Quasiconvexity in Relatively Hyperbolic Groups

Leonid Potyagailo (University of Lille 1)

November 5, 2012

1. Introduction

This is joint work with **Victor Gerasimov** (Belo Horisonté, Brazil)

Let G be a discrete group acting by homeomorphisms on a compact Hausdorff space (compactum) X.

We say that the action $G \curvearrowright X$ is **convergence** (or **3-discontinuous**) *if the induced action on the space of distinct triples* $\Theta^3 X = \left\{ \{x_1, x_2, x_3\} \mid x_i \in X \right\}$ *is discontinuous.*

This is joint work with **Victor Gerasimov** (Belo Horisonté, Brazil)

Let G be a discrete group acting by homeomorphisms on a compact Hausdorff space (compactum) X.

We say that the action $G \curvearrowright X$ is **convergence** (or **3-discontinuous**) *if the induced action on the space of distinct triples* $\Theta^3 X = \Big\{ \{x_1, x_2, x_3\} \mid x_i \in X \Big\}$ *is discontinuous.*

1. Introduction

This is joint work with **Victor Gerasimov** (Belo Horisonté, Brazil)

Let G be a discrete group acting by homeomorphisms on a compact Hausdorff space (compactum) X.

We say that the action $G \curvearrowright X$ is **convergence** (or **3-discontinuous**) *if the induced action on the space of distinct triples* $\Theta^3 X = \Big\{\{x_1, x_2, x_3\} \mid x_i \in X\Big\}$ *is discontinuous.*

The limit set $T = \mathbf{\Lambda} G$ is the set of accumulation points of a G-orbit.
$\mathbf{\Lambda}(G) = \emptyset$ if G is finite, $\mathbf{\Lambda}(G)$ is one or two points if the action is *parabolic* or *loxodromic*; or $\mathbf{\Lambda}(G)$ is uncountable if the action is *non-elementary*.
The set of all parabolic points we denote by Par.

The set $\Omega G = X \setminus T$ is the set of discontinuity which we denote by A.
We always suppose that G is infinite, so up to adding a discrete G-orbit we always have that $A \neq \emptyset$ and $|X| > 2$.

Furthermore A will be always a discrete and G-finite set (i.e. $|A/G| < \infty$) which we identify with with the vertex set Γ^0 of a G-graph Γ.

We also assume that $|\Gamma^1/G| < \infty$, i.e. Γ is *cofinite* and connected graph where $\Gamma^1 \subset \Theta^2 A = \{\{a_1, a_2\} \mid a_i \in A\}$.

The limit set $T = \mathbf{\Lambda} G$ is the set of accumulation points of a G-orbit.
$\mathbf{\Lambda}(G) = \emptyset$ if G is finite, $\mathbf{\Lambda}(G)$ is one or two points if the action is *parabolic* or *loxodromic*; or $\mathbf{\Lambda}(G)$ is uncountable if the action is *non-elementary*.
The set of all parabolic points we denote by Par.

The set $\Omega G = X \setminus T$ is the set of discontinuity which we denote by A.
We always suppose that G is infinite, so up to adding a discrete G-orbit we always have that $A \neq \emptyset$ and $|X| > 2$.

Furthermore A will be always a discrete and G-finite set (i.e. $|A/G| < \infty$) which we identify with with the vertex set Γ^0 of a G-graph Γ.

We also assume that $|\Gamma^1/G| < \infty$, i.e. Γ is *cofinite* and connected graph where $\Gamma^1 \subset \Theta^2 A = \{\{a_1, a_2\} \mid a_i \in A\}$.

The limit set $T = \mathbf{\Lambda}G$ is the set of accumulation points of a G-orbit.
$\mathbf{\Lambda}(G) = \emptyset$ if G is finite, $\mathbf{\Lambda}(G)$ is one or two points if the action is *parabolic* or *loxodromic*; or $\mathbf{\Lambda}(G)$ is uncountable if the action is *non-elementary*.
The set of all parabolic points we denote by Par.

The set $\mathbf{\Omega}G = X \setminus T$ is the set of discontinuity which we denote by A.
We always suppose that G is infinite, so up to adding a discrete G-orbit we always have that $A \neq \emptyset$ and $|X| > 2$.

Furthermore A will be always a discrete and G-finite set (i.e. $|A/G| < \infty$) which we identify with with the vertex set Γ^0 of a G-graph Γ.

We also assume that $|\Gamma^1/G| < \infty$, i.e. Γ is *cofinite* and connected graph where $\Gamma^1 \subset \Theta^2 A = \{\{a_1, a_2\} \mid a_i \in A\}$.

The limit set $T = \mathbf{\Lambda} G$ is the set of accumulation points of a G-orbit.

$\mathbf{\Lambda}(G) = \emptyset$ if G is finite, $\mathbf{\Lambda}(G)$ is one or two points if the action is *parabolic* or *loxodromic*; or $\mathbf{\Lambda}(G)$ is uncountable if the action is *non-elementary*.

The set of all parabolic points we denote by Par.

The set $\mathbf{\Omega} G = X \setminus T$ is the set of discontinuity which we denote by A.

We always suppose that G is infinite, so up to adding a discrete G-orbit we always have that $A \neq \emptyset$ and $|X| > 2$.

Furthermore A will be always a discrete and G-finite set (i.e. $|A/G| < \infty$) which we identify with with the vertex set Γ^0 of a G-graph Γ.

We also assume that $|\Gamma^1/G| < \infty$, i.e. Γ is *cofinite* and connected graph where $\Gamma^1 \subset \Theta^2 A = \{\{a_1, a_2\} \mid a_i \in A\}$.

The limit set $T = \mathbf{\Lambda}G$ is the set of accumulation points of a G-orbit.
$\mathbf{\Lambda}(G) = \emptyset$ if G is finite, $\mathbf{\Lambda}(G)$ is one or two points if the action is *parabolic* or *loxodromic*; or $\mathbf{\Lambda}(G)$ is uncountable if the action is *non-elementary*.
The set of all parabolic points we denote by Par.

The set $\mathbf{\Omega}G = X \setminus T$ is the set of discontinuity which we denote by A.
We always suppose that G is infinite, so up to adding a discrete G-orbit we always have that $A \neq \emptyset$ and $|X| > 2$.

Furthermore A will be always a discrete and G-finite set (i.e. $|A/G| < \infty$) which we identify with with the vertex set Γ^0 of a G-graph Γ.

We also assume that $|\Gamma^1/G| < \infty$, i.e. Γ is *cofinite* and connected graph where $\Gamma^1 \subset \Theta^2 A = \{\{a_1, a_2\} \mid a_i \in A\}$.

The limit set $T = \mathbf{\Lambda} G$ is the set of accumulation points of a G-orbit.
$\mathbf{\Lambda}(G) = \emptyset$ if G is finite, $\mathbf{\Lambda}(G)$ is one or two points if the action is *parabolic* or *loxodromic*; or $\mathbf{\Lambda}(G)$ is uncountable if the action is *non-elementary*.
The set of all parabolic points we denote by Par.

The set $\mathbf{\Omega} G = X \setminus T$ is the set of discontinuity which we denote by A.
We always suppose that G is infinite, so up to adding a discrete G-orbit we always have that $A \neq \emptyset$ and $|X| > 2$.

Furthermore A will be always a discrete and G-finite set (i.e. $|A/G| < \infty$) which we identify with with the vertex set Γ^0 of a G-graph Γ.

We also assume that $|\Gamma^1/G| < \infty$, i.e. Γ is *cofinite* and connected graph where $\Gamma^1 \subset \Theta^2 A = \{\{a_1, a_2\} \mid a_i \in A\}$.

