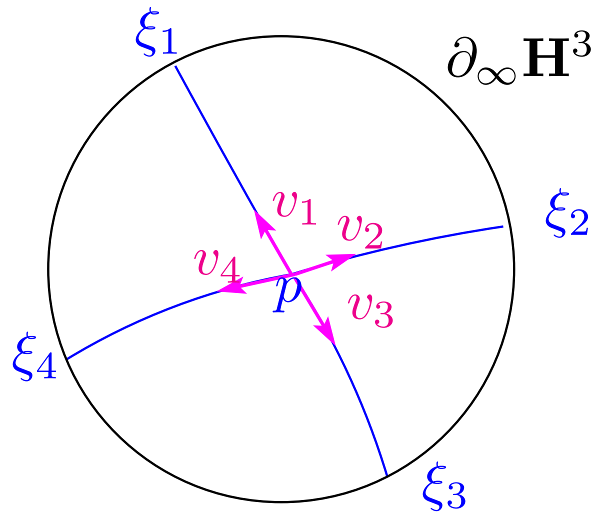

- $CM : (\mathbf{P}^1)^N/\Sigma_N \setminus \Delta_{N/2} \to \mathbf{H}^3$, $CM^{-1}(p) \cong \begin{cases} S_0 & \text{if } N \text{ odd} \\ S_0 \setminus \Delta_{N/2} & \text{if } N \text{ even} \end{cases}$

  where $S_0 = \{(v_1, \ldots, v_N) \in (T_p\mathbf{H}^3)^N \mid |v_i| = 1, \sum v_i = 0\}/\Sigma_N$,
  each unit vector $v_i \in T_p\mathbf{H}^3$ points to $\xi_i \in \partial_\infty \mathbf{H}^3 \cong \mathbf{P}^1$
  Properness follows because $SO(3)$ acts properly on S_0.

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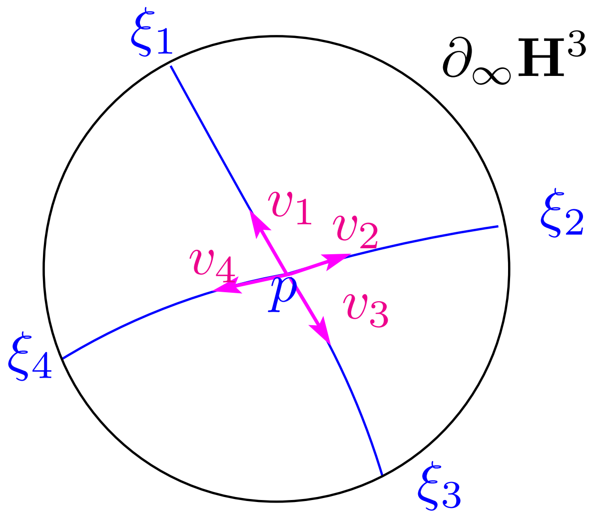

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- Fibration: $CM^{-1}(p) \to D/\rho_N(\pi_1 M) \to M$

Domain when $N = 2$

- $\rho_2 = \mathrm{Sym}^2 \circ \widetilde{hol} : \pi_1 M \to SL_3(\mathbf{C})$

$SL_2(\mathbf{C})$ cannot be properly on any $D \subset \mathbf{P}^2 \cong (\mathbf{P}^1)^2/\Sigma_2$.

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Theorem: $\rho_2(\pi_1 M)$ acts properly on

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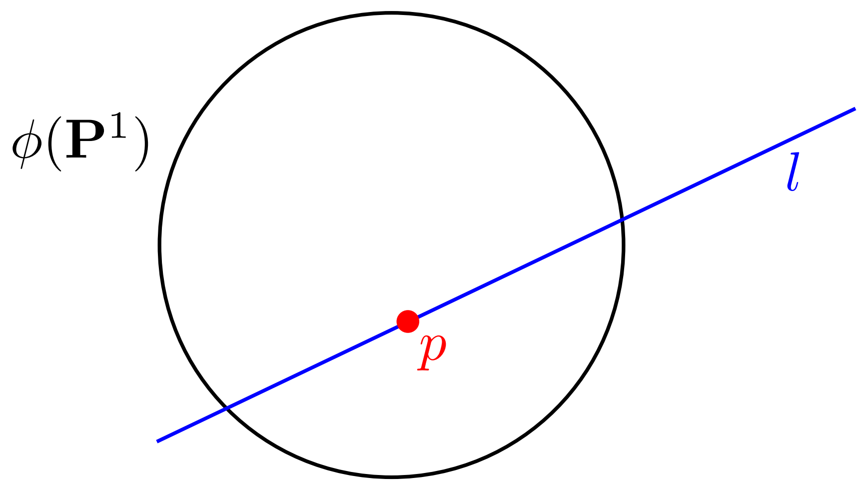

