# HEEGAARD FLOER HOMOLOGIES
# LECTURE NOTES

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## Contents



## 1. Introduction and overview

1.1. **A brief overview.** Heegaard Floer homology is a family of related invariants of objects in low-dimensional topology. The first of these invariants were introduced by Ozsváth-Szabó: invariants of closed 3-manifolds [OSz04d] and smooth 4-dimensional cobordisms [OSz06]. Later, Ozsváth-Szabó and, independently, Rasmussen introduced invariants of knots in 3-manifolds [OSz04b, Ras03]. (There are also several other invariants, including invariants of contact structures, more invariants of knots and 3-manifolds,

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and invariants of Legendrian and transverse knots.) The subject has had many applications; I will not even try to list them here, though we will see a few in the lectures.

In the first three of these lectures, we will focus on a generalization of one variant of these invariants: an invariant of sutured 3-manifolds, due to Juhász, called *sutured Floer homology* [Juh06]. The main goal will be to relate these invariants to ideas in more classical 3-manifold topology. In particular, we will sketch a proof that sutured Floer homology detects the genus of a knot. The proof uses Gabai's theory of sutured manifolds and sutured hierarchies, which we will review briefly in the first lecture.

In the fourth lecture, we go in a different direction: we will talk about the surgery exact sequence in Heegaard Floer homology. The goal is to sketch a (much studied) relationship between Heegaard Floer homology and Khovanov homology: a spectral sequence due to Ozsváth-Szabó [OSz05].

1.2. **A more precise overview.** Heegaard Floer homology assigns to each closed, oriented, connected 3-manifold $Y$ an abelian group $\widehat{HF}(Y)$, and $\mathbb{Z}[U]$-modules $HF^+(Y)$, $HF^-(Y)$ and $HF^\infty(Y)$. These are the homologies of chain complexes $\widehat{CF}(Y)$, $CF^+(Y)$, $CF^-(Y)$ and $CF^\infty(Y)$. These chain complexes are related by short exact sequences

$$0 \longrightarrow CF^-(Y) \longrightarrow CF^\infty(Y) \longrightarrow CF^+(Y) \longrightarrow 0$$

$$0 \longrightarrow CF^-(Y) \xrightarrow{\cdot U} CF^-(Y) \longrightarrow \widehat{CF}(Y) \longrightarrow 0$$

$$0 \longrightarrow \widehat{CF}(Y) \longrightarrow CF^+(Y) \xrightarrow{\cdot U} CF^+(Y) \longrightarrow 0$$

which, of course, induce long exact sequences in homology. In particular, either of $CF^+(Y)$ or $CF^-(Y)$ determines $\widehat{CF}(Y)$. (The complexes $CF^+(Y)$ and $CF^-(Y)$ also have equivalent information, though this does not quite follow from what we've said so far.) These invariants are defined in [OSz04d]. (Some people report finding it helpful to read [Lip06] in conjunction with [OSz04d].) It is now known, by work of Hutchings, Taubes, and Kutluhan-Lee-Taubes or Colin-Ghiggini-Honda, that these invariants correspond to certain Seiberg-Witten Floer homology groups.

Roughly, smooth, compact, connected 4-dimensional cobordisms between connected 3-manifolds induce chain maps on $\widehat{CF}$, $CF^\pm$ and $CF^\infty$, and composition of cobordisms corresponds to composition of maps. From the maps on $CF^\pm$ and the exact sequences above, one can recover the Seiberg-Witten invariant, or at least something very much like it. (See [OSz06].) Note, in particular, that $\widehat{CF}$ does not have enough information to recover the Seiberg-Witten invariant.

There is an extension of the Heegaard Floer homology groups to nullhomologous knots in 3-manifolds, called *knot Floer homology* [OSz04b, Ras03]. Given a knot $K$ in a 3-manifold $Y$ there is an induced filtration of $\widehat{CF}(Y)$, $CF^+(Y)$, and so on. In particular, we can define the *knot Floer homology groups* $\widehat{HFK}(Y,K)$, the homology of the associated graded complex to the filtration on $\widehat{CF}(Y)$. (So, there is a spectral sequence from $\widehat{HFK}(Y,K)$ to $\widehat{HF}(Y)$.)

The gradings in the subject are quite subtle. The chain complexes $\widehat{CF}(Y)$, $CF^+(Y)$, ..., decompose as direct sums according to spin$^c$-*structures* on $Y$, i.e.,

$$\widehat{CF}(Y) = \bigoplus_{\mathfrak{s}\in \mathrm{spin}^c(Y)} \widehat{CF}(Y,\mathfrak{s}).$$

(We will discuss spin$^c$ structures more in Section 3.4.1.) Each of the $\widehat{CF}(Y,\mathfrak{s})$ is relatively graded by some $\mathbb{Z}/n\mathbb{Z}$ (where $n$ is the divisibility of $c_1(\mathfrak{s})$). In particular, if $c_1(\mathfrak{s}) = 0$ (i.e.,

$\mathfrak{s}$ is *torsion*) then $\widehat{CF}(Y,\mathfrak{s})$ has a relative $\mathbb{Z}$ grading. Similarly, $\widehat{HFK}(Y,K)$ decomposes as a direct sum of groups, one per relative spin$^c$ structure on $(Y,K)$.

In the special case that $Y=S^3$, there is a canonical identification $\mathrm{spin}^c(S^3,K)\cong\mathbb{Z}$, and each $\widehat{HFK}(Y,K,\mathfrak{s})$ in fact has an absolute $\mathbb{Z}$-grading. That is, $\widehat{HFK}(Y,K)$ is a bigraded abelian group; we will write $\widehat{HFK}(S^3,K)=\widehat{HFK}(K)=\bigoplus_{i,j}\widehat{HFK}_i(K,j)$, where $j$ stands for the spin$^c$ grading. The grading $j$ is also called the *Alexander grading*, because

$$\sum_{i,j}(-1)^i t^j \operatorname{rank}\widehat{HFK}_i(K,j)=\Delta_K(t),$$

the (Conway normalized) Alexander polynomial of $K$.

The breadth of the Alexander polynomial $\Delta_K(t)$, or equivalently the degree of the symmetrized Alexander polynomial, gives a lower bound on the genus of $K$ (i.e., the minimal genus of any Seifert surface for $K$). One of the main goals of these lectures will be to sketch a proof of the following refinement:

**Theorem 1.1.** *[OSz04a] Given a knot $K$ in $S^3$,*

$$g(K)=\max\{j \mid \big(\bigoplus_i \widehat{HFK}_i(K,j)\big)\neq 0\}.$$

Rather than giving the original proof of Theorem 1.1, we will give a proof using an extension of $\widehat{HF}$ and $\widehat{HFK}$, due to Juhász, called *sutured Floer homology*. Sutured manifolds were introduced by Gabai in his work on foliations, fibrations, the Thurston norm, and knot genus [Gab83, Gab84, Gab86, Gab87]; we will review some aspects of this theory in the first lecture. Sutured Floer homology associates to each sutured manifold $(Y,\Gamma)$ satisfying certain conditions (called being *balanced*) a chain complex $SFC(Y,\Gamma)$ whose homology $SFH(Y,\Gamma)$ is an invariant of the sutured manifold. These chain complexes behave in a particular way under Gabai's *surface decompositions*, allowing us to prove Theorem 1.1.

In the last lecture, we turn to a different topic: the behavior of Heegaard Floer homology under knot surgery. The goal is to relate these lectures to the lecture series on Khovanov homology. In particular, we will sketch the origins of Ozsváth-Szabó's spectral sequence $\widetilde{Kh}(m(K))\Rightarrow\widehat{HF}(\Sigma(K))$ from the (reduced) Khovanov homology of the mirror of $K$ to the Heegaard Floer homology of the branched double cover of $K$ [OSz05].

1.3. **References for further reading.** There are a number of survey articles on Heegaard Floer homology. Two by Ozsváth-Szabó [OS05a, OS06a, OS06b] give nice introductions to the Heegaard Floer invariants of 3- and 4-manifolds and knots. Juhász's recent survey [Juh13] contains an introduction to sutured Floer homology, which is the main subject of these lectures. There are also some more focused surveys of other recent developments [Man14, LOT11].

Sutured Floer homology, as we will discuss it, is developed in a pair of papers by Juhász [Juh06, Juh08]. For a somewhat different approach to relating sutured manifolds and Floer theory, see the work of Ni (starting with [Ni09]).

## 2. Sutured manifolds, foliations and sutured hierarchies

### 2.1. The Thurston norm and foliations.

**Definition 2.1.** *Given a knot $K\subset S^3$, the* genus *of $K$ is the minimal genus of any Seifert surface for $K$ (i.e., of any embedded surface $F\subset S^3$ with $\partial F=K$).*

Thurston found a useful generalization of this notion to arbitrary 3-manifolds and, more generally, to link complements in arbitrary 3-manifolds:

**Definition 2.2.** *Given a 3-manifold $Y$ with boundary $\partial Y$ a disjoint union of tori, the* Thurston norm

$$x\colon H_2(Y, \partial Y) \to \mathbb{Z}$$

*is defined as follows. Given a compact, oriented surface $F$ (not necessarily connected, possibly with boundary) define the* complexity of $F$ *to be*

$$x(F) = \sum_{\chi(F_i) \leq 0} |\chi(F_i)|,$$

*where the sum is over the connected components $F_i$ of $F$.*

*Given an element $h \in H_2(Y, \partial Y)$ and a surface $F \subset Y$ with $\partial F \subset \partial Y$ we say that $F$* represents $h$ *if the inclusion map sends the fundamental class of $F$ in $H_2(F, \partial F)$ to $h$. Define*

$$x(h) = \min\{x(F) \mid F \text{ represents } h\}.$$

For this definition to make sense, we need to know the surface $F$ exists:

**Lemma 2.3.** *Any element $h \in H_2(Y, \partial Y)$ is represented by some surface $F$.*

*Idea of Proof.* The class $h$ is Poincaré dual to a class in $H^1(Y)$, which in turn is represented by a map $f_h\colon Y \to K(\mathbb{Z}, 1) = S^1$. The preimage of a regular value of $f_h$ represents $h$. See [Thu86] for more details. ☐

**Proposition 2.4.** *If $(Y, \partial Y)$ has no essential spheres ($Y$ is* irreducible*) or disks ($\partial Y$ is* incompressible*) then $x$ defines a pseudo-norm on $H_2(Y, \partial Y)$ (i.e., a norm except for the non-degeneracy axiom). If moreover $Y$ has no essential annuli or tori ($Y$ is* atoroidal*) then $x$ defines a norm on $H_2(Y, \partial Y)$, and induces a norm on $H_2(Y, \partial Y; \mathbb{Q})$.*

*Idea of Proof.* Again, see [Thu86] for details. The main points to check are that:

(1) $x(n \cdot h) = n \cdot x(h)$ for $n \in \mathbb{N}$.
(2) $x(h + k) \leq x(h) + x(k)$.

For the first point, a little argument shows that a surface representing $n \cdot h$ (with $h$ indivisible) necessarily has $n$ connected components, each representing $h$. The second is a little more complicated. Roughly, one takes surfaces representing $h$ and $k$ and does surgery on their circles and arcs of intersection to get a new surface representing $h + k$ without changing the Euler characteristic. (More precisely, one first has to eliminate intersections which are inessential on both surfaces, as doing surgery along them would create disjoint $S^2$ or $\mathbb{D}^2$ components.) ☐

*Example* 2.5. If $Y = S^3 \setminus \text{nbd}(K)$ is the exterior of a knot then $H_2(Y, \partial Y) \cong \mathbb{Z}$ and surfaces representing a generator for $H_2(Y, \partial Y)$ are Seifert surfaces for $K$. The Thurston norm of a generator is given by $2g(K) - 1$.

*Example* 2.6. Consider $Y = S^1 \times \Sigma_g$. Fix a collection of curves $\gamma_i$, $i = 1, \dots, 2g$, in $\Sigma$ giving a basis for $H_1(\Sigma)$. Then $H_2(Y) \cong \mathbb{Z}^{2g+1}$, with basis (the homology classes represented by) $S^1 \times \gamma_i$, $i = 1, \dots, 2g$, and $\{pt\} \times \Sigma$. We have $x([S^1 \times \gamma_i]) = 0$, from which it follows (why?) that $x$ is determined by $x([\{pt\} \times \Sigma_g])$. One can show using elementary algebraic topology that $x([\{pt\} \times \Sigma_g]) = 2g - 2$; see Exercise 1.

*Remark* 2.7. A norm is determined by its unit ball. The Thurston norm ball turns out to be a polytope defined by inequalities with integer coefficients [Thu86, Theorem 2].

*A priori*, the Thurston norm looks impossible to compute in general. Remarkably, however, it can be understood. The two key ingredients are foliations, which we discuss now, and a decomposition technique, due to Gabai, which we discuss next.