If G is finitely generated then w.l.o.g. Γ is the Cayley graph, if not we consider a graph of entourages on X for which all these properties are true too ([Gerasimov-P., Geometry, Groups, Dynamics], to appear).

The action $G \curvearrowright X$ is said **2-cocompact** *if* $\Theta^2 X/G$ *is compact where* $\Theta^2 X = \{\{x_1, x_2\} \mid x_i \in X\}$ *is the space of distinct pairs.*

Definition *(RH_{32}). A group G is said* **relatively hyperbolic** *(RH or RHG) if it admits a 3-discontinuous and 2-cocompact action (RH_{32}-action) on a compactum X.*

If G is finitely generated then w.l.o.g. Γ is the Cayley graph, if not we consider a graph of entourages on X for which all these properties are true too ([Gerasimov-P., Geometry, Groups, Dynamics], to appear).

The action $G \curvearrowright X$ is said **2-cocompact** *if* $\Theta^2 X/G$ *is compact where* $\Theta^2 X = \{\{x_1, x_2\} \mid x_i \in X\}$ *is the space of distinct pairs.*

Definition *(*RH_{32}*). A group G is said* **relatively hyperbolic** *(*RH *or* RHG*) if it admits a 3-discontinuous and 2-cocompact action (*RH_{32}*-action) on a compactum X.*

If G is finitely generated then w.l.o.g. Γ is the Cayley graph, if not we consider a graph of entourages on X for which all these properties are true too ([Gerasimov-P., Geometry, Groups, Dynamics], to appear).

The action $G \curvearrowright X$ is said **2-cocompact** *if* $\Theta^2 X/G$ *is compact where* $\Theta^2 X = \{\{x_1, x_2\} \mid x_i \in X\}$ *is the space of distinct pairs.*

Definition *(*RH_{32}*). A group* G *is said* **relatively hyperbolic** *(*RH *or* RHG*) if it admits a 3-discontinuous and 2-cocompact action (*RH_{32}*-action) on a compactum* X.

Remark

Advantages of the RH_{32} Definition:

a) Works in the case of non-metrizable spaces and uncountable groups;

b) All previously known other definitions of RHG *(due to M. Gromov, B. Farb, B. Bowditch and P. Tukia) are equivalent to it if* X *is metrizable or* G *is countable.*

c) It implies B. Bowditch's, D. Osin's and P. Tukia's definitions of RHG *in any case.*

Disadvantages of the RH_{32} Definition: *Not known yet.*

Remark

Advantages of the RH_{32} Definition:

a) Works in the case of non-metrizable spaces and uncountable groups;

b) All previously known other definitions of RHG *(due to M. Gromov, B. Farb, B. Bowditch and P. Tukia) are equivalent to it if* X *is metrizable or* G *is countable.*

c) It implies B. Bowditch's, D. Osin's and P. Tukia's definitions of RHG *in any case.*

Disadvantages of the RH_{32} Definition: *Not known yet.*

Remark

Advantages of the RH_{32} Definition:

a) Works in the case of non-metrizable spaces and uncountable groups;

b) All previously known other definitions of RHG *(due to M. Gromov, B. Farb, B. Bowditch and P. Tukia) are equivalent to it if* X *is metrizable or* G *is countable.*

c) It implies B. Bowditch's, D. Osin's and P. Tukia's definitions of RHG *in any case.*

Disadvantages of the RH_{32} Definition: *Not known yet.*

Remark

Advantages of the RH_{32} Definition:
a) Works in the case of non-metrizable spaces and uncountable groups;
b) All previously known other definitions of RHG *(due to M. Gromov, B. Farb, B. Bowditch and P. Tukia) are equivalent to it if* X *is metrizable or* G *is countable.*
c) It implies B. Bowditch's, D. Osin's and P. Tukia's definitions of RHG *in any case.*

Disadvantages of the RH_{32} Definition: *Not known yet.*

A parabolic point $p \in T$ is called **bounded parabolic** if $(T \setminus p)/\mathrm{St_G p}$ is compact where $\mathrm{St_G p}$ denotes the stabilizer of p in G.

It is known (Gerasimov, GAFA 09) that if the action is $\mathrm{RH_{32}}$ then every point of T is either bounded parabolic or conical.
One says that G is relatively hyperbolic with respect to the maximal finite set $\mathcal{P}$ of the non-conjugate stabilizers of the parabolic points.
If $\mathcal{P} = \emptyset$ then G is hyperbolic and in particular finitely generated.

A parabolic point $p \in T$ is called **bounded parabolic** if $(T \setminus p)/\mathrm{St_G p}$ is compact where $\mathrm{St_G p}$ denotes the stabilizer of p in G.

It is known (Gerasimov, GAFA 09) that if the action is $\mathrm{RH_{32}}$ then every point of T is either bounded parabolic or conical.
One says that G is relatively hyperbolic with respect to the maximal finite set $\mathcal{P}$ of the non-conjugate stabilizers of the parabolic points.
If $\mathcal{P} = \emptyset$ then G is hyperbolic and in particular finitely generated.

A parabolic point $p \in T$ is called **bounded parabolic** if $(T \setminus p)/\mathrm{St_G p}$ is compact where $\mathrm{St_G p}$ denotes the stabilizer of p in G.

It is known (Gerasimov, GAFA 09) that if the action is $\mathrm{RH_{32}}$ then every point of T is either bounded parabolic or conical.
One says that G is relatively hyperbolic with respect to the maximal finite set $\mathcal{P}$ of the non-conjugate stabilizers of the parabolic points.
If $\mathcal{P} = \emptyset$ then G is hyperbolic and in particular finitely generated.

A parabolic point $p \in T$ is called **bounded parabolic** if $(T \setminus p)/\mathrm{St_G p}$ is compact where $\mathrm{St_G p}$ denotes the stabilizer of p in G.

It is known (Gerasimov, GAFA 09) that if the action is RH_{32} then every point of T is either bounded parabolic or conical.
One says that G is relatively hyperbolic with respect to the maximal finite set $\mathcal{P}$ of the non-conjugate stabilizers of the parabolic points.
If $\mathcal{P} = \emptyset$ then G is hyperbolic and in particular finitely generated.

Convention. In this section we will assume that G is a finitely generated RHG.

A. $\alpha-$**quasiconvexity.**

Let $\alpha : \mathbb{N} \to \mathbb{R}_+$ be a non-decreasing function s.t. $\alpha(n) = \alpha_n \geq n$. A curve $\gamma : I \to \Gamma$ is called *α-distorted geodesic ($\alpha-$geodesic)* if $\mathrm{diam}(J) \leq \alpha(\mathrm{diam}(\gamma(\partial J))$ for every finite $J \subset I$.

Examples. $\alpha \equiv \mathrm{id} \iff \gamma$ is a geodesic;
$\alpha(n) = C \cdot n \iff \gamma$ is Lipschitz;
$\alpha(n) = C \cdot n + D \iff \gamma$ is quasigeodesic.

Convention. In this section we will assume that G is a finitely generated RHG.

A. $\alpha-$**quasiconvexity.**

Let $\alpha : \mathbb{N} \to \mathbb{R}_+$ be a non-decreasing function s.t. $\alpha(n) = \alpha_n \geq n$. A curve $\gamma : I \to \Gamma$ is called *α-distorted geodesic ($\alpha-$geodesic)* if $\mathrm{diam}(J) \leq \alpha(\mathrm{diam}(\gamma(\partial J))$ for every finite $J \subset I$.