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- $\mathrm{Sym}^2 \cong Ad \curvearrowright \mathfrak{sl}_2(\mathbf{C})$

  $\phi(\mathbf{P}^1) \leftrightarrow \{\mathbf{C}$–Killing form $= 0\} \leftrightarrow \{$infinitesimal parabolic isometries$\}$

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  map $a \in \mathfrak{sl}_2(\mathbf{C})) \setminus \phi(\mathbf{P}^1)$ to the geodesic invariant by $\exp(ta)$

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- $D \subset \mathcal{F} \subset \mathbf{P}^2 \times (\mathbf{P}^2)^* \cong \mathbf{P}^2 \times \mathbf{P}^2$ corresponds to ordered pairs of unoriented geodesics of $\mathbf{H}^3$ that meet perpendicularly.

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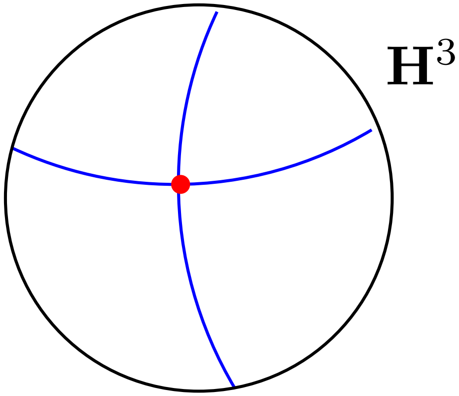
{ordered pairs of unoriented geodesics in $\mathbf{H}^3$ that meet perpendicularly}

$\downarrow$ intersection point of the geodesics

$\mathbf{H}^3$



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$$\Big\downarrow \; \text{intersection point of the geodesics}$$

$$\mathbf{H}^3$$

Fibration:

$$\begin{array}{ccccc} SO(3)/\begin{pmatrix} \pm 1 & & \\ & \pm 1 & \\ & & \pm 1 \end{pmatrix} & \longrightarrow & D/\rho_2(\pi_1 M) & \longrightarrow & M \\ \| & & & & \\ S^3/Q_8 & & & & \end{array}$$

Deformations of ρ_N

- M^3 hyperbolic, orientable and of finite type.
  $\overline{M^3}$ compact, $\partial\overline{M^3} \cong T^2_1 \sqcup \ldots \sqcup T^2_k \sqcup F^2_{g_1} \sqcup \cdots \sqcup F^2_{g_r}$.

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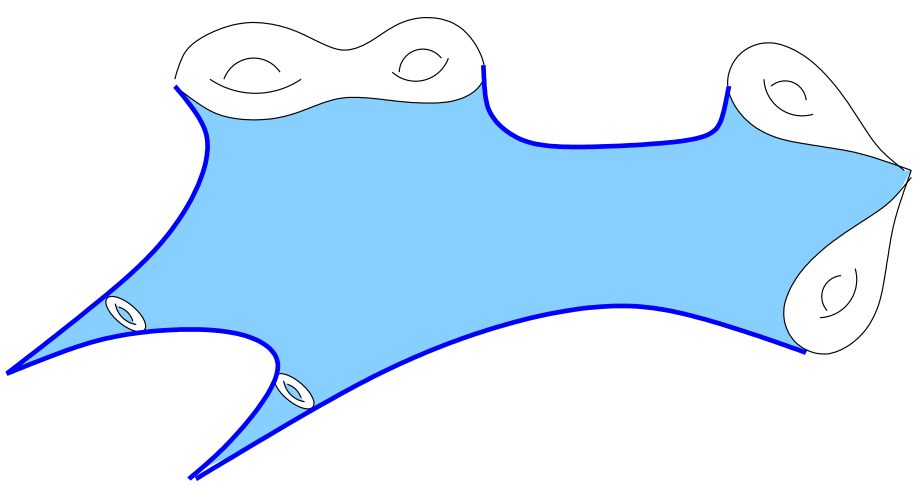
Thm (Menal-Ferrer, P.): $[\rho_N]$ is a smooth point of $X(M, SL_{N+1}(\mathbf{C}))$ of local dimension $kN + \sum(g_j - 1)((N+1)^2 - 1)$