**Definition 2.8.** *A* smooth, codimension-1 foliation $\mathcal{F}$ *of* $M$ *is a collection of disjoint, codimension-1 immersed submanifolds* $\{N_j \subset M\}_{j\in J}$ *so that each immersion is injective and every point in* $M$ *is in (exactly) one of the* $N_j$*. The* $N_j$ *are called the* leaves *of the foliation.*

We will only be interested in smooth, codimension-1 foliations, so we will refer to these simply as foliations. (Actually, there are good reasons to consider non-smooth foliations in this setting. Higher codimension foliations are also, of course, interesting.)

In a small enough neighborhood of any point, the $N_j$ look like pages of a book, though each $N_j$ may correspond to many pages. The standard examples are foliations of the torus $T^2 = [0,1] \times [0,1]/\sim$ by the curves $\{y = mx + b\}$ for fixed $m \in \mathbb{R}$ and $b$ allowed to vary. If $m$ is rational then the leaves are circles. If $m$ is irrational then the leaves are immersed copies of $\mathbb{R}$.

The tangent spaces to the leaves $N_j$ in a foliation $\mathcal{F}$ of $M^n$ define an $(n-1)$-plane field in $TM$; I will call this the *tangent space to* $\mathcal{F}$ and write it as $T\mathcal{F}$. An *orientation* of $\mathcal{F}$ is an orientation $T\mathcal{F}$, and a *co-orientation* is an orientation of the orthogonal complement $T\mathcal{F}^\perp$ of $T\mathcal{F}$. Since we are only interested in oriented 3-manifolds, the two notions are equivalent.

A curve is *transverse to* $\mathcal{F}$ if it is transverse $T\mathcal{F}$.

**Definition 2.9.** *A foliation* $\mathcal{F}$ *of* $M$ *is called* taut *if there is a curve* $\gamma$ *transverse to* $\mathcal{F}$ *such that* $\gamma$ *intersects every leaf of* $\mathcal{F}$*.*

**Theorem 2.10.** *[Thu86, Corollary 2, p. 119] Let* $\mathcal{F}$ *be a taut foliation of* $Y$ *so that for every component* $T$ *of* $\partial Y$ *either:*

- $T$ *is a leaf of* $\mathcal{F}$ *or*
- $T$ *is transverse to* $\mathcal{F}$ *and* $\mathcal{F} \cap T$ *is taut in* $T$*.*

*Then every compact leaf of* $\mathcal{F}$ *is genus minimizing.*

(The proof is not so easy.)

Remarkably, as we discuss next, Gabai shows that Theorem 2.10 can always be used to determine the Thurston norm.

## 2.2. Sutured manifolds.

**Definition 2.11.** *A* sutured manifold *is a 3-manifold* $Y$ *together with a decomposition of* $\partial Y$ *into three parts (codimension-0 submanifolds with boundary): the bottom part* $R_-$*, the top part* $R_+$*, and the vertical part* $\gamma$*. This decomposition is required to satisfy the properties that:*

(1) *Every component of* $\gamma$ *is either an annulus or a torus.*
(2) $\partial R_+ \cap \partial R_- = \emptyset$ *(so* $\partial\gamma = \partial R_+ \amalg \partial R_-$*).*
(3) *Each annulus in* $\gamma$ *shares one boundary component with* $R_+$ *and one boundary component with* $R_-$*.*
(4) *Orient* $R_+$ *(respectively* $R_-$*) using the orientation of* $Y$ *and the outward-pointing (respectively inward-pointing) normal vector. That is, the orientation of* $R_+$ *agrees with the standard orientation of* $\partial Y$*, and the orientation of* $R_-$ *agrees with* $-\partial Y$*. Then both* $R_+$ *and* $R_i$ *induce orientations of the cores of the annuli in* $\gamma$*, and we require that these orientations agree.* Fixed

*(See [Gab83, Section 2].)*

*Let* $T(\gamma)$ *denote the union of the toroidal components of* $\gamma$ *and* $A(\gamma)$ *the union of the annular components of* $\gamma$*. We will often denote a sutured manifold by* $(Y,\gamma)$*, since* $\gamma$ *determines* $R_\pm$*.*

*A sutured manifold is called* taut *if $Y$ is irreducible (every $S^2$ bounds a $\mathbb{D}^3$) and $R_+$ and $R_-$ are norm-minimizing in their homology classes.*

*A sutured manifold is called* balanced *if:*

(1) $T(\gamma) = \emptyset$.
(2) *$R_+$ and $R_-$ have no closed components.*
(3) *$Y$ has no closed components.*
(4) $\chi(R_+) = \chi(R_-)$.

*Let $\Gamma$ denote the cores of the annular components of $\gamma$. Then for a balanced sutured manifold, $(Y, \Gamma)$ determines the whole sutured structure, so we may refer to $(Y, \Gamma)$ as a sutured manifold.*

*Example* 2.12. Given a surface $R$ with boundary, consider $Y = [0, 1] \times R$. Make this into a sutured manifold by defining $R_+ = \{1\} \times R$, $R_- = \{0\} \times R$ and $\gamma = [0, 1] \times \partial R$. Sutured manifolds of this form are called *product sutured manifolds*.

Product sutured manifolds are taut and, if $R$ has no closed components, balanced.

*Example* 2.13. Let $Y$ be a closed, connected 3-manifold. We can view $Y$ as a somewhat trivial example of a sutured 3-manifold. This sutured 3-manifold may or may not be taut, but is not balanced.

We can also delete a ball $\mathbb{D}^3$ from $Y$ and place, say, a single annular suture on the resulting $S^2$ boundary. (So, $R_+ = \mathbb{D}^2$, $R_- = \mathbb{D}^2$, and $\gamma = [0, 1] \times S^1$.) This sutured manifold is not taut (unless $Y = S^3$)—a sphere parallel to the boundary does not bound a disk—but it is balanced.

*Example* 2.14. Let $Y$ be a closed, connected 3-manifold and let $K \subset Y$ be a knot. Consider $Y \setminus \mathrm{nbd}(K)$, the exterior of $K$. We can view this as a sutured manifold by defining $\gamma$ to be the whole torus boundary. This sutured manifold is not balanced.

More relevant to our later constructions, we can define a balanced sutured manifold by letting $\Gamma$ consist of $2n$ meridional circles, so $R_+$ and $R_-$ each consists of $n$ annuli. See Figure 1. (In my head, this looks like a knotted sea monster biting its own tail: $R_+$ is the part above the water.)

**Definition 2.15.** *A foliation $\mathcal{F}$ on $Y$ is* compatible with $\gamma$ *if*

(1) *$\mathcal{F}$ is transverse to $\gamma$.*
(2) *$R_+$ and $R_-$ are unions of leaves of $\mathcal{F}$, and the orientations of these leaves agree with the orientations of $R_\pm$.*

*(I think this is not a standard term.)*

*A foliation $\mathcal{F}$ on $(Y, \gamma)$ is* taut *if*

(1) *$\mathcal{F}$ is compatible with $\gamma$.*
(2) *$\mathcal{F}$ is taut.*
(3) *For each component $S$ of $\gamma$, $\mathcal{F} \cap S$ is taut, as a foliation of $S$.*

*Example* 2.16. Every product sutured manifold admits an obvious taut foliation, where the leaves are $\{t\} \times R$.

**Definition 2.17.** *We call a sutured manifold* rational homology trivial *or* RHT *if the homology group $H_2(Y)$ vanishes. (This is not a standard term.)*

*Example* 2.18. A knot complement in $S^3$ is RHT. If $Y$ is a closed 3-manifold then $Y \setminus \mathbb{D}^3$ is RHT if and only if $Y$ is a rational homology sphere.

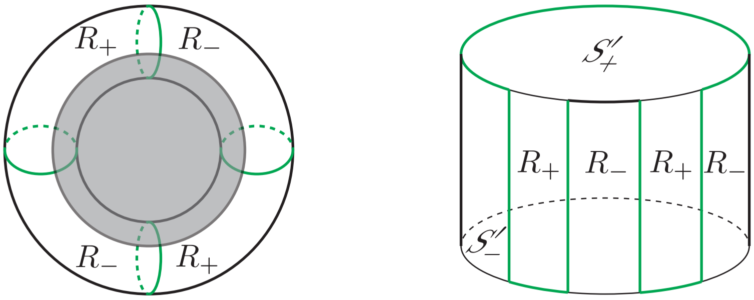



FIGURE 1. **A sutured manifold decomposition.** Left: the complement of the unknot, with four meridional sutures, together with a (gray) decomposing disk. Only the cores of the annular sutures are drawn, as green circles. Right: the result of performing a surface decomposition to this sutured manifold.

### 2.3. Surface decompositions and Gabai's theorem.

**Definition 2.19.** *[Gab83, Definition 3.1] A* decomposing surface *in a sutured manifold $(Y, \gamma)$ is a compact, oriented surface with boundary $(S, \partial S) \subset (Y, \partial Y)$ so that for every component $\lambda$ of $\partial S \cap \gamma$, either:*

(1) *$\lambda$ is a properly embedded, non-separating arc in $\gamma$, or*
(2) *$\lambda$ is a circle which is essential in the component of $\gamma$ containing $\lambda$.*

*We also require that in each torus component $T$ of $\gamma$, the orientations of all circles in $S \cap T$ agree, and in each annular component $A$ of $\gamma$, the orientation of all circles in $S \cap A$ agree with the orientation of the core of $A$.*

*Given a sutured manifold $(Y, \gamma)$ and a decomposing surface $S$ we can form a new sutured manifold $(Y', \gamma')$ as follows. Topologically, $Y' = Y \setminus \mathrm{nbd}(S)$. Let $S_+, S_- \subset \partial Y'$ denote the positive and negative pushoffs of $S$, respectively. Then $R'_+ = (R_+ \cap \partial Y') \cup S'_+$ (minus a neighborhood of its boundary), $R'_- = (R_- \cap \partial Y') \cup S'_-$ (minus a neighborhood of its boundary), and $\gamma'$ is the rest of $\partial Y'$ (cf. Exercise 2). We call this operation* sutured manifold decomposition *and write $(Y, \gamma) \overset{S}{\rightsquigarrow} (Y', \gamma')$.*

*Example* 2.20. If $K$ is a knot in $S^3$, say, $Y = S^3 \setminus \mathrm{nbd}(K)$, and $\gamma$ consists of $2n$ meridional sutures as in Example 2.14 then any Seifert surface for $K$ is a decomposing surface for $(Y, \gamma)$.

If $K$ is a fibered knot and $F$ is a Seifert surface for $K$ which is a fiber of the fibration then the result of doing a surface decomposition to the exterior $(Y, \gamma)$ of $K$ is a product sutured manifold. The case that $K$ is the unknot is illustrated in Figure 1.

It is maybe better to think of the inverse operation to surface decomposition; perhaps I will say that $(Y, \Gamma)$ is obtained from $(Y', \Gamma')$ by a *suture-compatible gluing* if $(Y', \Gamma')$ is obtained from $(Y, \Gamma)$ by surface decomposition. (This is not a standard term.) Unlike surface decomposition, suture-compatible gluing is not a well-defined operation: it depends on both a choice of subsurface $S' \subset \partial Y'$ and a choice of homeomorphism $S'_+ \cong S'_-$. I think this is why it is not talked about.

**Definition 2.21.** *We call a decomposing surface $S$ in a balanced sutured manifold $(Y, \gamma)$* balanced-admissible *if $S$ has no closed components and for every component $R$ of $R_+$ and $R_-$, the set of closed components of $S \cap R$ is a union of parallel curves (where each of these curves has orientation induced by the boundary of $S$), and if these curves are* *null-homotopic then they are oriented as the boundary of their interiors. (This is not a standard term.)*

**Lemma 2.22.** *If* $(Y, \Gamma) \overset{S}{\rightsquigarrow} (Y', \Gamma')$*,* $(Y, \Gamma)$ *is balanced, and* $S$ *is balanced-admissible then* $(Y', \Gamma')$ *is balanced.*

The proof is Exercise 4.

A particularly simple kind of sutured decomposition is the following:

**Definition 2.23.** *A* product disk *in* $(Y, \Gamma)$ *is a decomposing surface* $S$ *for* $(Y, \Gamma)$ *so that* $S \cong \mathbb{D}^2$ *and* $S \cap \Gamma$ *consists of two points. A* product decomposition *is a sutured decomposition* $(Y, \Gamma) \overset{S}{\rightsquigarrow} (Y', \Gamma')$ *where* $S$ *is a product disk.*

**Lemma 2.24.** *Suppose that* $(Y, \Gamma) \overset{S}{\rightsquigarrow} (Y', \Gamma')$*, where* $\partial S$ *is disjoint from any toroidal sutures of* $Y$*. Let* $\mathcal{F}'$ *be a foliation on* $(Y', \Gamma')$*. Then there is an induced foliation* $\mathcal{F}$ *on* $(Y, \Gamma)$ *with the property that* $S$ *is a leaf of* $\mathcal{F}$*.*

In other words, suture-compatible gluing takes foliations to foliations with $S$ as a leaf. This is the easy case in the proof of [Gab83, Theorem 5.1]; the proof is Exercise 5. The harder case, when $\partial S$ intersects some toroidal sutures, takes up most of the proof.