Examples. $\alpha \equiv \mathrm{id} \iff \gamma$ is a geodesic;
$\alpha(n) = C \cdot n \iff \gamma$ is Lipschitz;
$\alpha(n) = C \cdot n + D \iff \gamma$ is quasigeodesic.

Convention. In this section we will assume that G is a finitely generated RHG.

A. $\alpha-$**quasiconvexity.**

Let $\alpha : \mathbb{N} \to \mathbb{R}_+$ be a non-decreasing function s.t. $\alpha(n) = \alpha_n \geq n$. A curve $\gamma : I \to \Gamma$ is called *α-distorted geodesic ($\alpha-$geodesic)* if $\mathrm{diam}(J) \leq \alpha(\mathrm{diam}(\gamma(\partial J))$ for every finite $J \subset I$.

Examples. $\alpha \equiv \mathrm{id} \iff \gamma$ is a geodesic;
$\alpha(n) = C \cdot n \iff \gamma$ is Lipschitz;
$\alpha(n) = C \cdot n + D \iff \gamma$ is quasigeodesic.

Definition. For a set $F \subset \Gamma^0$ the set $H_\alpha(F) = \{\mathrm{Im}\gamma \mid \gamma : I \to \Gamma \text{ is } \alpha-\text{geodesic in } \Gamma : \partial\gamma \subset F\}$ is called α-hull.

Important partial case:
$H_\alpha(p) = \{\gamma : \mathbb{Z} \to \Gamma - \alpha - \text{geodesic } : \lim_{n\to\pm\infty} \gamma(n) = p \in T\}$ is α-horosphere at p and γ is an α-horocycle.

Property. *Horospheres and horocycles can only exist at parabolic points.*

Pf Follows from the facts that the boundary of the (α)-convex hull of a set coincides with the boundary of the set if the action has convergence property. QED.

NB Do not compare this situation with $\mathbb{H}^n$ as in our case the horospheres are defined on a discrete set.

Definition. For a set $F \subset \Gamma^0$ the set
$H_\alpha(F) = \{\mathrm{Im}\gamma \mid \gamma : I \to \Gamma \text{ is } \alpha-\text{geodesic in } \Gamma : \partial\gamma \subset F\}$ is called α-hull.
Important partial case:
$H_\alpha(p) = \{\gamma : \mathbb{Z} \to \Gamma - \alpha - \text{geodesic } : \lim_{n \to \pm\infty} \gamma(n) = p \in T\}$ is α-horosphere at p and γ is an α-horocycle.

Property. *Horospheres and horocycles can only exist at parabolic points.*
Pf Follows from the facts that the boundary of the (α)-convex hull of a set coincides with the boundary of the set if the action has convergence property. QED.

NB Do not compare this situation with $\mathbb{H}^n$ as in our case the horospheres are defined on a discrete set.

Definition. For a set $F \subset \Gamma^0$ the set
$H_\alpha(F) = \{\mathrm{Im}\gamma \mid \gamma : I \to \Gamma \text{ is } \alpha-\text{geodesic in } \Gamma : \partial\gamma \subset F\}$ is called α-hull.
Important partial case:
$H_\alpha(p) = \{\gamma : \mathbb{Z} \to \Gamma - \alpha - \text{geodesic} \ : \ \lim_{n\to\pm\infty} \gamma(n) = p \in T\}$ is α-horosphere at p and γ is an α-horocycle.

Property. *Horospheres and horocycles can only exist at parabolic points.*
Pf Follows from the facts that the boundary of the (α)-convex hull of a set coincides with the boundary of the set if the action has convergence property. QED.

NB Do not compare this situation with $\mathbb{H}^n$ as in our case the horospheres are defined on a discrete set.

Definition. For a set $F \subset \Gamma^0$ the set
$H_\alpha(F) = \{\mathrm{Im}\gamma \mid \gamma : I \to \Gamma \text{ is } \alpha - \text{geodesic in } \Gamma : \partial\gamma \subset F\}$ is called α-hull.

Important partial case:
$H_\alpha(p) = \{\gamma : \mathbb{Z} \to \Gamma - \alpha - \text{geodesic } : \lim_{n \to \pm\infty} \gamma(n) = p \in T\}$ is α-horosphere at p and γ is an α-horocycle.

Property. *Horospheres and horocycles can only exist at parabolic points.*
Pf Follows from the facts that the boundary of the (α)-convex hull of a set coincides with the boundary of the set if the action has convergence property. QED.

NB Do not compare this situation with $\mathbb{H}^n$ as in our case the horospheres are defined on a discrete set.

Definition. For a set $F \subset \Gamma^0$ the set
$H_\alpha(F) = \{\mathrm{Im}\gamma \mid \gamma : I \to \Gamma \text{ is } \alpha - \text{geodesic in } \Gamma : \partial\gamma \subset F\}$ is called α-hull.
Important partial case:
$H_\alpha(p) = \{\gamma : \mathbb{Z} \to \Gamma - \alpha - \text{geodesic } : \lim_{n \to \pm\infty} \gamma(n) = p \in T\}$ is α-horosphere at p and γ is an α-horocycle.

Property. *Horospheres and horocycles can only exist at parabolic points.*
Pf Follows from the facts that the boundary of the (α)-convex hull of a set coincides with the boundary of the set if the action has convergence property. QED.

NB Do not compare this situation with $\mathbb{H}^n$ as in our case the horospheres are defined on a discrete set.

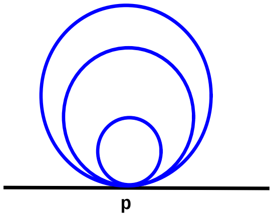


Figure: Horocycles at the parabolic point p.

Definition. *A set* $F \subset \Gamma^0$ *is* $\alpha-$*quasiconvex if* $H_\alpha(F) \subset N_r(F)$ *where* $N_r(\cdot)$ *is an r-neighborhood in* Γ.

Important partial case: F is H-invariant subset of Γ *for a subgroup H of G. Then H is* $\alpha-$*quasiconvex if F is.*

Definition. *A set* $F \subset \Gamma^0$ *is* $\alpha-$*quasiconvex if* $H_\alpha(F) \subset N_r(F)$ *where* $N_r(\cdot)$ *is an r-neighborhood in* Γ*.*

Important partial case: F *is H-invariant subset of* Γ *for a subgroup* H *of* G*. Then* H *is* $\alpha-$*quasiconvex if* F *is.*

Definition. *A set* $F \subset \Gamma^0$ *is* $\alpha-$*quasiconvex if* $H_\alpha(F) \subset N_r(F)$ *where* $N_r(\cdot)$ *is an r-neighborhood in* Γ*.*

Important partial case: F *is* H*-invariant subset of* Γ *for a subgroup* H *of* G*. Then* H *is* $\alpha-$*quasiconvex if* F *is.*

B. **Relarive quasiconvexity.**

Definition

A relative graph Δ is obtained from Γ by adding an edge between every pair of distinct points of each horosphere S_p $(p \in \mathrm{Par})$.

Since $\Gamma^0 = \Delta^0$ define for $F \subset \Gamma^0$ its relative convex hull as

$$H_{\mathrm{rel}}(F) = \Big\{ \mathrm{Im}(\delta) \mid \delta \text{ is a geodesic in } \Delta \wedge \partial\delta \subset F \Big\}.$$

Put

$$H_{\alpha,e}(F) = \Big\{ \gamma(i) \,|\, \gamma \text{ is alpha} - \text{geodesic } : \partial\gamma \subset F \wedge \mathrm{depth}_\alpha(\gamma(i)) \le e \Big\}$$

where $\mathrm{depth}_\alpha(i, \gamma)$ is the maximal $\varepsilon > 0$ such that $\gamma(]i - \varepsilon, i + \varepsilon[)$ belongs to an α-horosphere.