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- Idea of the proof: $H^1(\pi_1 M, \mathfrak{sl}_{N+1}(\mathbf{C})_{Ad\rho_N}) \cong T^{Zar} X(M, SL_{N+1}(\mathbf{C}))$

Show that $H^1(\pi_1 M, \mathfrak{sl}_{N+1}(\mathbf{C})_{Ad\rho_N})$ is the actual tangent space
(ie all infinitesimal deformations can be integrated)
and compute its dimension.

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  - Claim 1: $H^1(M; \mathfrak{sl}_{N+1})$ injects into $H^1(\partial \overline{M}; \mathfrak{sl}_{N+1})$

    Flat bundle: $\mathfrak{sl}_{N+1}(\mathbf{C}) \to E \to M$

    $\Omega^p(M;E) = p$-forms valued on E

    It has de Rham cohomology $\cong H^*(M; \mathfrak{sl}_{N+1})$

  - Claim $\frac{1}{2}$: L^2 closed forms in $\Omega^1(M;E)$ are exact

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  - <u>Claim $\frac{1}{2}$</u>: L^2 closed forms in $\Omega^1(M; E)$ are exact (in fact, a Thm)

Claim $\frac{1}{2}$ $\Rightarrow$ Claim 1 because $\ker(H^1(M; \mathfrak{sl}_{N+1}) \to H^1(\mathcal{N}(\partial\overline{M}); \mathfrak{sl}_{N+1}))$
is represented by closed forms with compact support

Flat bundle: $\mathfrak{sl}_{N+1}(\mathbf{C}) \to E \to M$

- Claim $\frac{1}{2}$: L^2 closed forms in $\Omega^p(M; E)$ are exact ($p = 1, 2$).

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  $\langle \Delta\alpha, \alpha \rangle \geq C\langle \alpha, \alpha \rangle$, $\forall \alpha \in \Omega^p(M; E)$ with comp. support $p = 1, 2$

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- It implies Claim 1: $H^1(M; \mathfrak{sl}_{N+1})$ injects into $H^1(\partial\overline{M}; \mathfrak{sl}_{N+1})$

- Conclusion: There is an infinite sequence of obstructions in H^2 to integrate an infinitesimal vector in H^1.

– The obstructions vanish by naturality and because:

  $H^2(M; \mathfrak{sl}_{N+1})$ injects into $H^2(\partial\overline{M}; \mathfrak{sl}_{N+1})$

  $X(\pi_1\partial_i\overline{M}, SL_{N+1})$ is smooth at the restriction of ρ_N

– Artin's theorem: formal integrability implies actual integrability

– T^{Zar} = actual tangent space $\Rightarrow$ smooth point

- M^3 hyperbolic, orientable and of finite type.
  $\overline{M^3}$ compact, $\partial \overline{M^3} \cong T^2_1 \sqcup \ldots \sqcup T^2_k \sqcup F^2_{g_1} \sqcup \cdots \sqcup F^2_{g_r}$.

$$\rho_N = \mathrm{Sym}^N \circ \widetilde{hol} : \pi_1 M \to SL_2(\mathbf{C}) \to SL_{N+1}(\mathbf{C})$$

- $\mathrm{Sym}^N(SL_2(\mathbf{C})) \subset G$, where $G = \begin{cases} SO(N+1, \mathbf{C}) \text{ for } N+1 \text{ odd} \\ Sp(N+1, \mathbf{C}) \text{ for } N+1 \text{ even} \end{cases}$

Theorem(Menal-Ferrer, P): $[\rho_N]$ is a <u>smooth</u> point of $X(M, G)$ of local dimension $k\, \mathrm{rank}(G) + \sum (g_j - 1) \dim(G)$

