Rapid Decay and 3-manifolds

François Gautero

Université de Nice - Sophia Antipolis,
Laboratoire J.A. Dieudonné (UMR CNRS 6621)

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Group algebra

G discrete group. We denote by $|.|$ the length function on G associated to a given generating set $S = S^{-1}$:

For $\gamma \in G$, $|\gamma|$ is the minimal number of elements in S needed to write γ.

$\mathbb{C}G = \{f\colon G \to \mathbb{C}\,;\, |\mathrm{supp}(f)| < \infty\}$.

This is the set of all the sums $\sum_{\gamma \in G} a_\gamma \gamma$ with $a_\gamma \in \mathbb{C}$ and $a_\gamma = 0$ outside a finite number of elements $\gamma \in G$.

Convolution product :

$$(f * g)(\gamma) = \sum_{\mu \in G} f(\mu) g(\mu^{-1}\gamma)$$

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- l^2-norm : $||f||_2 = \sqrt{\sum_{\gamma \in G} |f(\gamma)|^2}$
- Operator-norm : $||f||_* = \sup_{||g||_2 = 1} ||f * g||_2$
- Weighted-norm : $||f||_s = \sqrt{\sum_{\gamma \in G} |f(\gamma)|^2 (1 + |\gamma|)^{2s}}$

Rapid Decay

Definition (Haagerup, Jolissaint)

A discrete group G satisfies the property of Rapid Decay if and only if there exist two positive constants C and s such that for any function f in $\mathbb{C}G$ we have

$$||f||_* \leq C||f||_s$$

Characterizations of property (RD)

1. There exists a polynome P such that for any $r > 0$ and for any $f \in \mathbb{R}^+G$ with $\mathrm{supp(f)} \subset B_r(e)$ we have $||f||_* \leq P(r)||f||_2$.

2. Let χ_m be the characteristic function of the sphere of radius m centered at e.
   There exist $C, r > 0$ such that for any non-negative integers k, l, m and for any functions $f, g \in \mathbb{C}G$ with $\mathrm{supp}(f) \subset \partial B_k(e)$ and $\mathrm{supp}(g) \subset \partial B_l(e)$ we have

$$||(f * g)\chi_m||_2 \leq C||f||_r||g||_2 \text{ if } |k - l| \leq m \leq k + l$$

and $||(f * g)\chi_m||_2 = 0$ otherwise.

Some groups with (RD)

1. Finite groups.
2. Groups with polynomial growth (Jolissaint).
3. Free and hyperbolic groups (Haagerup, de la Harpe).
4. Discrete, finite co-volume subgroups of the isometry group of a complete, simply-connected Riemannian manifold with pinched negative curvature (Chatterji, Drutu-Sapir).
5. Mapping-tori $G \rtimes_\alpha \mathbb{Z}$ where G has property (RD) and α is a polynomially growing automorphism (Jolissaint).
6. Any mapping-torus of a free group (G.-Lustig).

An obstruction

A discrete group G is amenable if and only if there exists a finitely additive probability measure μ which is *left-invariant* : for any $\gamma \in G$ and for any measurable set $M \subset G$, $\mu(\gamma M) = \mu(M)$.

Proposition (Jolissaint)

An amenable group has property (RD) if and only if it has polynomial growth.

Consequence

The group $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$, where A is an Anosov map, does not have property (RD).

Anosov map : 2×2 integer matrix with eigenvalues $\lambda > 1$ and $\frac{1}{\lambda}$.
For instance

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Question

Under which condition(s)

the fundamental group

of a closed 3-manifold

has property (RD) ?

Answer

Theorem (G.)

Work in progress The fundamental group of a closed irreducible 3-manifold has property (RD) if and only if M^3 is not finitely covered by the suspension of an Anosov of the torus.

Remark

$\Rightarrow$ Jolissaint.

Topology and Geometry of 3-manifolds I

Theorem (3-manifold decomposition)

Let M^3 be a closed, irreducible, orientable 3-manifold. There exists a finite collection $T_1, \cdots, T_n$ of disjointly embedded incompressible tori, such that the closure of each connected component of $M^3 \setminus \bigcup_{i=1}^{n} T_i$ is :

1. *either a compact 3-manifold of hyperbolic interior the boundary of which is a union of tori,*
2. *or a graph-manifold.*

Topologie and Geometry of 3-manifolds II

Definition

A graph-manifold is an orientable irreducible 3-manifold whose boundary is a (possibly empty) union of tori and which admits a finite collection $T_1, \cdots, T_n$ *of disjointly embedded incompressible tori such that the closure of each connected component of* $M^3 \setminus \bigcup_{i=1}^{n} T_i$ *is a Seifert manifold.*

Definition

A Seifert-manifold is an orientable 3-manifold which is a disjoint union of circles C_α*. Each circle* C_α *in the interior admits a neighborhood* $N(C_\alpha)$ *which is homeomorphic to the suspension of the disc under a rotation with angle* $2\pi p/q$*. This homeomorphism maps* C_α *to the orbit of the center of the disc and each circle* $C_\beta \subset N(C_\alpha)$ *to some orbit of this suspension.*

Examples

Example (of a Seifert manifold)

The mapping-torus $S \times [0,1]/(x,1) \sim (h(x),0)$ of an orientation-preserving periodic homeomorphism h of a compact orientable surface S.

Example (of a graph-manifold)

The mapping-torus $S \times [0,1]/(x,1) \sim (h(x),0)$ of an orientation-preserving linearly growing homeomorphism h of a compact orientable surface S.
The manifold $\mathbb{T}^2 \rtimes_A \mathbb{S}^1$ where A is an Anosov map, for instance

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Notion of relative hyperbolicity introduced by Gromov and elaborated by Farb, Bowditch, Osin.

Definition

G is strongly hyperbolic relative to a subgroup H if and only if G acts isometrically and properly discontinuously on a complete Gromov hyperbolic geodesic metric space X such that the quotient X/G is quasi isometric to a ray and H stabilizes the end of this ray.

Theorem (Druti-Sapir)

Let G be a group which is strongly hyperbolic relative to a finite collection of subgroups with property (RD). Then G also has property (RD).

Geometry of groups and 3-manifolds II

Theorem (Dahmani, G.)

The fundamental group of an orientable irreducible 3-manifold is strongly hyperbolic relative to the subgroups associated to the maximal graph-submanifolds.

Proposition (G.)

The fundamental group of a graph-manifold M^3 has property (RD) if and only if M^3 is not finitely covered by the suspension of an Anosov of the torus.

Quick, and false, proof

1. The previous proposition holds for 3-manifolds which fiber over the circle.
2. By Kapovich-Leeb the fundamental group of any graph-manifold is quasi isometric to the fundamental group of a 3-manifold which fibers over the circle, hence the conclusion.

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3. BUT....
   It is not known whether property (RD) is a quasi isometry invariant.

Conclusion

Theorem (G.)

The fundamental group of a closed irreducible 3-manifold M^3 has property (RD) if and only if M^3 is not finitely covered by the suspension of an Anosov of the torus.

Proof

$\Rightarrow$ Jolissaint.
$\Leftarrow$ Since (RD) passes to finite covers, we can assume M^3 is orientable. $\pi_1(M^3)$ is strongly hyperbolic relative to the subgroups associated to the maximal graph-submanifolds. Since none of these components is finitely covered by the suspension of an Anosov of the torus, they have property (RD). By Drutu-Sapir, $\pi_1(M^3)$ also has property (RD).