The "relativization" of the notion of (α)-quasiconvexity is the following:

B. **Relarive quasiconvexity.**

Definition

A relative graph Δ is obtained from Γ by adding an edge between every pair of distinct points of each horosphere S_p ($p \in \mathrm{Par}$).

Since $\Gamma^0 = \Delta^0$ define for $F \subset \Gamma^0$ its relative convex hull as

$$H_{\mathrm{rel}}(F) = \Big\{ \mathrm{Im}(\delta) \mid \delta \text{ is a geodesic in } \Delta \wedge \partial\delta \subset F \Big\}.$$

Put

$$H_{\alpha,e}(F) = \Big\{ \gamma(i) \,|\, \gamma \text{ is alpha} - \text{geodesic } : \partial\gamma \subset F \wedge \mathrm{depth}_\alpha(\gamma(i)) \leq e \Big\}$$

where $\mathrm{depth}_\alpha(i,\gamma)$ is the maximal $\varepsilon > 0$ such that $\gamma(]i-\varepsilon, i+\varepsilon[)$ belongs to an α-horosphere.
The "relativization" of the notion of (α)-quasiconvexity is the following:

B. Relarive quasiconvexity.

Definition

A relative graph Δ is obtained from Γ by adding an edge between every pair of distinct points of each horosphere S_p ($p \in \mathrm{Par}$).

Since $\Gamma^0 = \Delta^0$ define for $F \subset \Gamma^0$ its relative convex hull as

$$H_{\mathrm{rel}}(F) = \Big\{ \mathrm{Im}(\delta) \mid \delta \text{ is a geodesic in } \Delta \wedge \partial\delta \subset F \Big\}.$$

Put

$$H_{\alpha,e}(F) = \Big\{ \gamma(i) \,|\, \gamma \,\mathrm{is\ alpha-geodesic}\, : \, \partial\gamma \subset F \wedge \mathrm{depth}_\alpha(\gamma(i)) \leq e \Big\}$$

where $\mathrm{depth}_\alpha(i, \gamma)$ is the maximal $\varepsilon > 0$ such that $\gamma(]i - \varepsilon, i + \varepsilon[)$ belongs to an α-horosphere.

The "relativization" of the notion of (α)-quasiconvexity is the following:

B. **Relarive quasiconvexity.**

Definition

A relative graph Δ is obtained from Γ by adding an edge between every pair of distinct points of each horosphere S_p ($p \in \mathrm{Par}$).

Since $\Gamma^0 = \Delta^0$ define for $F \subset \Gamma^0$ its relative convex hull as

$$H_{\mathrm{rel}}(F) = \Big\{ \mathrm{Im}(\delta) \mid \delta \text{ is a geodesic in } \Delta \wedge \partial\delta \subset F \Big\}.$$

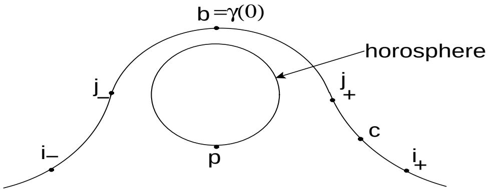
Put

$$H_{\alpha,e}(F) = \Big\{ \gamma(i) \,|\, \gamma \text{ is alpha} - \text{geodesic } : \partial\gamma \subset F \wedge \mathrm{depth}_\alpha(\gamma(i)) \leq e \Big\}$$

where $\mathrm{depth}_\alpha(i,\gamma)$ is the maximal $\varepsilon > 0$ such that $\gamma(]i-\varepsilon, i+\varepsilon[)$ belongs to an α-horosphere.
The "relativization" of the notion of (α)-quasiconvexity is the following:

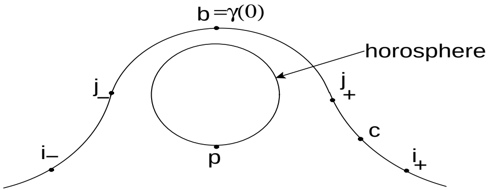
Definition

1. $F \subset \Gamma^0$ *is relatively quasiconvex if* $H_{\mathrm{rel}}(F) \subset N_r(F)$.
2. $F \subset \Gamma^0$ *is relatively* $\alpha-$*quasiconvex if* $H_{\alpha,e}(F) \subset N_r(F)$ *for some* $e > 0$.



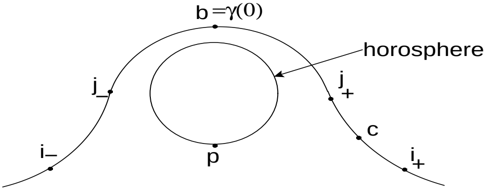
Definition

1. $F \subset \Gamma^0$ *is relatively quasiconvex if* $H_{\mathrm{rel}}(F) \subset N_r(F)$.
2. $F \subset \Gamma^0$ *is relatively* $\alpha-$*quasiconvex if* $H_{\alpha,e}(F) \subset N_r(F)$ *for some* $e > 0$.



Definition

1. $F \subset \Gamma^0$ *is relatively quasiconvex if* $H_{\mathrm{rel}}(F) \subset N_r(F)$.
2. $F \subset \Gamma^0$ *is relatively* $\alpha-$*quasiconvex if* $H_{\alpha,e}(F) \subset N_r(F)$ *for some* $e > 0$.



The following Proposition establishes the link between these two notions:

Proposition

Let $G \curvearrowright X$ be a 3-discontinuous and 2-cocompact action. Then there exists a quadratic polynomial α such that any lift γ of any geodesic δ in the relative graph is α-distorted in Γ. Moreover, $\mathrm{depth}_\alpha(v, \gamma)$ is uniformly bounded for every $v \in \mathrm{Im}\delta$.

The lift of a geodesic from the relative graph to the absolute one is obtained by replacing every "horospherical" edge in the relative graph Δ by a geodesic path in the absolute graph Γ.

The following Proposition establishes the link between these two notions:

Proposition

Let $G \curvearrowright X$ be a 3-discontinuous and 2-cocompact action. Then there exists a quadratic polynomial α such that any lift γ of any geodesic δ in the relative graph is α-distorted in Γ. Moreover, $\mathrm{depth}_\alpha(v, \gamma)$ is uniformly bounded for every $v \in \mathrm{Im}\delta$.

The lift of a geodesic from the relative graph to the absolute one is obtained by replacing every "horospherical" edge in the relative graph Δ by a geodesic path in the absolute graph Γ.

C. **Visible quasiconvexity.**

We use the following construction due to W. Floyd in the case of geometrically finite Kleinian groups (Inventionnes, 1980).

Let $f : \mathbb{N} \to \mathbb{R}_{>0}$ be a positive scalar function satisfying:

1. $\exists\ \lambda > 0\ \forall n\ : \lambda \leq f_n/f_{n+1} \leq 1$,
2. $\Sigma_n f_n < \infty$

For a fixed $v \in \Gamma^0$ define a new metric $\delta_{v,f} = \delta_v$ on Γ as the path metric obtained by the rescaling of the length of every edge $e \in \Gamma^1$ to be $f(d(v,e))$.
Let $\overline{\Gamma}_v$ be the Cauchy completion of the space (Γ, δ_v) called Floyd completion.
The set $\partial_v\Gamma = \overline{\Gamma}_v \setminus \Gamma$ is called Floyd boundary.

C.**Visible quasiconvexity.**

We use the following construction due to W. Floyd in the case of geometrically finite Kleinian groups (Inventionnes, 1980).