- Assume that M has finite volume and one single cusp.
- Symmetric functions σ_i. For $A \in SL_{N+1}$:

$$A^{N+1} - \sigma_1(A)A^N + \sigma_2(A)A^{N-1} - \cdots + (-1)^N \sigma_N(A)A + (-1)^{N+1} Id$$

Local coordinates

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Theorem (Menal-Ferrer, P): For any peripheral $\gamma \in \pi_1(\partial \overline{M}) \setminus \{1\}$
$(\sigma_1^\gamma, \ldots \sigma_N^\gamma) : X(M, SL_{N+1}\mathbf{C}) \to \mathbf{C}^N$ is a local diffeo around $[\rho_N]$

- For $N = 1$, the trace of a peripheral curve is a local diffeomorphism for $X(M, SL_2(\mathbf{C}))$ (Bromberg, M. Kapovich).
- Sketch proof: view infinitesimal deformations as differential forms in

$$\Omega^1(T^2 \times [0, \infty), \mathbf{E})$$

which ones are L^2 and which ones are not.

Global coordinates: $SL_2(\mathbf{C})$

Theorem (Tietze): $F_2 = \langle x_1, x_2 \mid \rangle$ free group or rank 2. The map

$$\begin{aligned} X(F^2, SL_2(\mathbf{C})) &\rightarrow \mathbf{C}^3 \\ \chi &\mapsto (\chi(x_1), \chi(x_2), \chi(x_1 x_2)) \end{aligned}$$

is an isomorphism

- Then use $\mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}) = \mathrm{tr}(A)\,\mathrm{tr}(B)$ to compute

$$X(\Gamma, SL_2(\mathbf{C}))$$

where $\Gamma = \langle a, b \mid r_i(a, b) = 1 \rangle$ (eg $\Gamma = \pi_1(S^3 \setminus$ 2-bridge link$)$)

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- Question: does a similar method work for $SL_3(\mathbf{C})$?

Global coordinates: $X(F_2, SL_3(\mathbf{C}))$

- S. Lawton: $F_2 = \langle \mathbf{x}_1, \mathbf{x}_2 \mid \rangle$ free group of rank 2, $SL_3(\mathbf{C})$:

$X(F_2, SL_3(\mathbf{C})) \cong \mathbf{C}[t_{(1)}, \ldots, t_{(-5)}]/(t_{(5)} + t_{(-5)} = P, t_{(5)}\, t_{(-5)} = Q)$

where:

$$
\begin{array}{ll}
t_{(1)} \mapsto \operatorname{tr}(\mathbf{x}_1) & t_{(-1)} \mapsto \operatorname{tr}(\mathbf{x}_1^{-1}) \\
t_{(2)} \mapsto \operatorname{tr}(\mathbf{x}_2) & t_{(-2)} \mapsto \operatorname{tr}(\mathbf{x}_2^{-1}) \\
t_{(3)} \mapsto \operatorname{tr}(\mathbf{x}_1\mathbf{x}_2) & t_{(-3)} \mapsto \operatorname{tr}(\mathbf{x}_1^{-1}\mathbf{x}_2^{-1}) \\
t_{(4)} \mapsto \operatorname{tr}(\mathbf{x}_1\mathbf{x}_2^{-1}) & t_{(-4)} \mapsto \operatorname{tr}(\mathbf{x}_1^{-1}\mathbf{x}_2) \\
t_{(5)} \mapsto \operatorname{tr}(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_1^{-1}\mathbf{x}_2^{-1}) & t_{(-5)} \mapsto \operatorname{tr}(\mathbf{x}_2\mathbf{x}_1\mathbf{x}_2^{-1}\mathbf{x}_1^{-1}).
\end{array}
$$

$$
\begin{aligned}
P = & t_{(1)}t_{(-1)}t_{(2)}t_{(-2)} - t_{(1)}t_{(2)}t_{(-3)} - t_{(-1)}t_{(-2)}t_{(3)} - t_{(1)}t_{(-2)}t_{(-4)} \\
& - t_{(-1)}t_{(2)}t_{(4)} + t_{(1)}t_{(-1)} + t_{(2)}t_{(-2)} + t_{(3)}t_{(-3)} + t_{(4)}t_{(-4)} - 3,
\end{aligned}
$$