**Theorem 2.25.** *[Gab83, Theorems 4.2 and 5.1] Let* $(Y, \gamma)$ *be a taut sutured manifold. Then there is a sequence of surface decompositions*

$$(Y, \gamma) = (Y_1, \gamma_1) \overset{S_1}{\rightsquigarrow} (Y_2, \gamma_2) \overset{S_2}{\rightsquigarrow} \cdots \overset{S_{n-1}}{\rightsquigarrow} (Y_n, \gamma_n)$$

*so that* $(Y_n, \gamma_n)$ *is a product sutured manifold, and so that moreover there is an induced taut foliation on* $(Y, \gamma)$*.*

*Comments on Proof.* Gabai's proof of existence of the sequence of decompositions (*sutured hierarchy*), Theorem 4.2 in his paper, is an intricate induction; even saying what it is an induction on is not easy. Once one has the hierarchy, one uses Lemma 2.24 and its harder cousin for decomposing surfaces intersecting $T(\gamma)$ to reassemble the obvious foliation of the product sutured manifold $(Y_n, \gamma_n)$ to a foliation for $(Y, \gamma)$; this part is Theorem 5.1 in his paper. □

In fact, Theorem 2.25 has two modest refinement:

**Proposition 2.26.** *( [Sch89, Theorem 4.19], see also [Juh08, Theorem 8.2]) With notation as in Theorem 2.25, if* $(Y, \gamma)$ *is balanced then we can assume the surfaces* $S_i$ *are all balanced-admissible.*

**Definition 2.27.** *A balanced-admissible decomposing surface* $S$ *is called* good *if every component of* $\partial S$ *intersects both* $R_+$ *and* $R_-$*. (This is Juhász's term [Juh08, Definition 4.6].)*

**Proposition 2.28.** *[Juh08, Lemma 4.5] Any balanced-admissible decomposing surface* $S$ *is isotopic to a good decomposing surface* $S'$ *so that decomposing along* $S$ *and decomposing along* $S'$ *give the same result. In particular, in Proposition 2.26, we can assume the decomposing surfaces are all good.*

2.4. **Suggested exercises.**

(1) Show, using algebraic topology, that in $S^1 \times \Sigma_g$, the fiber $\Sigma_g$ is a minimal genus representative of its homology class.
(2) Give an explicit description of $\gamma'$ from Definition 2.19.
(3) Prove that if $(Y, \gamma) \overset{S}{\rightsquigarrow} (Y', \gamma')$ and $(Y', \gamma')$ is taut then either $Y$ is taut of $Y = \mathbb{D}^2 \times S^1$ and $S$ is a disk. (This is [Gab83, Lemma 3.5].)

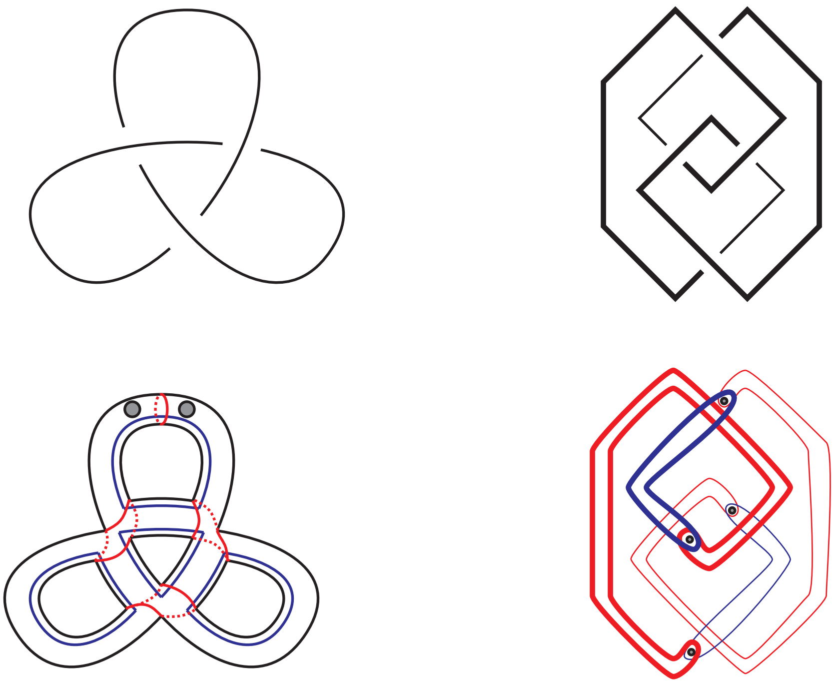


FIGURE 2. **Diagrams for knot complements from knot diagrams.** Left: the usual diagram for the trefoil and a corresponding sutured Heegaard diagram for its exterior. The gray dots are holes in the Heegaard surface. Right: a 2-bridge presentation of the trefoil and a corresponding sutured Heegaard diagram. The surface $\Sigma$ is $S^2$ minus 4 disks. In this Heegaard diagram, the thin red and blue circles are not part of the diagram.

(4) Prove Lemma 2.22.
(5) Prove Lemma 2.24.

## 3. Heegaard diagrams and holomorphic disks

Except as noted, the definitions and theorems in this lecture are all due to Juhász [Juh06] (building on earlier work of Ozsváth-Szabó, Rasmussen, and others). Many of the examples predate his work, but I will state them in his language.

Throughout this lecture, sutured manifold will mean *balanced* sutured manifold.

### 3.1. Heegaard diagrams for sutured manifolds.

**Definition 3.1.** *A* sutured Heegaard diagram *is a surface $\Sigma$ with boundary and tuples $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_n\}$ and $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_n\}$ of pairwise disjoint circles so that the result $F_-$ (respectively $F_+$) of performing surgery on the $\alpha$-circles (respectively $\beta$-circles) has no closed components.*

*A sutured Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ specifies a sutured 3-manifold $Y(\mathcal{H})$ as follows:*

- *As a topological space, $Y(\mathcal{H})$ is obtained from a thickened copy $\Sigma \times [0,1]$ of $\Sigma$ by attaching 2-dimensional 1-handles along the $\alpha_i \times \{0\}$ and the $\beta_i \times \{1\}$.*
- *The boundary of $Y(\mathcal{H})$ is $F_- \cup \big((\partial\Sigma) \times [0,1]\big) \cup F_+$. We let $R_- = F_-$, $R_+ = F_+$ and $\Gamma = (\partial\Sigma) \times \{1/2\}$.*

*Example* 3.2. Fix a knot $K \subset S^3$ and a knot diagram $D$ for $K$ with $n$ crossings. Consider the 3-manifold $Y = S^3 \setminus \mathrm{nbd}(K)$. We can find a Heegaard diagram for $Y$ with $2n$ meridional sutures as on the left of Figure 2. More generally, given an $n$-bridge presentation

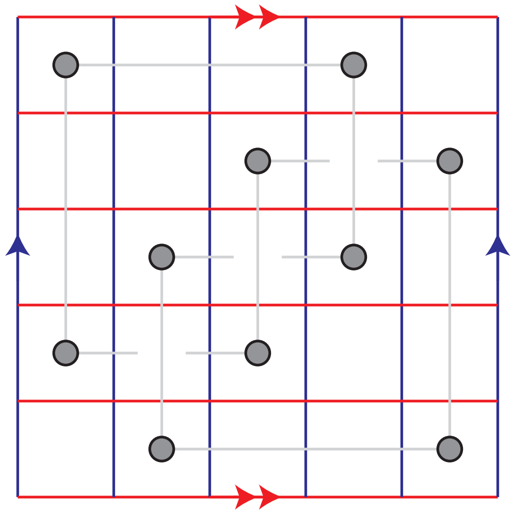


FIGURE 3. **A toroidal grid diagram for the trefoil.** The left and right edges of the diagram are identified. The knot itself is shown in light gray.

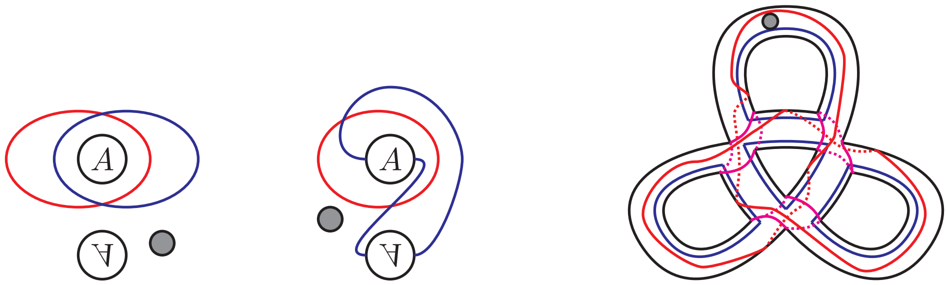



FIGURE 4. **Heegaard diagrams for 3-manifolds with $S^2$ boundary.** Left: a Heegaard diagram for $S^2 \times S^1$. Center: a Heegaard diagram for $\mathbb{R}P^3$. Right: a Heegaard diagram for a surgery on the trefoil. The $\alpha$ circles are red, $\beta$ circles are blue, and the labeled empty circles indicate handles. (So, the first two pictures lie on punctured tori, and the third on a punctured surface of genus 2.

of $K$, there is a corresponding sutured Heegaard diagram for $K$ with $2n$ sutures; see the right of Figure 2.

These kinds of Heegaard diagram are exploited in [OSS09, OS09] to give a cube of resolutions description of knot Floer homology.

*Example* 3.3. An $n \times n$ *toroidal grid diagram* is a special kind of sutured Heegaard diagram in which the $\alpha$-circles (respectively $\beta$-circles) are $n$ horizontal (respectively vertical) circles on a torus with $2n$ disks removed. (Each horizontal (respectively vertical) annulus between two adjacent $\alpha$-circles (respectively $\beta$-circles) should have two punctures.) See Figure 3. A toroidal grid diagram represents the complement of a link in $S^3$, with meridional sutures on the link components. Toroidal grid diagrams have received a lot of attention because, as we will discuss later, their Heegaard Floer invariants have nice combinatorial descriptions [MOS09]; see Section 3.3.3.

*Example* 3.4. Suppose $Y$ is a closed manifold. Fix a *Heegaard splitting* for $Y$, i.e., a decomposition $Y = \mathsf{H}_1 \cup_\Sigma \mathsf{H}_2$, where the $\mathsf{H}_i$ are handlebodies. We can obtain Heegaard

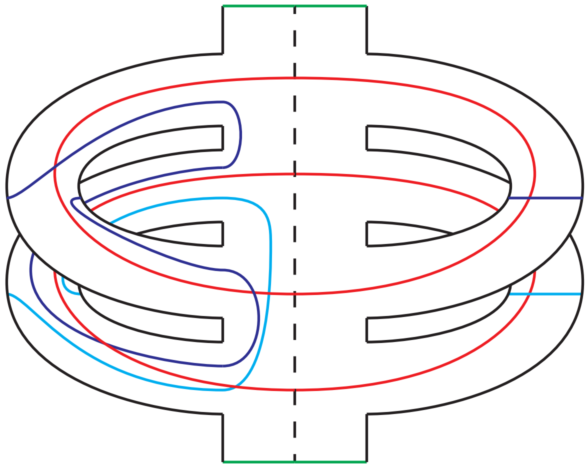


FIGURE 5. **Sutured Heegaard diagrams for fibered knot complements.** This is a Heegaard diagram for the genus 1, fibered knot with monodromy $ab^{-1}$. The $\alpha$-circles are in red and the $\beta$-circles are in blue. The two black arcs in the boundary are meant to be glued together in the obvious way.

diagrams for $Y \setminus \mathbb{D}^3$ as follows. Suppose $\Sigma$ has genus $g$. Fix pairwise-disjoint circles $\alpha_1, \dots, \alpha_g \subset \Sigma$ so that:

- Each $\alpha_i$ bounds a disk in $\mathsf{H}_1$ and
- The $\alpha_i$ are linearly independent in $H_1(\Sigma)$.

Fix circles $\beta_i$ with the same property, but with $\mathsf{H}_2$ in place of $\mathsf{H}_1$. Let $\Sigma'$ be the result of deleting a disk $D$ from $\Sigma$ (chosen so that $D$ is disjoint from the $\alpha_i$ and $\beta_i$). Then $(\Sigma', \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ is a sutured Heegaard diagram for $Y \setminus \mathbb{D}^3$ (with a single suture on the $S^2$ boundary). See Figure 4 for some examples.

In the early days of the subject, these were the only kinds of diagrams considered in Heegaard Floer homology.