Let $f : \mathbb{N} \to \mathbb{R}_{>0}$ be a positive scalar function satisfying:

1. $\exists\ \lambda > 0\ \forall n\ : \lambda \leq f_n/f_{n+1} \leq 1$,
2. $\Sigma_n f_n < \infty$

For a fixed $v \in \Gamma^0$ define a new metric $\delta_{v,f} = \delta_v$ on Γ as the path metric obtained by the rescaling of the length of every edge $e \in \Gamma^1$ to be $f(d(v, e))$.

Let $\overline{\Gamma}_v$ be the Cauchy completion of the space (Γ, δ_v) called Floyd completion.

The set $\partial_v \Gamma = \overline{\Gamma}_v \setminus \Gamma$ is called Floyd boundary.

C. **Visible quasiconvexity.**

We use the following construction due to W. Floyd in the case of geometrically finite Kleinian groups (Inventionnes, 1980).

Let $f : \mathbb{N} \to \mathbb{R}_{>0}$ be a positive scalar function satisfying:

1. $\exists\ \lambda > 0\ \forall n\ : \lambda \leq f_n/f_{n+1} \leq 1,$
2. $\Sigma_n f_n < \infty$

For a fixed $v \in \Gamma^0$ define a new metric $\delta_{v,f} = \delta_v$ on Γ as the path metric obtained by the rescaling of the length of every edge $e \in \Gamma^1$ to be $f(d(v,e))$.

Let $\overline{\Gamma}_v$ be the Cauchy completion of the space (Γ, δ_v) called Floyd completion.

The set $\partial_v\Gamma = \overline{\Gamma}_v \setminus \Gamma$ is called Floyd boundary.

C.**Visible quasiconvexity.**

We use the following construction due to W. Floyd in the case of geometrically finite Kleinian groups (Inventionnes, 1980).

Let $f : \mathbb{N} \to \mathbb{R}_{>0}$ be a positive scalar function satisfying:

1. $\exists\ \lambda > 0\ \forall n\ : \lambda \le f_n / f_{n+1} \le 1$,
2. $\Sigma_n f_n < \infty$

For a fixed $v \in \Gamma^0$ define a new metric $\delta_{v,f} = \delta_v$ on Γ as the path metric obtained by the rescaling of the length of every edge $e \in \Gamma^1$ to be $f(d(v,e))$.

Let $\overline{\Gamma}_v$ be the Cauchy completion of the space (Γ, δ_v) called Floyd completion.

The set $\partial_v \Gamma = \overline{\Gamma}_v \setminus \Gamma$ is called Floyd boundary.

C. **Visible quasiconvexity.**

We use the following construction due to W. Floyd in the case of geometrically finite Kleinian groups (Inventionnes, 1980).

Let $f : \mathbb{N} \to \mathbb{R}_{>0}$ be a positive scalar function satisfying:

1. $\exists\ \lambda > 0\ \forall n\ : \lambda \leq f_n/f_{n+1} \leq 1$,
2. $\Sigma_n f_n < \infty$

For a fixed $v \in \Gamma^0$ define a new metric $\delta_{v,f} = \delta_v$ on Γ as the path metric obtained by the rescaling of the length of every edge $e \in \Gamma^1$ to be $f(d(v, e))$.

Let $\overline{\Gamma}_v$ be the Cauchy completion of the space (Γ, δ_v) called Floyd completion.

The set $\partial_v \Gamma = \overline{\Gamma}_v \setminus \Gamma$ is called Floyd boundary.

Known fact: If G is RH then there exists an equivariant map $F : \overline{\Gamma}_v \to X$ ([Gerasimov, GAFA, 2012]).

Furthermore F is injective on conical points and $F^{-1}(p) = \partial_v(\mathrm{St}_G p)$ is the Floyd boundary of the stabilizer of $p \in \mathrm{Par}$ ([Gerasimov-P, to appear in JEMS]; [Gerasimov-P, to appear in GGD]). □

Using the map F we transfer the metric δ_v to a metric on X which we call shortcut metric and denote it by $\overline{\delta}_w$.

$$\overline{\delta}_w(x, y) = \inf \delta_v(F^{-1}(x), F^{-1}(y)),\ w = F(v).$$

Known fact: If G is RH then there exists an equivariant map $F : \overline{\Gamma}_v \to X$ ([Gerasimov, GAFA, 2012]).

Furthermore F is injective on conical points and $F^{-1}(p) = \partial_v(\mathrm{St}_G p)$ is the Floyd boundary of the stabilizer of $p \in \mathrm{Par}$ ([Gerasimov-P, to appear in JEMS]; [Gerasimov-P, to appear in GGD]). $\square$

Using the map F we transfer the metric δ_v to a metric on X which we call shortcut metric and denote it by $\overline{\delta}_w$.

$$\overline{\delta}_w(x, y) = \inf \delta_v(F^{-1}(x), F^{-1}(y)),\ w = F(v).$$

Known fact: If G is RH then there exists an equivariant map $F : \overline{\Gamma}_v \to X$ ([Gerasimov, GAFA, 2012]).

Furthermore F is injective on conical points and $F^{-1}(p) = \partial_v(\mathrm{St}_G p)$ is the Floyd boundary of the stabilizer of $p \in \mathrm{Par}$ ([Gerasimov-P, to appear in JEMS]; [Gerasimov-P, to appear in GGD]). □

Using the map F we transfer the metric δ_v to a metric on X which we call shortcut metric and denote it by $\overline{\delta}_w$.

$$\overline{\delta}_w(x, y) = \inf \delta_v(F^{-1}(x), F^{-1}(y)), \; w = F(v).$$

Known fact: If G is RH then there exists an equivariant map $F : \overline{\Gamma}_v \to X$ ([Gerasimov, GAFA, 2012]).

Furthermore F is injective on conical points and $F^{-1}(p) = \partial_v(\mathrm{St}_G p)$ is the Floyd boundary of the stabilizer of $p \in \mathrm{Par}$ ([Gerasimov-P, to appear in JEMS]; [Gerasimov-P, to appear in GGD]). □

Using the map F we transfer the metric δ_v to a metric on X which we call shortcut metric and denote it by $\overline{\delta}_w$.

$$\overline{\delta}_w(x, y) = \inf \delta_v(F^{-1}(x), F^{-1}(y)),\ w = F(v).$$

For $F \subset X$ and $\varepsilon > 0$ the set

$$\mathrm{Vis}_\varepsilon(F) = \{v \in \Gamma^0 \mid \mathrm{diam}_{\overline{\delta}_w}(F) \geq \varepsilon\}$$

is visibility hull.

Definition

Quasiconvex visibility $F \subset \Gamma^0$ *is visibly quasiconvex if*

$$\forall \varepsilon > 0 \; \exists r \; \mathrm{Vis}_\varepsilon(F) \subset N_r(F).$$

The following property clarifies this notion (originally due to A. Karlsson for the case of geodesics in a Cayley graph):

Property of $\overline{\delta}_w$ (Karlsson Lemma). Suppose $\Sigma_n \alpha_{2n+1} f_n < \infty$ then

$$\forall \varepsilon > 0 \, \exists r = K(\varepsilon) \, \forall \alpha\text{-geodesic}\, \gamma \subset \Gamma \; \mathrm{diam}_{\overline{\delta}_w}(\partial\gamma) \geq \varepsilon \implies d(v, \gamma) \leq r.$$

For $F \subset X$ and $\varepsilon > 0$ the set

$$\mathrm{Vis}_\varepsilon(F) = \{v \in \Gamma^0 \mid \mathrm{diam}_{\overline{\delta}_w}(F) \geq \varepsilon\}$$

is visibility hull.