- $\dim(X(F_2, SL_3(\mathbf{C}))) = 8$, $t_{(5)}$ and $t_{(-5)}$ solutions of $z^2 - Pz + Q = 0$

Global coordinates: $X(F_2, SL_3(\mathbf{C}))$

$$
\begin{aligned}
Q = & 9 - 6t_{(1)}t_{(-1)} - 6t_{(2)}t_{(-2)} - 6t_{(3)}t_{(-3)} - 6t_{(4)}t_{(-4)} + t_{(1)}^3 + t_{(2)}^3 + t_{(3)}^3 \\
& + t_{(4)}^3 + t_{(-1)}^3 + t_{(-2)}^3 + t_{(-3)}^3 + t_{(-4)}^3 - 3t_{(-4)}t_{(-3)}t_{(-1)} - 3t_{(4)}t_{(3)}t_{(1)} \\
& - 3t_{(-4)}t_{(2)}t_{(3)} - 3t_{(4)}t_{(-2)}t_{(-3)} + 3t_{(-4)}t_{(-2)}t_{(1)} + 3t_{(4)}t_{(2)}t_{(-1)} \\
& + 3t_{(1)}t_{(2)}t_{(-3)} + 3t_{(-1)}t_{(-2)}t_{(3)} + t_{(-2)}t_{(-1)}t_{(2)}t_{(1)} + t_{(-3)}t_{(-2)}t_{(3)}t_{(2)} \\
& + t_{(-4)}t_{(-1)}t_{(4)}t_{(1)} + t_{(-4)}t_{(-2)}t_{(4)}t_{(2)} + t_{(-3)}t_{(-1)}t_{(3)}t_{(1)} + t_{(-3)}t_{(-4)}t_{(3)}t_{(4)} \\
& + t_{(-4)}^2t_{(-3)}t_{(-2)} + t_{(4)}^2t_{(3)}t_{(2)} + t_{(-1)}^2t_{(-2)}t_{(-4)} + t_{(1)}^2t_{(2)}t_{(4)} + t_{(1)}t_{(-2)}^2t_{(-3)} \\
& + t_{(-1)}t_{(2)}^2t_{(3)} + t_{(-4)}t_{(-3)}t_{(1)}^2 + t_{(4)}t_{(3)}t_{(-1)}^2 + t_{(-4)}t_{(2)}t_{(-3)}^2 + t_{(4)}t_{(-2)}t_{(3)}^2 \\
& + t_{(-1)}^2t_{(-3)}t_{(2)} + t_{(1)}^2t_{(3)}t_{(-2)} + t_{(-4)}t_{(1)}t_{(2)}^2 + t_{(4)}t_{(-1)}t_{(-2)}^2 + t_{(-4)}t_{(3)}t_{(-2)}^2 \\
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& + t_{(1)}t_{(4)}t_{(-3)}^2 - 2t_{(-3)}^2t_{(-2)}t_{(-1)} - 2t_{(3)}^2t_{(2)}t_{(1)} - 2t_{(-4)}^2t_{(-1)}t_{(2)} \\
& - 2t_{(4)}^2t_{(1)}t_{(-2)} + t_{(-1)}^2t_{(-2)}^2t_{(-3)} + t_{(1)}^2t_{(2)}^2t_{(3)} + t_{(-4)}t_{(-1)}^2t_{(2)}^2 \\
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& - t_{(-4)}t_{(-2)}t_{(-1)}t_{(1)}^2 - t_{(4)}t_{(2)}t_{(1)}t_{(-1)}^2 - t_{(-1)}t_{(-2)}^3t_{(1)} - t_{(-1)}t_{(2)}^3t_{(1)} \\
& - t_{(-1)}^3t_{(-2)}t_{(2)} - t_{(1)}^3t_{(-2)}t_{(2)} - t_{(-4)}t_{(-3)}t_{(-2)}t_{(-1)}t_{(2)} - t_{(4)}t_{(3)}t_{(2)}t_{(1)}t_{(-2)} \\
& - t_{(-1)}t_{(1)}t_{(2)}t_{(-4)}t_{(3)} - t_{(-1)}t_{(1)}t_{(-2)}t_{(4)}t_{(-3)} + t_{(-2)}t_{(-1)}^2t_{(1)}^2t_{(2)} + t_{(-1)}t_{(-2)}^2t_{(2)}^2t_{(1)}.
\end{aligned}
$$