*Example* 3.5. Suppose $K$ is a fibered knot in $Y$, with fiber surface $F$ and monodromy $\phi\colon F \to F$. Divide $\partial F$ into two sub-arcs, $A$ and $B$, so that $\partial F = A \cup B$ and $A \cap B = \partial A = \partial B$. Choose $\phi$ so that $\phi(A) = A$ and $\phi(B) = B$.

Choose $2k$ disjoint, embedded arcs $a_1, \dots, a_{2k}$ in $F$ with boundary in $A$, giving a basis for $H_1(F, \partial F)$. Let $b_1, \dots, b_{2k}$ be a set of dual arcs to $a_1, \dots, a_{2k}$, with boundary in $B$. (That is, $a_i$ and $b_i$ intersect transversely in a single point and $a_i \cap b_j = \emptyset$ if $i \neq j$.)

Let $\Sigma = [F \cup (-F)] \setminus \mathrm{nbd}(A \cap B)$ be the result of gluing together two copies of $F$ and deleting a neighborhood of the endpoints $A$. Let $\alpha_i = a_i \cup a_i$ and let $\beta_i = b_i \cup \phi(b_i)$. Then $(\Sigma, \alpha_1, \dots, \alpha_{2k}, \beta_1, \dots, \beta_{2k})$ is a sutured Heegaard diagram for $Y \setminus \mathrm{nbd}(K)$, with two meridional sutures along $K$. See Figure 5.

To see this, let $f\colon (Y \setminus \mathrm{nbd}(K)) \to S^1$ be the fibration. Write $S^1 = [0, \pi] \cup_\partial [\pi, 2\pi]$. We can think of $\Sigma$ as

$$\big(f^{-1}(0)\big) \cup \big(f^{-1}(\pi)\big) \cup \big([0,\pi] \times \partial\,\mathrm{nbd}(A)\big) \cup \big([\pi, 2\pi] \times \partial\,\mathrm{nbd}(B)\big).$$

Use the monodromy along $[0, \pi]$ to identify $F = f^{-1}(0)$ and $-F = f^{-1}(\pi)$. Then each $\alpha_i$ bounds a disk in $f^{-1}([0, \pi])$, and each $\beta_i$ bounds a disk in $f^{-1}([\pi, 2\pi])$.

Notice that the sutured manifolds specified by a Heegaard diagram are balanced. (We could have specified unbalanced ones by allowing the number of $\alpha$ and $\beta$ circles to be

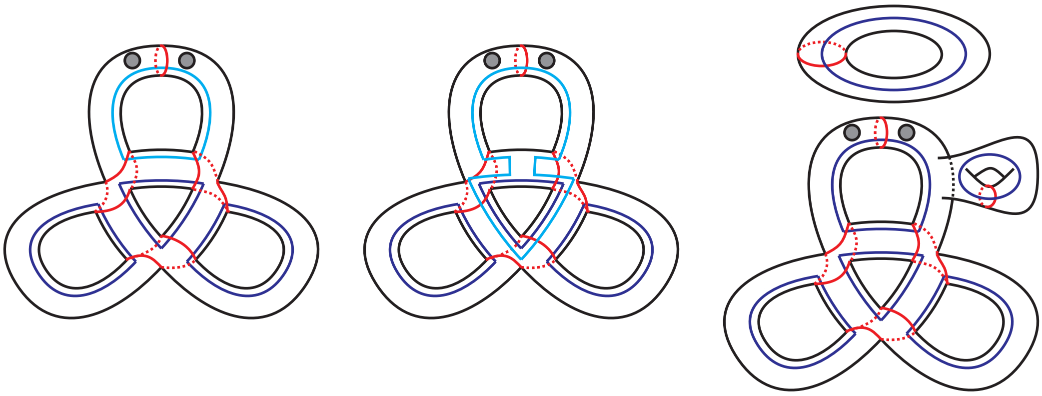


FIGURE 6. **Heegaard moves.** Left: a sutured Heegaard diagram for the trefoil complement. Center: the result of a handleslide among the $\beta$ circles. Right: the standard diagram used in the stabilization move, and the result of a stabilization.

different and dropping our restriction on closed components, but we will not be able to define invariants of such unbalanced diagrams.

**Theorem 3.6.** *Any balanced sutured manifold* $(Y, \Gamma)$ *is represented by a sutured Heegaard diagram.*

*Proof sketch.* We will build a Morse function $f: Y \to \mathbb{R}$ with certain properties and use $f$ to construct the Heegaard diagram. Specifically, we want a Morse function $f$ so that:

1. $f: Y \to [0,3]$.
2. $f^{-1}(0) = R_-^\circ$ and $f^{-1}(3) = R_+^\circ$, where $R_\pm^\circ = R_\pm \setminus \text{nbd}(\Gamma)$.
3. $f$ has no critical points of index 0 or 3.
4. $f$ is *self-indexing*, i.e., for any $p \in \mathit{Crit}(f)$, $f(p) = \text{ind}(p)$.
5. $f|_{\text{nbd}(\Gamma)\subset \partial Y}: \text{nbd}(\Gamma) \cong [0,3] \times \Gamma \to [0,3]$ is projection.

To construct such a Morse function, first define $f$ by hand in a neighborhood of $\partial Y$. Extend $f$ to a Morse function on all of $Y$; this is possible since Morse functions are generic. Finally, move around / cancel critical points to achieve points points (3) and (4); see [Mil65] for a discussion of how to do that.

Fix also a metric $g$, so that $(\nabla f)|_{\text{nbd}(\Gamma)}$ is tangent to $\partial Y$.

Now, the Heegaard diagram is given as follows:

- $\Sigma = f^{-1}(3/2)$.
- The $\alpha$-circles are the ascending (stable) spheres of the index 1 critical points.
- The $\beta$-circles are the descending (unstable) spheres of the index 2 critical points.

It follows from standard results in Morse theory that the resulting Heegaard diagram represents the original sutured manifold; see [Mil65] or [Mil63]. □

We will associate an abelian group $\mathit{SFH}(\mathcal{H})$ to each sutured Heegaard diagram $\mathcal{H}$. To prove that these groups depend only on $Y(\mathcal{H})$ (which we will not actually do), it is useful to have a set of moves connecting any two sutured Heegaard diagrams:

**Theorem 3.7.** *If* $\mathcal{H}$ *and* $\mathcal{H}'$ *represent homeomorphic sutured manifolds then* $\mathcal{H}$ *and* $\mathcal{H}'$ *can be made homeomorphic by a sequence of the following moves:*

- *Isotopies of* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$.
- *Handleslides of one* $\alpha$*-circle over another or one* $\beta$*-circle over another. (See Figure 6.)*

- *Stabilizations and destabilizations, i.e., taking the connected sum with the diagram in Figure 6.*

Again, the proof I know uses Morse theory.

*Remark* 3.8. If one wants to study maps on sutured Floer homology associated to cobordisms [Juh09], one needs a more refined statement than Theorem 3.7. See [JT12].

3.2. **Holomorphic disks in the symmetric product and** $SFH$**.** Brief version:

**Definition 3.9.** *Fix a sutured Heegaard diagram* $(\Sigma, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n)$*. Then* $SFH(\mathcal{H})$ *is the Lagrangian intersection Floer homology of*

$$T_\alpha = \alpha_1 \times \cdots \times \alpha_n,\ T_\beta = \beta_1 \times \cdots \times \beta_n \subset \mathrm{Sym}^n(\Sigma),$$

*the* $n^{th}$ *symmetric product of* $\Sigma$*.*

Longer version:

3.2.1. *Generators.* As its name suggests, the Lagrangian intersection Floer homology is the homology of a complex $SFC(\mathcal{H})$ generated by the intersection points between $T_\alpha$ and $T_\beta$:

$$SFC(\mathcal{H}) = \mathbb{F}_2\langle T_\alpha \cap T_\beta \rangle.$$

Unpacking the definition, a point in $T_\alpha \cap T_\beta$ is an $n$-tuple of points $\{x_i\}_{i=1}^n$, where $x_i \in \alpha_i \cap \beta_{\sigma(i)}$ for some permutation $\sigma \in S_n$.

3.2.2. *Differential.* The differential, unfortunately, is harder: it counts holomorphic disks. Recall that an *almost complex structure* on $M$ is a map $J\colon TM \to TM$ so that $J^2 = -\mathbb{I}$. For instance, given a complex manifold, multiplication by $i$ on the tangent spaces is an almost complex structure.

To count holomorphic disks, one must work with an appropriate almost complex structure $J$ on $\mathrm{Sym}^n(\Sigma)$:

(1) The manifold $\mathrm{Sym}^n(\Sigma)$ can be given a reasonably natural smooth structure, and in fact has a symplectic form $\omega$. Moreover, the form $\omega$ can be chosen so that $T_\alpha$ and $T_\beta$ are Lagrangian.[1] In order to know that the moduli spaces of holomorphic disks are compact (or have nice compactifications) one wants $J$ to be *compatible* with $\omega$, in the sense that $\omega(v, Jw)$ is a Riemannian metric.
(2) One wants $J$ to be generic enough that the moduli spaces of holomorphic disks are transversely cut out.

In practice, one can often work with a *split* almost complex structure. That is, fix an almost complex structure $j$ on $\Sigma$. The almost complex structure $j$ induces an almost complex structure $j^{\times n}$ on $\Sigma^{\times n}$. There is a unique almost complex structure $\mathrm{Sym}^n(j)$ on $\mathrm{Sym}^n(\Sigma)$ so that the projection map $\Sigma^{\times n} \to \mathrm{Sym}^n(\Sigma)$ is $(j^{\times n}, \mathrm{Sym}^n(j))$-holomorphic.

The point of choosing a complex structure is so that we can talk about holomorphic disks in $\mathrm{Sym}^n(\Sigma)$: a continuous map $u\colon \mathbb{D}^2 \to \mathrm{Sym}^n(\Sigma)$ is *$J$-holomorphic* if $J \circ du = du \circ j$ at all interior points of $\mathbb{D}^2$, where $j$ is the almost complex structure on $\mathbb{D}^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$ induced by the complex structure on $\mathbb{C}$.

**Definition 3.10.** *Given* $\mathbf{x}, \mathbf{y} \in T_\alpha \cap T_\beta$*, let* $\mathcal{M}(\mathbf{x}, \mathbf{y})$ *be the set of non-constant* $J$*-holomorphic disks* $u\colon \mathbb{D}^2 \to \mathrm{Sym}^n(\Sigma)$ *so that*

- $u(-i) = \mathbf{x}$,
- $u(+i) = \mathbf{y}$,
- $u(\{z \in \partial\mathbb{D}^2 \mid \Re(z) \geq 0\}) \subset T_\alpha$ *and*

[1] The original formulation of Heegaard Floer avoided using this fact, by a short but clever argument.

- $u(\{z \in \partial\mathbb{D}^2 \mid \Re(z) \leq 0\}) \subset T_\beta$.

*There is an $\mathbb{R}$-action on $\mathcal{M}(\mathbf{x}, \mathbf{y})$, coming from the 1-parameter family of conformal transformations of $\mathbb{D}^2$ fixing $\pm i$. (If we identify $\mathbb{D}^2 \setminus \{\pm i\}$ with $[0,1] \times \mathbb{R}$, this $\mathbb{R}$-action is simply translation in $\mathbb{R}$.)*

**Definition 3.11.** *Suppose that $\mathcal{H}$ represents a RHT sutured 3-manifold. Then define $\partial\colon SFC(Y) \to SFC(Y)$ by*

$$\partial(\mathbf{x}) = \sum_{\mathbf{y}} (\#\mathcal{M}(\mathbf{x}, \mathbf{y})/\mathbb{R})\, \mathbf{y}.$$

*Here, $\#$ denotes the number of elements modulo 2, and if $\mathcal{M}(\mathbf{x}, \mathbf{y})/\mathbb{R}$ is infinite then we declare $\#\mathcal{M}(\mathbf{x}, \mathbf{y})/\mathbb{R} = 0$.*

At first glance, this definition looks hard to use: how does one understand a holomorphic disk in $\mathrm{Sym}^g(\Sigma)$? Somewhat miraculously, these disks often can be understood, as we will see in the next section.

If $\mathcal{H}$ represents a non-RHT sutured 3-manifold, one needs a slightly more complicated definition. Maps $\mathbb{D}^2 \to \mathrm{Sym}^g(\Sigma)$ decompose into homotopy classes (corresponding to elements of $H_2(Y)$), and $\mathcal{M}(\mathbf{x}, \mathbf{y})$ is a disjoint union over homotopy classes $\phi$, $\mathcal{M}(\mathbf{x}, \mathbf{y}) = \amalg_\phi \mathcal{M}^\phi(\mathbf{x}, \mathbf{y})$. One then defines $\partial(\mathbf{x}) = \sum_{\mathbf{y}} \sum_\phi \left(\#\mathcal{M}^\phi(\mathbf{x}, \mathbf{y})/\mathbb{R}\right) \mathbf{y}$, with the same convention about $\#$ as before. One also needs to add a requirement on the sutured Heegaard diagram, called *admissibility*, which ensure that $\#\mathcal{M}^\phi(\mathbf{x}, \mathbf{y}) = 0$ for all but finitely-many homotopy classes $\phi$. (Admissibility is needed to get well-defined invariants even if the counts happen to be finite for other reasons.)