Definition

Quasiconvex visibility $F \subset \Gamma^0$ *is visibly quasiconvex if*

$$\forall \varepsilon > 0\ \exists r\ \mathrm{Vis}_\varepsilon(F) \subset N_r(F).$$

The following property clarifies this notion (originally due to A. Karlsson for the case of geodesics in a Cayley graph):

Property of $\overline{\delta}_w$ (Karlsson Lemma). Suppose $\Sigma_n \alpha_{2n+1} f_n < \infty$ then

$$\forall \varepsilon > 0\ \exists r = K(\varepsilon)\ \forall \alpha\text{−geodesic}\ \gamma \subset \Gamma\ \mathrm{diam}_{\overline{\delta}_w}(\partial\gamma) \geq \varepsilon \implies d(v, \gamma) \leq r.$$

For $F \subset X$ and $\varepsilon > 0$ the set

$$\mathrm{Vis}_\varepsilon(F) = \{v \in \Gamma^0 \mid \mathrm{diam}_{\overline{\delta}_w}(F) \geq \varepsilon\}$$

is visibility hull.

Definition

Quasiconvex visibility $F \subset \Gamma^0$ *is visibly quasiconvex if*

$$\forall \varepsilon > 0 \; \exists r \; \mathrm{Vis}_\varepsilon(F) \subset N_r(F).$$

The following property clarifies this notion (originally due to A. Karlsson for the case of geodesics in a Cayley graph):

Property of $\overline{\delta}_w$ (Karlsson Lemma). Suppose $\Sigma_n \alpha_{2n+1} f_n < \infty$ then

$$\forall \varepsilon > 0 \; \exists r = K(\varepsilon) \; \forall \alpha\text{−geodesic}\; \gamma \subset \Gamma \; \mathrm{diam}_{\overline{\delta}_w}(\partial\gamma) \geq \varepsilon \implies d(v, \gamma) \leq r.$$

For $F \subset X$ and $\varepsilon > 0$ the set

$$\mathrm{Vis}_\varepsilon(F) = \{v \in \Gamma^0 \mid \mathrm{diam}_{\overline{\delta}_w}(F) \geq \varepsilon\}$$

is visibility hull.

Definition

Quasiconvex visibility $F \subset \Gamma^0$ *is visibly quasiconvex if*

$$\forall \varepsilon > 0 \; \exists r \; \mathrm{Vis}_\varepsilon(F) \subset N_r(F).$$

The following property clarifies this notion (originally due to A. Karlsson for the case of geodesics in a Cayley graph):

Property of $\overline{\delta}_w$ (Karlsson Lemma). Suppose $\Sigma_n \alpha_{2n+1} f_n < \infty$ then

$$\forall \varepsilon > 0 \, \exists r = K(\varepsilon) \, \forall \alpha\mathrm{-geodesic} \, \gamma \subset \Gamma \, \mathrm{diam}_{\overline{\delta}_w}(\partial\gamma) \geq \varepsilon \implies d(v, \gamma) \leq r.$$

The following is one of the main results :

Theorem A. *Let a finitely generated discrete group G act 3-discontinuously and 2-cocompactly on a compactum X. The following properties of a subset F of the discontinuity domain of the action are equivalent:*
— F is relatively quasiconvex;
— F is visibly quasiconvex;
— F is relatively $\alpha-$quasiconvex where α is a quadratic polynomial with big enough coefficients. □

The proof uses the Proposition above and Karlsson Lemma.

The following is one of the main results :

Theorem A. *Let a finitely generated discrete group G act 3-discontinuously and 2-cocompactly on a compactum X. The following properties of a subset F of the discontinuity domain of the action are equivalent:*
— F is relatively quasiconvex;
— F is visibly quasiconvex;
— F is relatively $\alpha-$quasiconvex where α is a quadratic polynomial with big enough coefficients. □

The proof uses the Proposition above and Karlsson Lemma.

To formulate a corollary introduce one more type quasiconvexity (due to B. Bowditch)

Definition

A subgroup H of G is **dynamically quasiconvex** *if for any two closed disjoint subsets K and L of $T = \Lambda(G)$ the set* $\{g \in G \ : \ g(\Lambda(H)) \cap K \neq \emptyset \ \wedge g(\Lambda(H)) \cap L \neq \emptyset\}$ *is at most finite.*

The following is a Corollary from Theorem A:

Corollary

If $H < G$ and $F \subset \Gamma^0$ is H-invariant and H-finite ($|F/H| < \infty$) then the quasiconvex visibility of F is equivalent to the dynamical quasiconvexity of H.
In particular a finitely generated subgroup H of the relatively hyperbolic group G is relatively quasiconvex if and only if it is dynamically quasiconvex. □

The second part of the Corollary answers a question of D. Osin [Mem of AMS, 2006].

To formulate a corollary introduce one more type quasiconvexity (due to B. Bowditch)

Definition

A subgroup H of G is **dynamically quasiconvex** *if for any two closed disjoint subsets K and L of $T = \mathbf{\Lambda}(G)$ the set $\{g \in G \ : \ g(\mathbf{\Lambda}(H)) \cap K \neq \emptyset \ \wedge g(\mathbf{\Lambda}(H)) \cap L \neq \emptyset\}$ is at most finite.*

The following is a Corollary from Theorem A:

Corollary

If $H < G$ and $F \subset \Gamma^0$ is H-invariant and H-finite ($|F/H| < \infty$) then the quasiconvex visibility of F is equivalent to the dynamical quasiconvexity of H.
In particular a finitely generated subgroup H of the relatively hyperbolic group G is relatively quasiconvex if and only if it is dynamically quasiconvex. □

The second part of the Corollary answers a question of D. Osin [Mem of AMS, 2006].

To formulate a corollary introduce one more type quasiconvexity (due to B. Bowditch)

Definition

A subgroup H of G is **dynamically quasiconvex** *if for any two closed disjoint subsets K and L of $T = \mathbf{\Lambda}(G)$ the set $\{g \in G \ : \ g(\mathbf{\Lambda}(H)) \cap K \neq \emptyset \ \wedge g(\mathbf{\Lambda}(H)) \cap L \neq \emptyset\}$ is at most finite.*

The following is a Corollary from Theorem A:

Corollary

If $H < G$ and $F \subset \Gamma^0$ is H-invariant and H-finite ($|F/H| < \infty$) then the quasiconvex visibility of F is equivalent to the dynamical quasiconvexity of H.

In particular a finitely generated subgroup H of the relatively hyperbolic group G is relatively quasiconvex if and only if it is dynamically quasiconvex. □

The second part of the Corollary answers a question of D. Osin [Mem of AMS, 2006].

To formulate a corollary introduce one more type quasiconvexity (due to B. Bowditch)

Definition

A subgroup H of G is **dynamically quasiconvex** *if for any two closed disjoint subsets K and L of $T = \mathbf{\Lambda}(G)$ the set $\{g \in G \ : \ g(\mathbf{\Lambda}(H)) \cap K \neq \emptyset \ \wedge g(\mathbf{\Lambda}(H)) \cap L \neq \emptyset\}$ is at most finite.*

The following is a Corollary from Theorem A:

Corollary

If $H < G$ and $F \subset \Gamma^0$ is H-invariant and H-finite ($|F/H| < \infty$) then the quasiconvex visibility of F is equivalent to the dynamical quasiconvexity of H.
In particular a finitely generated subgroup H of the relatively hyperbolic group G is relatively quasiconvex if and only if it is dynamically quasiconvex. $\square$

The second part of the Corollary answers a question of D. Osin [Mem of AMS, 2006].