Global coordinates: $X(F_2, SL_3(\mathbf{C}))$

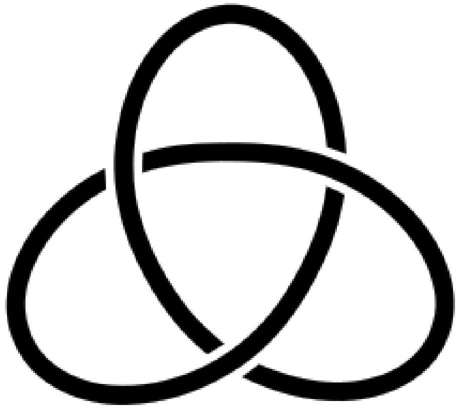
... it is nontrivial to compute examples!

Trefoil and $SL_3(\mathbf{C})$

- $\Gamma = \pi_1(S^3 - \text{Trefoil}) \cong \langle a, b \mid a^2 = b^3 \rangle$
- $\rho : \Gamma \to SL_3(\mathbf{C})$ is irreducible

  if $\rho(\Gamma)$ has no proper invariant subspace of $\mathbf{C}^3$

- Want to compute $X(\Gamma, SL_3(\mathbf{C}))^{irr}$.



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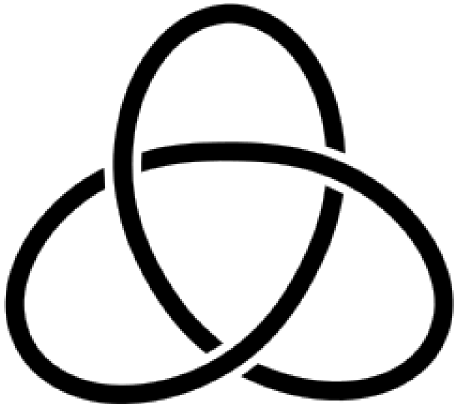
  Hence $\rho(a^2) = \rho(b^3) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$, $\begin{pmatrix} \omega & & \\ & \omega & \\ & & \omega \end{pmatrix}$ or $\begin{pmatrix} \omega^2 & & \\ & \omega^2 & \\ & & \omega^2 \end{pmatrix}$, with $\omega^3 = 1$
- In fact $\rho(a^2) = \rho(b^3) = \mathrm{Id}$, otherwise ρ is reducible (computation).



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- Look at the action on $\mathbf{P}^2$:

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  $SL_3(\mathbf{C})$ acts simply transitively on quadruples of generic points

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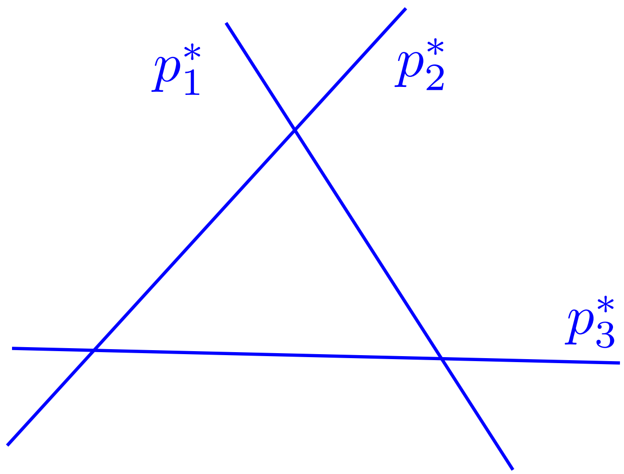

p_1^*, p_2^* and p_3^* are lines of reducible representations, intersecting at totally reducible reps and permuted by the center of $SL_3(\mathbf{C})$

Question: compute $X(\pi_1(M^3), SL_{N+1}(\mathbf{C}))$ for $N+1>2$

and for you preferred manifold M^3

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Thanks a lot for your attention