### 3.3. **First computations of sutured Floer homology.**

3.3.1. *Some $n = 1$ examples.* If $n = 1$ we're just looking at disks in $\mathrm{Sym}^1(\Sigma) = \Sigma$.

**Lemma 3.12.** *The 0-dimensional moduli spaces of holomorphic disks in $(\Sigma, \alpha \cup \beta)$ correspond to isotopy classes of orientation-preserving immersions $\mathbb{D} \to \Sigma$ (with boundary as specified in Definition 3.10), with $90^\circ$ corners at $x$ and $y$.*

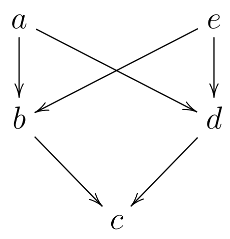
(This follows from the Riemann mapping theorem—exercise.)

Here are some examples.

Consider the diagram in Figure 7. This represents $S^3 \setminus \mathbb{D}^3 = \mathbb{D}^3$, with a single suture on the boundary $S^2$. The complex $SFC(\mathcal{H})$ has five generators, $a, b, c, d, e$. The differential is given by

$$\partial(a) = b + d \qquad \partial(b) = c \qquad \partial(c) = 0$$
$$\partial(d) = c \qquad \partial(e) = b + d$$

or graphically




So,

$$SFH(\mathbb{D}^3) \cong \mathbb{F}_2.$$

See Figure 7 for a hint of why $\partial^2 = 0$, and [LOT12, Section 3.1] for further discussion of this point.

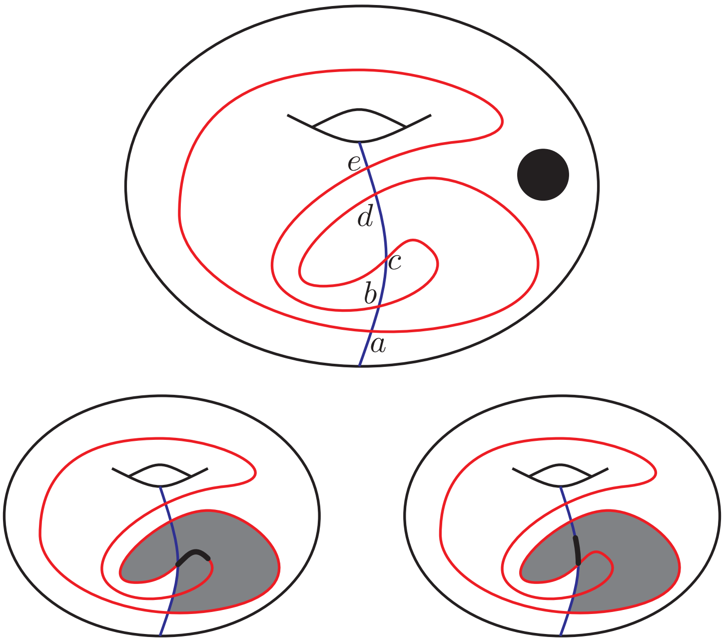



FIGURE 7. **A genus-1 Heegaard diagram for $S^3 \setminus \mathbb{D}^3$.** Top: the diagram, with generators labeled. The big, black disk indicates a hole in $\Sigma$. Bottom: a hint of why $\partial^2 = 0$. This diagram is adapted from [LOT12].

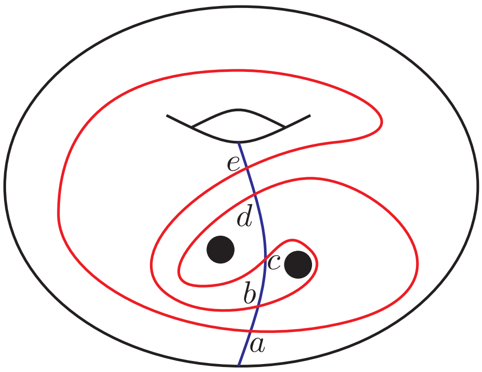



FIGURE 8. **A diagram for the figure-8 complement.** The two black disks indicate holes in $\Sigma$.

Note that the fact that the maps must be orientation-preserving mean that the disk from $a$ to $b$ can *not* be read backwards as a disk from $b$ to $a$.

Next, consider the Heegaard diagram in Figure 8. This is the same as Figure 7, except with different holes. With the new holes, the differential becomes trivial: the disks we counted before now have holes in them. So,

$$SFH(S^3 \setminus 4_1, \Gamma) \cong (\mathbb{F}_2)^5.$$

These examples can be generalized to compute the Floer homology of (the complement of) any 2-bridge knot or, more generally, any (1,1)-knot.

3.3.2. *A stabilized diagram for* $\mathbb{D}^3$. Consider the diagram $\mathcal{H}$ in Figure 9. This diagram again represents $\mathbb{D}^3$, but now has genus 2. The complex $SFC(\mathcal{H})$ has three generators: $\{r, v\}$, $\{s, v\}$ and $\{t, v\}$. (Notice that one of the $\alpha$-circles is disjoint from one of the

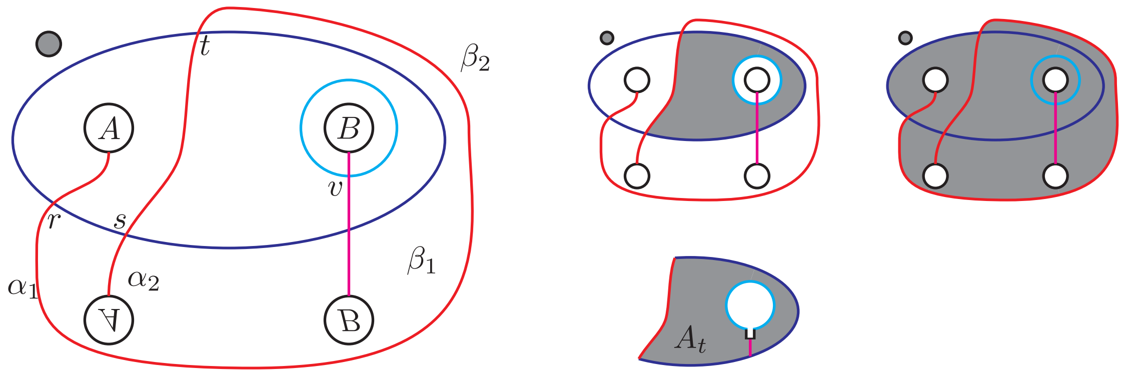



FIGURE 9. **A more complicated diagram for $\mathbb{D}^3$.** The $\alpha$-circles are in red and the $\beta$-circles are in blue; intersection points which form parts of generators are labeled. On the right are two interesting domains, the first from $\{t, v\}$ to $\{s, v\}$ and the second from $\{r, v\}$ to $\{t, v\}$. The annulus $A_t$ is also shown.

$\beta$-circles, reducing the number of generators.) Since $SFH(\mathcal{H}) = SFH(\mathbb{D}^3) = \mathbb{F}_2$, the differential must be nontrivial.

There are no obvious bigons in the diagram (or in $\mathrm{Sym}^1(\Sigma)$), but there is a disk in $\mathrm{Sym}^2(\Sigma)$. Consider the shaded region $A$ in the middle picture in Figure 9. Topologically, $A$ is an annulus; it inherits a complex structure from the complex structure on $\Sigma$. I want to produce a holomorphic map $\mathbb{D}^2 \to \mathrm{Sym}^2(A)$ giving a term $\{s, v\}$ in $\partial\{t, v\}$. Consider the result $A_t$ of cutting $A$ along $\alpha_2$ starting at $v$ for a distance $t$. The key point is the following:

**Lemma 3.13.** *There is (algebraically) one length of cut so that $A_t$ admits a holomorphic involution $\tau$ which takes $\alpha$-arcs to $\alpha$-arcs (and $\beta$-arcs to $\beta$-arcs and corners to corners).*

This is an adaptation of the proof of [OSz04d, Lemma 9.4]. See Exercise 4.

Given Lemma 3.13, we can construct the map $u\colon \mathbb{D}^2 \to \mathrm{Sym}^2(A)$ as follows. The quotient $A_t/\tau$ is analytically isomorphic to $\mathbb{D}^2$, via an isomorphism taking the image of $t$ and one copy of $v$ to $-i$ and the image of $s$ and the other copy of $v$ to $+i$ (and hence the $\alpha$-arc to the right half of $\partial\mathbb{D}^2$). This gives a 2-fold branched cover $u_{\mathbb{D}}\colon A_t \to \mathbb{D}^2$. Now, the map $u$ sends a point $x \in \mathbb{D}^2$ to $u_{\mathbb{D}}^{-1}(x) \in \mathrm{Sym}^2(A)$. It is immediate that $u$ is holomorphic.

This example illustrates an important principal: any holomorphic disk $u\colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to (\mathrm{Sym}^g(\Sigma), T_\alpha \cup T_\beta)$ has a shadow in $\Sigma$, in the form of an element of $H_2(\Sigma, \boldsymbol{\alpha} \cup \boldsymbol{\beta})$ (i.e., a cellular 2-chain). This shadow is called the *domain* of the disk $u$. The multiplicity of the domain $D(u)$ at a point $p \in \Sigma$ is given by the intersection number $u \cdot [\{p\} \times \mathrm{Sym}^{g-1}(\Sigma)]$.

Note that the domain has multiplicity 0 near $\partial\Sigma$. Moreover, it follows from *positivity of intersections* [MW95] that the coefficients in the domain of a holomorphic $u$ are always non-negative (at least if one works with an almost complex structure on $\mathrm{Sym}^g(\Sigma)$ which is close to a split one, or agrees with a split one on an appropriate subset of $\mathrm{Sym}^g(\Sigma)$; in Heegaard Floer theory one always makes this restriction). Finally, the domain has a particular kind of behavior near the generators connected by $u$: if $u$ connects $\mathbf{x}$ to $\mathbf{y}$ then $\partial(\partial D(u) \cap \boldsymbol{\alpha}) = \mathbf{y} - \mathbf{x} = -\partial(\partial D(u) \cap \boldsymbol{\beta})$.

From these observations, it is fairly easy to see that the only other possible domain of a holomorphic curve is shown on the far right of Figure 9. This domain connects $\{r, v\}$ to $\{t, v\}$. But a curve in this homotopy class would violate $\partial^2 = 0$, so the algebraic number of such curves is 0.

It turns out that one can read the dimension of the moduli space of disks from the domain $D(u)$: see [Lip06, Corollary 4.10].

Of course, in general, computations are more complicated: domains do not need to be planar (the domain in the right of Figure 9 is not planar), and branched covers of degree greater than 2 are harder to analyze. Because direct computations are so hard, there has been a lot of interest in both theoretical and practical techniques for computing Heegaard Floer homology.

3.3.3. *Grid diagrams.* Consider a toroidal grid diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, and let $n$ be the number of $\alpha$-circles (which is, of course, also the number of $\beta$-circles). Since each $\alpha_i$ intersects each $\beta_j$ in a single point, the generators $\{x_i \in \alpha_i \cap \beta_{\sigma(i)}\}$ correspond to the permutations $\sigma \in S_n$. (This correspondence is not quite canonical, since we are using the indexing of the $\alpha$-circles and $\beta$-circles.)

Next, consider two generators $\mathbf{x}$ and $\mathbf{y}$ such that:

- $\mathbf{x} \cap \mathbf{y}$ consists of $(n-2)$ points.
- There is a rectangle $r$ in $\Sigma$ so that the lower-left and upper-right corners of $r$ are $\mathbf{x} \setminus \mathbf{y}$, and the upper-left and lower-right corners of $r$ are $\mathbf{y} \setminus \mathbf{x}$. (This is a meaningful statement.)
- The interior of $r$ is disjoint from $\mathbf{x}$.

We will say that $\mathbf{x}$ and $\mathbf{y}$ are *connected by an empty rectangle*, and call $r$ an *empty rectangle from* $\mathbf{x}$ *to* $\mathbf{y}$.