To formulate a corollary introduce one more type quasiconvexity (due to B. Bowditch)

Definition

A subgroup H of G is **dynamically quasiconvex** *if for any two closed disjoint subsets K and L of $T = \mathbf{\Lambda}(G)$ the set $\{g \in G \ : \ g(\mathbf{\Lambda}(H)) \cap K \neq \emptyset \ \wedge g(\mathbf{\Lambda}(H)) \cap L \neq \emptyset\}$ is at most finite.*

The following is a Corollary from Theorem A:

Corollary

If $H < G$ and $F \subset \Gamma^0$ is H-invariant and H-finite ($|F/H| < \infty$) then the quasiconvex visibility of F is equivalent to the dynamical quasiconvexity of H.
In particular a finitely generated subgroup H of the relatively hyperbolic group G is relatively quasiconvex if and only if it is dynamically quasiconvex. □

The second part of the Corollary answers a question of D. Osin [Mem of AMS, 2006].

3. Quasiconvexity. Part 2 : Hierarchy of subgroup "goodness" properties (decreasing order).

Different subgroups of a fixed relatively hyperbolic (or even convergence) group can be ordered according to their undistortedness in the ambiant group. We provide a short hierarchy of these properties starting with the strongest one (absolute quasiconvexity) and finishing by so called dynamical boundness (definition follows).

First Level (**Cocompactness outside the limit set**). A subgroup H of a RHG G is weakly $\alpha-$quasiconvex if every two points of an H-orbit can be joined by an $\alpha-$geodesic belonging to a fixed bounded neighborhood of H-orbit.
If the word "weakly" is omitted then this is true for any $\alpha-$geodesic and so this is the (absolute) $\alpha-$quasiconvexity.

3. Quasiconvexity. Part 2 : Hierarchy of subgroup "goodness" properties (decreasing order).

Different subgroups of a fixed relatively hyperbolic (or even convergence) group can be ordered according to their undistortedness in the ambiant group. We provide a short hierarchy of these properties starting with the strongest one (absolute quasiconvexity) and finishing by so called dynamical boundness (definition follows).

First Level (**Cocompactness outside the limit set**). A subgroup H of a RHG G is weakly $\alpha-$quasiconvex if every two points of an H-orbit can be joined by an $\alpha-$geodesic belonging to a fixed bounded neighborhood of H-orbit.
If the word "weakly" is omitted then this is true for any $\alpha-$geodesic and so this is the (absolute) $\alpha-$quasiconvexity.

The following result characterizes the Level 1 of the quasiconvexity:

Theorem B. *Let a finitely generated group G act 3-discontinuously and 2-cocompactly on a compactum X. Then there exists a constant $\lambda_0 \in]1, +\infty[$ such that the following properties of a subgroup H of G are equivalent:*

1. *H is weakly $\alpha-$quasiconvex for some distortion function α for which $\alpha(n) \leqslant \lambda_0^n$ $(n \in \mathbb{N})$, and for every $p \in \mathrm{Par}$ the subgroup $H \cap \mathrm{St}_G p$ is either finite or has finite index in $\mathrm{St}_G p$;*
2. *the space $(X \setminus \mathbf{\Lambda} H)/H$ is compact;*
3. *for every distortion function α bounded by λ_0^n $(n \in \mathbb{N})$, every H-invariant H-finite set $E \subset A$ is $\alpha-$quasiconvex and for every $p \in \mathrm{Par}$ the subgroup $H \cap \mathrm{St}_G p$ is either finite or has finite index in $\mathrm{St}_G p$.*

□

Corollary 1. *Cocompactness outside the limit set is equivalent to the strongest quasiconvexity such that the subgroup either avoid a parabolic subgroup of the ambiant group or intersects it in a finite index.*

Moreover the "weak" and absolute quasiconvexities are equivalent.

Corollary 2 (of the method). *Suppose that $G \curvearrowright X$ is RH_{32}-action. Let H be a subgroup of G acting cocompactly on $X \setminus \mathbf{\Lambda} H$ then if G is finitely presented then H is finitely presented too.*

Corollary 1. *Cocompactness outside the limit set is equivalent to the strongest quasiconvexity such that the subgroup either avoid a parabolic subgroup of the ambiant group or intersects it in a finite index.*
Moreover the "weak" and absolute quasiconvexities are equivalent.

Corollary 2 (of the method). *Suppose that $G \curvearrowright X$ is RH_{32}-action. Let H be a subgroup of G acting cocompactly on $X \setminus \Lambda H$ then if G is finitely presented then H is finitely presented too.*

Corollary 1. *Cocompactness outside the limit set is equivalent to the strongest quasiconvexity such that the subgroup either avoid a parabolic subgroup of the ambiant group or intersects it in a finite index.*
Moreover the "weak" and absolute quasiconvexities are equivalent.

Corollary 2 (of the method). *Suppose that $G \curvearrowright X$ is RH_{32}-action. Let H be a subgroup of G acting cocompactly on $X \setminus \mathbf{\Lambda} H$ then if G is finitely presented then H is finitely presented too.*

Last level of "subgroup goodness".

Definition

A subgroup H of G is called dynamically bounded if every infinite set of elements $S \subset G$ contains an infinite subset S_0 such that $\mathbf{\Lambda} G \setminus \bigcup_{s \in S_0} s(\mathbf{\Lambda} H)$ has a non-empty interior.

The following implications show the hierarchy between different types of subgroup properties:

Cocompactness on $X \setminus \mathbf{\Lambda} H \implies$
Strong absolute Quasiconvexity of $H \implies$
Relative Quasiconvexity of $H \implies$
H is **dynamically bounded.**

Last level of "subgroup goodness".

Definition

A subgroup H of G is called dynamically bounded if every infinite set of elements $S \subset G$ contains an infinite subset S_0 such that $\mathbf{\Lambda} G \setminus \bigcup_{s \in S_0} s(\mathbf{\Lambda} H)$ has a non-empty interior.

The following implications show the hierarchy between different types of subgroup properties:

Cocompactness on $X \setminus \mathbf{\Lambda} H \implies$
Strong absolute Quasiconvexity of $H \implies$
Relative Quasiconvexity of $H \implies$
H is **dynamically bounded.**

We finish this section by some questions and speculations.

Remark 1. In the case $G = \mathrm{Isom}\mathbb{H}^3$ C. McMullen showed that the most of the points of the limit set of any known geometrically infinite Kleinian group $H < G$ do not satisfy dynamical boundness condition.

The following is intriguing

Question

Suppose G is a RHG and $H < G$ is a finitely generated subgroup such that $\Lambda H \subsetneq \Lambda G$. Is H dynamically bounded ?

Remark 2. One cannot drop the condition of finite generatedness of H as any lattice admits an infinitely generated subgroup which is not dynamically bounded (this was pointed out to us by M. Kapovich).

We finish this section by some questions and speculations.

Remark 1. In the case $G = \mathrm{Isom}\mathbb{H}^3$ C. McMullen showed that the most of the points of the limit set of any known geometrically infinite Kleinian group $H < G$ do not satisfy dynamical boundness condition.

The following is intriguing

Question

Suppose G is a RHG and $H < G$ is a finitely generated subgroup such that $\boldsymbol{\Lambda} H \subsetneqq \boldsymbol{\Lambda} G$. Is H dynamically bounded ?

Remark 2. One cannot drop the condition of finite generatedness of H as any lattice admits an infinitely generated subgroup which is not dynamically bounded (this was pointed out to us by M. Kapovich).