Given an empty rectangle $r$, we can find a holomorphic disk (with respect to the split complex structure) with domain $r$ as follows. First, there is a unique holomorphic 2-fold branched cover $u_{\mathbb{D}} \colon r \to \mathbb{D}^2$ sending the $\mathbf{x}$-corners of $r$ to $-i$ and the $\mathbf{y}$-corners of $r$ to $+i$; see Exercise 14. (This map automatically sends the $\alpha$-boundary of $r$ to the right half of $\partial\mathbb{D}^2$ and the $\beta$-boundary to the left half.) Since the preimage of any point in $\mathbb{D}^2$ is two points in $r$ (counted with multiplicity—the branch point is a multiplicity-2 point), we can view $(u_{\mathbb{D}})^{-1}$ as a map $\mathbb{D}^2 \to \mathrm{Sym}^2(r)$. There is an inclusion $\mathrm{Sym}^2(r) \hookrightarrow \mathrm{Sym}^2(\Sigma) \hookrightarrow \mathrm{Sym}^g(\Sigma)$, where the second inclusion sends $p$ to $p \times (\mathbf{x} \cap \mathbf{y})$. (Remember: $p$ is a pair of points in $\Sigma$, and $\mathbf{x} \cap \mathbf{y}$ is an $(n-2)$-tuple of points in $\Sigma$, so $p \times (\mathbf{x} \cap \mathbf{y})$ is an $n$-tuple of points in $\Sigma$. Forgetting the ordering gives a point in $\mathrm{Sym}^n(\Sigma)$.)

Amazingly, these are the only relevant holomorphic curves in the grid diagram:

**Theorem 3.14.** *[MOS09] The rigid holomorphic disks in a toroidal grid diagram correspond exactly to the empty rectangles. In particular, the differential on $SFC(\mathcal{H})$ counts empty rectangles in $(\Sigma, \boldsymbol{\alpha} \cup \boldsymbol{\beta})$.*

The proof turns out not to be especially hard: it uses an index formula and some combinatorics to show that the domain of a rigid holomorphic curve in a toroidal grid diagram must be a rectangle. The result, however, is both surprising and useful.

A similar construction is possible for other 3-manifolds [SW10]. There is also a forthcoming textbook about grid diagrams and Floer homology [OSS].

3.4. **First properties.**

**Theorem 3.15.** *The map $\partial \colon SFC(\mathcal{H}) \to SFC(\mathcal{H})$ satisfies $\partial^2 = 0$.*

This follows from "standard techniques". The differential $\partial$ is defined by counting 0-dimensional moduli spaces of disks. The coefficient of $\mathbf{z}$ in $\partial^2(\mathbf{x})$ is given by $\# \amalg_{\mathbf{y}} \mathcal{M}(\mathbf{x}, \mathbf{y}) \times \mathcal{M}(\mathbf{y}, \mathbf{z})$. One shows that $\mathcal{M}(\mathbf{x}, \mathbf{z})$ is the interior of a compact 1-manifold with boundary $\amalg_{\mathbf{y}} \mathcal{M}(\mathbf{x}, \mathbf{y}) \times \mathcal{M}(\mathbf{y}, \mathbf{z})$; it follows that $\amalg_{\mathbf{y}} \mathcal{M}(\mathbf{x}, \mathbf{y}) \times \mathcal{M}(\mathbf{y}, \mathbf{z})$ consists of an even number of points. The proof that $\mathcal{M}(\mathbf{x}, \mathbf{z})$ has the desired structure boils down to three parts:

(1) A transversality statement, that for a generic almost complex structure, $\mathcal{M}(\mathbf{x}, \mathbf{z})$ is a smooth manifold.
(2) A compactness statement, that any sequence of disks in $\mathcal{M}(\mathbf{x}, \mathbf{z})$ converges either to a holomorphic disk or a broken holomorphic disk.
(3) A gluing statement, that near any broken holomorphic disk one can find an honest holomorphic disk (and, in fact, that near a broken disk the space of honest disks is a 1-manifold).

**Theorem 3.16.** *Up to isomorphism, $SFH(\mathcal{H})$ depends only on the (isomorphism class of the) sutured 3-manifold $(Y, \Gamma)$ represented by $\mathcal{H}$.*

The proof, which is similar to the invariance proof in [OSz04d], is broken into three parts: invariance under isotopies and change of almost complex structure; invariance under handleslides; and invariance under stabilization. Stabilization is easy: it suffices to stabilize near a boundary component, in which case the two complexes are isomorphic. Isotopy invariance follows from standard techniques in Floer theory: one considers moduli spaces of disks with boundary on a family of moving Lagrangians. Handleslide invariance is a little more complicated—one uses counts of certain holomorphic triangles (rather than bigons) to define the relevant maps—but fits rather nicely with the modern philosophy of Fukaya categories.

3.4.1. *Decomposition according to* $\mathrm{spin}^c$ *structures.* Notice in the example of $S^3 \setminus (4_1)$ that there were generators not connected by any topological disk (immersed or otherwise). This relates to the notion of $\mathrm{spin}^c$-structures.

**Definition 3.17.** *Fix a sutured manifold $(Y, \Gamma)$. Call a vector field $v$ on $Y$* well-behaved *if:*

- *$v$ is non-vanishing.*
- *On $R_+$, $v$ points out of $Y$.*
- *On $R_-$, $v$ points into $Y$.*
- *Along $\Gamma$, $v$ is tangent to $\partial Y$ (and points from $R_-$ to $R_+$).*

(The term "well-behaved" is not standard.)

**Definition 3.18.** *Fix $Y$ connected and a ball $\mathbb{D}^3$ in the interior of $Y$. We say well-behaved vector fields $v$ and $w$ on $Y$ are* homologous *if $v|_{Y \setminus \mathbb{D}^3}$ and $w|_{Y \setminus \mathbb{D}^3}$ are isotopic (through well-behaved vector fields). This is (obviously) an equivalence relation. Let $\mathrm{spin}^c(Y, \Gamma)$ denote the set of homology classes of vector fields; we refer to elements of $\mathrm{spin}^c(Y, \Gamma)$ as* $\mathrm{spin}^c$-structures *on $Y$. For $Y$ disconnected we define $\mathrm{spin}^c(Y, \Gamma) = \prod_i \mathrm{spin}^c(Y_i, \Gamma_i)$, where the product is over the connected components of $Y$.*

Juhász's Definition 3.18 is inspired by Turaev's work [Tur97] (and, of course, the analogous construction in the closed case from [OSz04d]).

**Lemma 3.19.** *$\mathrm{spin}^c(Y, \Gamma)$ is a torseur for (affine copy of) $H_1(Y) \cong H^2(Y, \partial Y)$.*

The first reason $\mathrm{spin}^c$ structures are of interest to us is the following:

**Lemma 3.20.** *There is a map $\mathfrak{s}\colon T_\alpha \cap T_\beta \to \mathrm{spin}^c(Y)$ with the property that $\mathfrak{s}(\mathbf{x}) = \mathfrak{s}(\mathbf{y})$ if and only if $\mathbf{x}$ and $\mathbf{y}$ can be connected by a bigon (Whitney disk) in $(\mathrm{Sym}^g(\Sigma), T_\alpha, T_\beta)$.*

The map $\mathfrak{s}$ is not hard to construct from the Morse theory picture. Start with the gradient vector field $\nabla f$. A generator $\mathbf{x}$ specifies an $n$-tuple $\{\eta_i\}$ of flow lines connecting the index 1 and 2 critical points. The vector field $\nabla f|_{Y \setminus \mathrm{nbd}\{\eta_i\}}$ extends to a non-vanishing vector field on all of $Y$ (easy exercise), which in turn specifies the $\mathrm{spin}^c$-structure $\mathfrak{s}(\mathbf{x})$.

**Corollary 3.21.** *$SFH(Y,\Gamma)$ decomposes as a direct sum over* spin$^c$ *structures on $Y$:*

$$SFH(Y,\Gamma) = \bigoplus_{\mathfrak{s}\in\mathrm{spin}^c(Y,\Gamma)} SFH(Y,\Gamma,\mathfrak{s}).$$

In fact, $SFH(Y,\Gamma)$ has a grading by homotopy classes of well-behaved vector fields. There is a free $\mathbb{Z}$-action on the set of homotopy classes of well-behaved vector fields, so that the quotient is the set of spin$^c$-structures. If $Y$ is RHT, this action is free, which we can abbreviate as:

$$0 \to \mathbb{Z} \to \{\text{well-behaved vector fields}\}/\text{isotopy} \to \mathrm{spin}^c(Y,\Gamma) \to 0.$$

The differential on $SFC(Y,\Gamma)$ changes the "$\mathbb{Z}$-component" of this grading by 1 (and leaves the "spin$^c(Y,\Gamma)$ component" unchanged, of course). See [RH11, RH12] for more details.

3.4.2. *Definition of $\widehat{HF}$ and $\widehat{HFK}$.* A few important special cases predated sutured Floer homology, and so have their own names:

- For $Y$ a closed 3-manifold, $\widehat{HF}(Y) \coloneqq SFH(Y\setminus\mathbb{D}^3,\Gamma)$, where $\Gamma$ consists of a single circle on $S^2$. This is one of Ozsváth-Szabó's original *Heegaard Floer homology* groups, from [OSz04d].
- For $K$ a nullhomologous knot in a closed manifold $Y$, $\widehat{HFK}(Y,K) \coloneqq SFH(Y\setminus \mathrm{nbd}(K),\Gamma)$, where $\Gamma$ consists of two meridional sutures. The group $\widehat{HFK}(Y,K)$ is (one variant of) the *knot Floer homology group* of $K$, and was introduced by Ozsváth-Szabó [OSz04b] and Rasmussen [Ras03]. In the special case $Y = S^3$, $\widehat{HFK}(Y,K)$ is often denoted simply by $\widehat{HFK}(K)$.

  For $Y = S^3$, spin$^c(Y\setminus\mathrm{nbd}(K),\Gamma)\cong\mathbb{Z}$ (canonically). So, $\widehat{HFK}(K)$ decomposes:

  $$\widehat{HFK}(K) = \bigoplus_j \widehat{HFK}(K,j).$$

  The integer $j$ is called the *Alexander grading.*

  There is also a $\mathbb{Z}$-valued homological grading, the *Maslov grading.* Further,

  $$\sum_{i,j}(-1)^i t^j \dim \widehat{HFK}_i(K,j) = \Delta_K(t),$$

  the Alexander polynomial of $K$. (Here, $i$ denotes the Maslov grading.)
- For $L$ a link in $Y$ each of whose components is nullhomologous, $\widehat{HFL}(Y,L) \coloneqq SFH(Y\setminus\mathrm{nbd}(L),\Gamma)$, where $\Gamma$ consists of two meridional sutures on each component of $\partial\,\mathrm{nbd}(L)$. Again, in the special case $Y = S^3$, one often writes simply $\widehat{HFL}(L)$. The group $\widehat{HFL}(Y,L)$ is (one variant of) the *link Floer homology* of $L$, and was introduced in [OSz08].

Some other, less well-studied variants also predated sutured Floer homology. For example, (one variant of) Eftekhary's *longitude Floer homology* [Eft05] corresponds to the sutured Floer homology of a knot complement with two longitudinal sutures.

3.4.3. *Product sutured manifolds.* If $(Y,\Gamma)$ is a product sutured manifold then we can take $\Sigma = R_- = R_+$, with 0 $\alpha$ and $\beta$ circles. In this rather degenerate case, $\mathrm{Sym}^0(\Sigma)$ is a single point, and $T_\alpha$ and $T_\beta$ are each a single point as well, giving $SFC(Y,\Gamma) = \mathbb{F}_2$ with trivial differential. There is also a unique spin$^c$ structure on $(Y,\Gamma)$. Thus:

**Lemma 3.22.** *For $(Y,\Gamma)$ a product sutured manifold, $SFH(Y,\Gamma) = \mathbb{F}_2$, supported in the unique* spin$^c$ *structure on $(Y,\Gamma)$.*

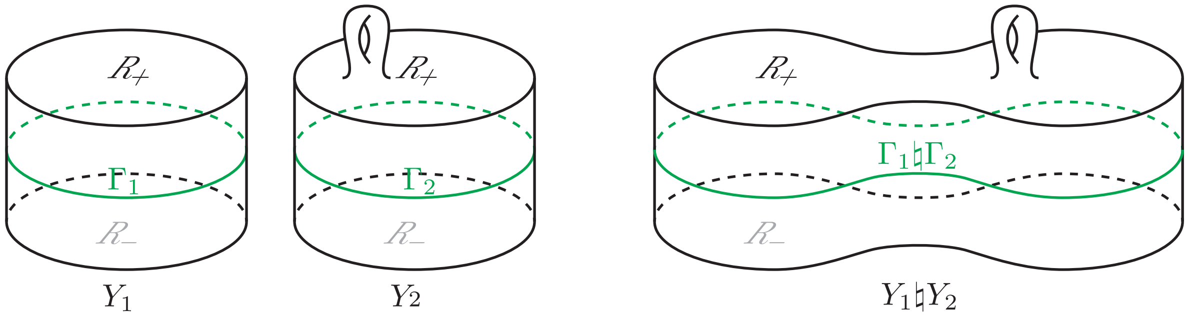



FIGURE 10. **The sutured manifold structure on the boundary sum.**

(If you are uncomfortable with $\mathrm{Sym}^0$, stabilize the diagram once. The computation remains trivial.)