We finish this section by some questions and speculations.

Remark 1. In the case $G = \mathrm{Isom}\mathbb{H}^3$ C. McMullen showed that the most of the points of the limit set of any known geometrically infinite Kleinian group $H < G$ do not satisfy dynamical boundness condition.

The following is intriguing

Question

Suppose G is a RHG and $H < G$ is a finitely generated subgroup such that $\mathbf{\Lambda} H \subsetneq \mathbf{\Lambda} G$. Is H dynamically bounded ?

Remark 2. One cannot drop the condition of finite generatedness of H as any lattice admits an infinitely generated subgroup which is not dynamically bounded (this was pointed out to us by M. Kapovich).

4. Applications.

We provide several applications for infinitely generated (and not in general countable) **RHG.**

Anti-Convention. In this Section we do not assume an **RH** group to be countable.

Recall that due to B. Bowditch a cofinite graph Γ (i.e. $|\Gamma^1/G| < \infty$) is called *fine* if for every $n \in \mathbb{N}$ and for every edge e the set of simple loops in Γ passing through e of length n is finite.

We prove the following:

Proposition. *Let $G \curvearrowright X$ be a* **RH**$_{32}$*-action. Then there exists a hyperbolic, G-cofinite graph Γ whose vertex stabilizers are all finite except the vertices corresponding to the parabolic points for the action $G \curvearrowright X$. Furthermore the graph Γ is fine.*

This Proposition generalizes A. Yaman's theorem to the case of uncountable **RH** groups. Furthermore we provide a self-contained independent argument in finitely generated case too.

4. Applications.

We provide several applications for infinitely generated (and not in general countable) **RHG.**

Anti-Convention. In this Section we do not assume an **RH** group to be countable.

Recall that due to B. Bowditch a cofinite graph Γ (i.e. $|\Gamma^1/G| < \infty$) is called *fine* if for every $n \in \mathbb{N}$ and for every edge e the set of simple loops in Γ passing through e of length n is finite.

We prove the following:

Proposition. *Let $G \curvearrowright X$ be a* $\mathbf{RH}_{32}$*-action. Then there exists a hyperbolic, G-cofinite graph Γ whose vertex stabilizers are all finite except the vertices corresponding to the parabolic points for the action $G \curvearrowright X$. Furthermore the graph Γ is fine.*

This Proposition generalizes A. Yaman's theorem to the case of uncountable **RH** groups. Furthermore we provide a self-contained independent argument in finitely generated case too.

We provide several applications for infinitely generated (and not in general countable) **RHG.**

Anti-Convention. In this Section we do not assume an **RH** group to be countable.

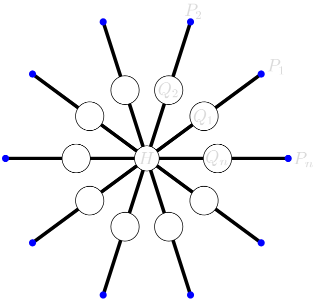
Recall that due to B. Bowditch a cofinite graph Γ (i.e. $|\Gamma^1/G| < \infty$) is called *fine* if for every $n \in \mathbb{N}$ and for every edge e the set of simple loops in Γ passing through e of length n is finite.

We prove the following:

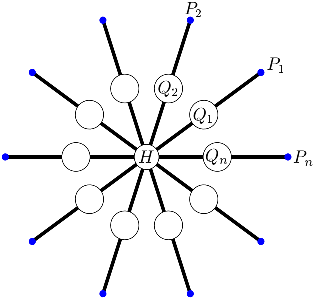
Proposition. *Let $G \curvearrowright X$ be a* $\mathbf{RH}_{32}$*-action. Then there exists a hyperbolic, G-cofinite graph Γ whose vertex stabilizers are all finite except the vertices corresponding to the parabolic points for the action $G \curvearrowright X$. Furthermore the graph Γ is fine.*

This Proposition generalizes A. Yaman's theorem to the case of uncountable **RH** groups. Furthermore we provide a self-contained independent argument in finitely generated case too.

The proof for infinitely generated groups is based on our Theorem [Gerasimov-P, GGD, to appear] claiming that any RH group G admits a star-graph of groups decomposition whose central vertex group is a finitely generated **RH** group H and all other vertex groups are maximal parabolic subgroups P_i $(i = 1, ..., n)$.



The proof for infinitely generated groups is based on our Theorem [Gerasimov-P, GGD, to appear] claiming that any RH group G admits a star-graph of groups decomposition whose central vertex group is a finitely generated **RH** group H and all other vertex groups are maximal parabolic subgroups P_i $(i = 1, ..., n)$.



One can show that the converse statement is also true: the above graph of groups gives an example of RH group G with respect to uncountable parabolic subgroups P_i.

Another application. Using our methods we have

Theorem C. *Let G be a group admitting non-elementary RH_{32}-actions on compacta X and Y such that every parabolic subgroup for the action on X is parabolic for the action on Y. Then there exists an G-equivariant map $F : X \to Y$. Furthermore there exists a compactum Z with a RH_{32}-action of G and two G-equivariant maps $F_1 : Z \to X$ and $F_2 : Z \to Y$.* □

Remarks. *1. In case when G is countable and X and Y are metrizable spaces the Theorem above is due to Y. Matsuda, A. Oguni and S. Yamagata (preprint, 2012)*
We provide an independent and self-contained argument valid for non-metrizable spaces and uncountable groups.

2. The existence of such universal space Z for any two RH_{32}-actions was open so called "pullback problem" (formulated first in [Gerasimov, GAFA, 2009]). It is true if G is finitely generated and follows from the existence of the Floyd map. However we provide a counter-example for the existence of such a space if G is infinitely generated (in fact a countable free group).

Remarks. *1. In case when G is countable and X and Y are metrizable spaces the Theorem above is due to Y. Matsuda, A. Oguni and S. Yamagata (preprint, 2012)*
We provide an independent and self-contained argument valid for non-metrizable spaces and uncountable groups.

2. The existence of such universal space Z for any two RH_{32}-actions was open so called "pullback problem" (formulated first in [Gerasimov, GAFA, 2009]). It is true if G is finitely generated and follows from the existence of the Floyd map. However we provide a counter-example for the existence of such a space if G is infinitely generated (in fact a countable free group).

Remarks. *1. In case when G is countable and X and Y are metrizable spaces the Theorem above is due to Y. Matsuda, A. Oguni and S. Yamagata (preprint, 2012)*
We provide an independent and self-contained argument valid for non-metrizable spaces and uncountable groups.

2. The existence of such universal space Z for any two RH_{32}*-actions was open so called "pullback problem" (formulated first in [Gerasimov, GAFA, 2009]). It is true if G is finitely generated and follows from the existence of the Floyd map. However we provide a counter-example for the existence of such a space if G is infinitely generated (in fact a countable free group).*

Most of the above results are in the preprint

"Quasiconvexity in the relatively hyperbolic groups"

ArXiv:1103.1211v2

The last part with Theorem C will be in a coming preprint.

Most of the above results are in the preprint

"Quasiconvexity in the relatively hyperbolic groups"

ArXiv:1103.1211v2

The last part with Theorem C will be in a coming preprint.

Most of the above results are in the preprint

"Quasiconvexity in the relatively hyperbolic groups"

ArXiv:1103.1211v2

The last part with Theorem C will be in a coming preprint.

Most of the above results are in the preprint

"Quasiconvexity in the relatively hyperbolic groups"

ArXiv:1103.1211v2

The last part with Theorem C will be in a coming preprint.

END OF THE TALK