3.4.4. *Product decompositions, disjoint unions, boundary sums and excess $S^2$ boundary components.* In the next lecture we will discuss how sutured Floer homology behaves under surface decompositions; this behavior is key to its utility. As a simple special case, however, consider a product decomposition $(Y, \Gamma) \overset{D}{\rightsquigarrow} (Y', \Gamma')$. One can find a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ for $(Y, \Gamma)$ with the following properties:

(1) $D \cap \Sigma$ consists of a single arc $\delta$ such that
(2) $\delta$ is disjoint from $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

(See [Juh06, Lemma 9.13].) Cutting $\Sigma$ along $\delta$ gives a sutured Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ for $(Y', \Gamma')$. With respect to these sutured Heegaard diagrams, there is an obvious correspondence between generators of $SFC(Y, \Gamma)$ and $SFC(Y', \Gamma')$. Moreover, since the domain of any holomorphic curve has multiplicity 0 near $\partial\Sigma$, it follows that this identification intertwines the differentials on $SFC(\mathcal{H})$ and $SFC(\mathcal{H}')$. (A little argument, using positivity of intersections, is needed here.) Thus:

**Proposition 3.23.** *If $(Y, \Gamma)$ and $(Y', \Gamma')$ are related by a product decomposition then $SFH(Y, \Gamma) \cong SFH(Y', \Gamma')$.*

In a slightly different direction, suppose we have sutured manifolds $(Y_1, \Gamma_1)$ and $(Y_2, \Gamma_2)$. The disjoint union $(Y_1 \amalg Y_2, \Gamma_1 \amalg \Gamma_2)$ is again a sutured manifold. Moreover, if $\mathcal{H}_i$ is a Heegaard diagram for $(Y_i, \Gamma_i)$ then $\mathcal{H}_1 \amalg \mathcal{H}_2$ is a Heegaard diagram for $(Y_1 \amalg Y_2, \Gamma_1 \amalg \Gamma_2)$. The symmetric product $\mathrm{Sym}^{g_1+g_2}(\Sigma_1 \amalg \Sigma_2)$ decomposes as $\amalg_{i+j=g_1+g_2} \mathrm{Sym}^i(\Sigma_1) \times \mathrm{Sym}^j(\Sigma_2)$, but the Heegaard tori lie in the component $\mathrm{Sym}^{g_1}(\Sigma_1) \times \mathrm{Sym}^{g_2}(\Sigma_2)$. So (choosing an appropriate almost complex structure), we get an isomorphism of chain complexes

$$SFC(\mathcal{H}_1 \amalg \mathcal{H}_2) \cong SFC(\mathcal{H}_1) \otimes SFC(\mathcal{H}_2).$$

Thus:

**Proposition 3.24.** $SFH(Y_1 \amalg Y_2, \Gamma_1 \amalg \Gamma_2) \cong SFH(Y_1, \Gamma_1) \otimes SFH(Y_2, \Gamma_2)$.

Next, suppose that $(Y_1, \Gamma_1)$ and $(Y_2, \Gamma_2)$ are sutured manifolds and $\mathcal{H}_1$ and $\mathcal{H}_2$ are associated Heegaard diagrams. Fix a point $p_i \in \partial\Sigma_i$, corresponding to a point $q_i \in \partial Y_i$. Then we can form the boundary sum $\mathcal{H}_1 \natural \mathcal{H}_2$ of $\mathcal{H}_1$ and $\mathcal{H}_2$ at the points $p_1$ and $p_2$. The diagram $\mathcal{H}_1 \natural \mathcal{H}_2$ represents the boundary sum of $Y_1$ and $Y_2$, which inherits a sutured manifold structure; see Figure 10. The manifold $Y_1 \natural Y_2$ differs from the disjoint union $Y_1 \amalg Y_2$ by a product decomposition, so:

**Corollary 3.25.** $SFH(Y_1 \natural Y_2, \Gamma) \cong SFH(Y_1, \Gamma_1) \otimes SFH(Y_2, \Gamma_2)$.

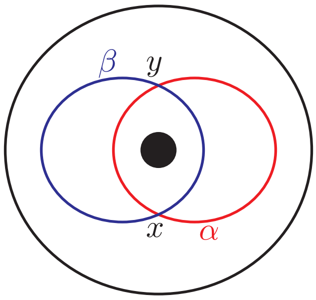



FIGURE 11. **A Heegaard diagram for** $(Y_1, \Gamma_1)$**.** The surface $\Sigma$ is an annulus, and there is a single $\alpha$ circle and a single $\beta$ circle running around the hole.

(Of course, this is also easy to prove directly.)

Now, consider the special case of $(Y_1 = [0,1] \times S^2, \Gamma_1)$, where $\Gamma_1$ consists of a single suture on each boundary component. A Heegaard diagram for $(Y_1, \Gamma_1)$ is shown in Figure 11. Here, $\mathit{SFC}(Y_1, \Gamma_1)$ has two generators, $x$ and $y$, and there are two disks from $x$ to $y$. (This computation is easy, since we are in the first symmetric product.) Consequently, $\partial(x) = 2y = 0$, so $\mathit{SFH}(Y_1, \Gamma_1) = (\mathbb{F}_2)^2$.

Notice that taking the boundary sum with $(Y_1, \Gamma_1)$ has the effect of introducing a new $S^2$ boundary component, with a single suture. So, we have:

**Corollary 3.26.** *Let* $(Y, \Gamma)$ *be a sutured manifold and let* $(Y', \Gamma')$ *be the result of deleting a* $\mathbb{D}^3$ *from the interior of* $Y$ *and placing a single suture on the resulting boundary component. Then* $\mathit{SFH}(Y', \Gamma') \cong \mathit{SFH}(Y, \Gamma) \otimes (\mathbb{F}_2)^2$.

*Remark* 3.27. At first glance, one might expect that $\mathit{SFH}(Y_1, \Gamma_1)$ vanishes, as one can find a Heegaard diagram in which $\alpha$ and $\beta$ are disjoint. Note that $(Y_1, \Gamma_1)$ is not RHT, so one is in the more complicated situation described at the end of Section 3.2.2. The need to work with an admissible Heegaard diagram is the reason $\mathit{SFH}(Y_1, \Gamma_1) \neq 0$; but see also Exercise 11.

## 3.5. Excess meridional sutures.

**Proposition 3.28.** *If* $(Y', \Gamma')$ *is obtained from* $(Y, \Gamma)$ *by replacing a suture on a toroidal boundary component with three parallel sutures then* $\mathit{SFH}(Y', \Gamma') \cong \mathit{SFH}(Y, \Gamma) \otimes (\mathbb{F}_2)^2$.

**Corollary 3.29.** *If* $\mathcal{H}$ *is a grid diagram for* $K$ *with* $n$ $\alpha$*-circles then*

$$\mathit{SFH}(\mathcal{H}) \cong \widehat{\mathit{HFK}}(S^3, K) \otimes (\mathbb{F}_2)^{2n}.$$

Probably this follows from [Juh08, Lemma 8.9] or [Juh08, Proposition 8.6], though I have not thought it through carefully. This is probably a good exercise. See also [OSz08] and [MOS09].

## 3.6. Suggested exercises.

(1) Convince yourself that Figure 8 does, in fact, represent the complement of the figure-eight knot, with two meridional sutures.
(2) Convince yourself that Figure 9 represents $\mathbb{D}^3$ (with one suture on the boundary).
(3) Generalize Example 3.5 to the case of fibered links. What if we want a diagram for $Y \setminus \mathbb{D}^3$, where $Y$ is the 3-manifold in which the knot (or link) $K$ lies, rather than $Y \setminus \mathrm{nbd}(K)$?
(4) Prove Lemma 3.13.

(5) Show that, given a link $L$ in $S^3$, there is a toroidal grid diagram representing $S^3 \setminus \text{nbd}(L)$, with some number of meridional sutures on each component of $L$. Explicitly find toroidal grid diagrams for $(p, q)$ torus knots.
(6) Use grid diagrams to compute *SFH* for the complement of the unknot with 4 meridional sutures and 6 meridional sutures, and the complement of the Hopf link with 4 meridional sutures on each component.
(7) Compute *SFH* for the complements of some other 2-bridge knots.
(8) State Lemma 3.12 precisely, and prove it.
(9) Prove Lemma 3.19.
(10) The group $\text{spin}^c(3)$ is isomorphic to $U(2)$. There is a map $\text{spin}^c(3) = U(2) \to SO(3)$ given by dividing out by $S^1 = \left\{\left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{smallmatrix}\right)\right\}$.
The usual definition of a $\text{spin}^c$ structure is a principal $\text{spin}^c(3)$-bundle $P$ over $Y$, and a bundle map from $P$ to the bundle of frames of $Y$, respecting the actions of $\text{spin}^c(3)$ and $SO(3)$ in the obvious sense. (This uses the homomorphism $\text{spin}^c(3) \to SO(3)$ above.) Identify this definition with Definition 3.18.
(11) Suppose that $(Y, \Gamma)$ is a sutured manifold so that one component of $\partial Y$ is a sphere with $n > 1$ sutures. Prove that $SFH(Y, \Gamma) = 0$. (Warning: to give an honest proof, you probably need to know something about admissibility conditions.)
(12) Find a genus 1 Heegaard diagram for the lens space $L(p, q)$ (or, from the sutured perspective, $L(p, q) \setminus \mathbb{D}^3$). Use this diagram to compute $\widehat{HF}(L(p, q)) = SFH(L(p, q) \setminus \mathbb{D}^3)$.
(13) Which surgery on the trefoil is shown in Figure 4?
(14) Let $r$ be a rectangle in the plane, i.e., a topological disk with boundary consisting of four smooth arcs. Show that there is a unique holomorphic 2-fold branched cover $r \to \mathbb{D}^2$ sending the corners to $\pm i$. (Hint: start by applying the Riemann mapping theorem. Then use the fact that branched double covers $r \to \mathbb{D}^2$ correspond to involutions of $r$.)

## 4. Surface decompositions and sutured Floer homology

Recall that to each balanced sutured manifold $(Y, \Gamma)$ we have associated an $\mathbb{F}_2$-vector space $SFH(Y, \Gamma)$. Moreover, $SFH(Y, \Gamma)$ is a direct sum over (relative) $\text{spin}^c$-structures on $(Y, \Gamma)$,

$$SFH(Y, \Gamma) = \bigoplus_{\mathfrak{s} \in \text{spin}^c(Y, \Gamma)} SFH(Y, \Gamma, \mathfrak{s}).$$

**Theorem 4.1.** *[Juh08, Theorem 1.3] Let $(Y, \Gamma)$ be a balanced sutured and $(Y, \Gamma) \overset{S}{\rightsquigarrow} (Y', \Gamma')$ a sutured manifold decomposition. Suppose that $S$ is good (Definition 2.27). Then*

$$SFH(Y', \Gamma') \cong \bigoplus_{\mathfrak{s} \in O(S)} SFH(Y, \Gamma, \mathfrak{s}).$$

(In fact, Theorem 4.1 holds with "good" replaced by "balanced-admissible".)

The notation $O(S)$ needs explanation. A $\text{spin}^c$ structure is called *outer* with respect to $S$ if it can be represented by a (non-vanishing) vector field $v$ which is never equal to $-\nu_S$, the (negative) normal vector field to $S$. $O(S)$ denotes the set of outer $\text{spin}^c$ structures. (This definition can be rephrased in terms of relative Chern classes; see [Juh08].)

A key step in proving Theorem 4.1 is to study Heegaard diagrams adapted to the surface decomposition:

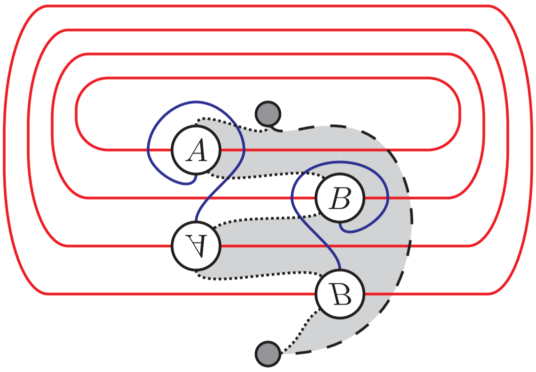



FIGURE 12. **A sutured Heegaard diagram adapted to a decomposing surface.** This is a Heegaard diagram for the complement of the figure 8 knot, and $S(P)$ is a minimal-genus Seifert surface for the Figure 8 knot. The polygon $P$ is shaded. $A$ is the single dashed arc and $B$ is the single dotted arc.

**Definition 4.2.** *Fix a sutured manifold* $(Y, \Gamma)$ *and a decomposing surface* $S$ *in* $Y$*. By a* Heegaard diagram for $(Y, \Gamma)$ adapted to $S$ *we mean a sutured Heegaard diagram* $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ *for* $(Y, \Gamma)$ *together with a subsurface* $P \subset \Sigma$ *with the following properties:*

(1) *The boundary of* $P$ *is the union* $A \cup B$ *where* $A$ *and* $B$ *are disjoint unions of smooth arcs.*
(2) $\partial A = \partial B = A \cap B \subset \partial\Sigma$.
(3) $A \cap \boldsymbol{\beta} = \emptyset$ *and* $B \cap \boldsymbol{\alpha} = \emptyset$.
(4) *Let* $S(P) = P \cup A \times [1/2, 1] \cup B \times [0, 1/2] \subset Y(\mathcal{H}) = Y$*. Then* $S(P)$ *is isotopic to* $S$*, where each intermediate surface in the isotopy is a decomposing surface.*

*See Figure 12.*

**Proposition 4.3.** *[Juh08, Proposition 4.4] Let* $(Y, \Gamma)$ *be a balanced sutured manifold and* $S$ *a good decomposing surface for* $(Y, \Gamma)$*. Then there is a sutured Heegaard diagram for* $(Y, \Gamma)$ *adapted to* $S$*.*

The proof is similar to (but somewhat more intricate than) the proof of Theorem 3.6. We will return to the proof of Theorem 4.1 in Section 4.2.

4.1. **Application: knot genus, (Thurston norm, fiberedness).** We recall Theorem 1.1:

**Theorem 4.4.** *[OSz04a, Theorem 1.2]* $\widehat{HFK}(S^3, K)$ *detects the genus of* $K$*. Specifically*

$$g(K) = \max\{j \mid \widehat{HFK}_*(K, j) \neq 0\}.$$

Similarly:

**Theorem 4.5.** *( [HN10, Theorem 2.2], building on [OSz04a, Theorem 1.1]) For* $Y^3$ *closed,* $\widehat{HF}(Y)$ *detects the Thurston norm: for* $h \in H_2(Y)$*,*

$$x(h) = \max\{\langle c_1(\mathfrak{s}), h\rangle \mid \widehat{HF}(Y, \mathfrak{s}) \neq 0\}.$$

Here, $c_1(\mathfrak{s})$ denotes the first Chern class of the spin$^c$-structure $\mathfrak{s}$ (which is the same as the Euler class of the 2-plane field orthogonal to $\mathfrak{s}$, if we think of $\mathfrak{s}$ as a vector field). Ozsváth-Szabó proved this result for a twisted version of Heegaard Floer homology; Hedden-Ni deduce the untwisted statement using the universal coefficient theorem.

**Theorem 4.6.** *( [Ni07, Theorem 1.1], building on [Ghi08, Theorem 1.4]) $\widehat{HFK}(K)$ detects fibered knots: $S^3 \setminus K$ fibers over $S^1$ if and only if $\sum_i \dim \widehat{HFK}_i(K,j) = 1$.*

Ni's result is, in fact, more general than Theorem 4.6. There is an analogous statement for closed 3-manifolds; see [Ni09].

In the rest of this section, we will sketch a proof of Theorem 1.1. First, the easy direction:

**Proposition 4.7.** *[OSz04b] Fix a Seifert surface $F$ for a knot $K$ in $S^3$. Then $\widehat{HFK}_*(K,j) = 0$ if $j < -g(F)$ or $j > g(F)$ (where $g(F)$ is the genus of $F$).*

*Proof sketch.* We can view $F$ as a good decomposing surface for $(S^3 \setminus \mathrm{nbd}(K), \Gamma)$, where $\Gamma$ consists of two meridional sutures. Choose a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, P)$ adapted to $F$. It turns out that the Alexander grading of a generator $\mathbf{x} \in \widehat{HFK}(K)$ is given by $|\mathbf{x} \cap P| - g(F)$, where $|\mathbf{x} \cap P|$ denotes the number of points in $\mathbf{x} \cap P$ and $g(F)$ is the genus of $F$. (This is, in fact, fairly close to the original definition of the Alexander grading in [OSz04b].) It follows that the Alexander grading is bounded below by $-g(F)$. For the upper bound we use a symmetry: $\widehat{HFK}_i(K,j) \cong \widehat{HFK}_{i-2j}(K,-j)$ [OSz04b, Proposition 3.10]. □

*Remark* 4.8. In the special case of fibered knots, Proposition 4.7 can also be proved using the Heegaard diagram from Example 3.5. (See also [HKM09].) That construction can be generalized to give a diagram for Proposition 4.7 in general. The resulting diagrams are, I think, examples of the ones used in this proof (i.e., they are sutured Heegaard diagrams adapted to the Seifert surface). One can also prove Proposition 4.7 using grid diagrams; see [OSS].

In the proof of Proposition 4.7, there are no generators of the chain complex $SFC$ in Alexander grading $< g$. For the diagrams discussed in the previous paragraph, there are also no generators in Alexander grading $> g$. My guess is that this will be true in general (for diagrams adapted to a Seifert surface), but I have not thought it through; perhaps you can prove it (it should not be hard if true) or give a counterexample.

*Proof of Theorem 1.1.* After Proposition 4.7, it remains to show that $\widehat{HFK}_*(K, -g(K)) \neq 0$. Let $Y_0$ denote the exterior of $K$ and let $\Gamma_0$ be two meridional sutures on $\partial Y$. Fix a minimal-genus Seifert surface $F$ for $K$. View $F$ as a decomposing surface for $Y_0$ (with $\partial F$ intersecting each suture once). Let $(Y_1, \Gamma_1)$ be the result of a surface decomposition of $(Y_0, \Gamma_0)$ along $F$. Since $F$ was minimal genus, the resulting sutured manifold is taut. By Theorem 4.1, $SFH(Y_1, \Gamma_1) \cong \bigoplus_{\mathfrak{s} \in O(F)} SFH(Y, \Gamma, \mathfrak{s})$. A short argument, similar to the argument omitted in the proof of Proposition 4.7, shows that $\bigoplus_{\mathfrak{s} \in O(F)} SFH(Y, \Gamma, \mathfrak{s}) = \widehat{HFK}_*(K, -g(K))$.

So, by Theorem 4.1, it suffices to show that $SFH(Y_1, \Gamma_1)$ is nontrivial. By Proposition 2.28, we can find a sequence of sutured manifold decompositions

$$(Y_1, \Gamma_1) \overset{S_1}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (Y_n, \Gamma_n)$$

where each $S_i$ is good and $(Y_n, \Gamma_n)$ is a product sutured manifold. By Lemma 3.22, $SFH(Y_n, \Gamma_n) = \mathbb{F}_2$. So, applying Theorem 4.1 $n$ times, $(Y_1, \Gamma_1)$ has an $\mathbb{F}_2$ summand. □

Juhász's proof, which we have sketched, of Theorem 1.1 is quite different from Ozsváth-Szabó's original proof. Theorem 4.6 can also be proved using sutured Floer homology, though the argument is more intricate, and close in spirit to Ni's original proof. Apparently, at the time of writing there is no known proof of Theorem 4.5 via sutured Floer homology.

4.2. **Sketch of proof of Theorem 4.1.** We will sketch the proof from [GW10], rather than Juhász's original proof from [Juh08]. Juhász's original proof, which uses Sarkar-Wang's *nice diagrams* [SW10], is technically simpler. Grigsby-Wehrli's proof has the advantage that it is more natural (in a sense they make precise). It is also closer in spirit to bordered Heegaard Floer theory, a subject I am particularly interested in.

I find it somewhat easier to think about the argument in the "cylindrical" formulation of Heegaard Floer homology [Lip06]. This generalizes the description of holomorphic maps $\mathbb{D}^2 \to \mathrm{Sym}^2(\Sigma)$ used in Sections 3.3.2 and 3.3.3. Specifically:

**Proposition 4.9.** *With respect to a split complex structure on* $\mathrm{Sym}^g(\Sigma)$*, there is a correspondence between holomorphic maps*

$$v\colon (\mathbb{D}^2, \partial\mathbb{D}^2 \cap \{\Re(z) \geq 0\}, \partial\mathbb{D}^2 \cap \{\Re(z) \leq 0\}) \to (\mathrm{Sym}^g(\Sigma), T_\alpha, T_\beta) \tag{4.10}$$

*and diagrams*

$$\begin{array}{ccc} (S, \partial_a S, \partial_b S) & \xrightarrow{\quad u_\Sigma \quad} & (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}) \\ \Big\downarrow{\scriptstyle u_{\mathbb{D}}} & & \\ (\mathbb{D}^2, \partial\mathbb{D}^2 \cap \{\Re(z) \geq 0\}, \partial\mathbb{D}^2 \cap \{\Re(z) \leq 0\}) & & \end{array} \tag{4.11}$$

*where $S$ is a Riemann surface with boundary $\partial S = \partial_a S \cup \partial_b S$; $u_\Sigma$ and $u_{\mathbb{D}}$ are holomorphic; and $u_{\mathbb{D}}$ is a $g$-fold branched cover.*

*Sketch of proof.* Given a diagram of the form (4.11) we get a map $\mathbb{D}^2 \to \mathrm{Sym}^g(\Sigma)$ by sending a point $p \in \mathbb{D}^2$ to $u_\Sigma(u_{\mathbb{D}}^{-1}(p))$ (which is $g$ points in $\Sigma$, counted with multiplicity, or equivalently a point in $\mathrm{Sym}^g(\Sigma)$). To go the other way, note that there is a branched cover $\pi\colon \Sigma \times \mathrm{Sym}^{g-1}(\Sigma) \to \mathrm{Sym}^g(\Sigma)$ gotten by forgetting the ordering between the $(g-1)$-tuple of points in $\Sigma$ and the one additional point. This is a $g$-fold branched cover. Given $v\colon \mathbb{D}^2 \to \mathrm{Sym}^g(\Sigma)$ as in Formula (4.10) we can pull back the branched cover $\pi$ to get a branched cover $u_{\mathbb{D}}\colon S \to \mathbb{D}^2$. The surface $S$ comes equipped with a map to $\Sigma \times \mathrm{Sym}^{g-1}(\Sigma)$, and projecting to $\Sigma$ gives $u_\Sigma$:

$$\begin{array}{ccccc} & & & & \Sigma \\ & & & {\scriptstyle u_\Sigma} \nearrow & \uparrow {\scriptstyle \pi_\Sigma} \\ S & \longrightarrow & \Sigma \times \mathrm{Sym}^{g-1}(\Sigma) & & \\ \Big\downarrow{\scriptstyle u_{\mathbb{D}}} & & \Big\downarrow{\scriptstyle \pi} & & \\ \mathbb{D}^2 & \xrightarrow{\ v\ } & \mathrm{Sym}^{g-1}(\Sigma). & & \end{array}$$

It is fairly straightforward to prove that both constructions give holomorphic maps (of the specified forms) and that the two constructions are inverses of each other. See [Lip06, Section 13] for more details (though this idea is not due to me). □

**Lemma 4.12.** *Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, P)$ be a sutured Heegaard diagram adapted to a decomposing surface. If $\mathbf{y}$ occurs as a term in $\partial(\mathbf{x})$ then $|\mathbf{x} \cap P| = |\mathbf{y} \cap P|$.*

Lemma 4.12 follows from various results about spin$^c$-structures, but it is also fairly easy to prove directly; see Exercise 7. In fact, a slightly stronger statement holds: if there is a domain connecting $\mathbf{x}$ to $\mathbf{y}$ then $|\mathbf{x} \cap P| = |\mathbf{y} \cap P|$.

**Lemma 4.13.** *With notation as in Lemma 4.12, a generator $\mathbf{x}$ represents an outer* spin$^c$*-structure if and only if $\mathbf{x} \cap P = \emptyset$.*

*Proof of Theorem 4.1.* Fix a Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ for $(Y, \Gamma)$ adapted to $S$ (so there is a distinguished surface $P \subset \Sigma$). Let $\mathcal{H}' = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ be the sutured Heegaard diagram obtained as follows. Topologically,

$$\Sigma' = \big(\Sigma \setminus \mathrm{int}(P)\big) \amalg P \amalg P / \sim$$

where $\sim$ identifies the subset $A$ of $\partial\big(\Sigma \setminus \mathrm{int}(P)\big)$ with $A$ in the boundary of the first copy $P_A$ of $P$, and the subset $B$ of $\partial\big(\Sigma \setminus \mathrm{int}(P)\big)$ with $B$ in the boundary of the second copy $P_B$ of $P$. There is a projection map $\pi\colon \Sigma' \to \Sigma$, which is 2-to-1 on $P$ and 1-to-1 elsewhere. There is a unique lift $\boldsymbol{\alpha}'$ of the curves $\boldsymbol{\alpha}$ from $\Sigma$ to $\Sigma'$: the lifted curves are disjoint from $P_B$. Similarly, there is a unique lift $\boldsymbol{\beta}'$ of the curves $\boldsymbol{\beta}$, and this lift is disjoint from $P_A$.

Let

$$SFC_P(\mathcal{H}) = \langle \{\mathbf{x} \mid \mathbf{x} \cap P = \emptyset\} \rangle \subset SFC(\mathcal{H}).$$

By Lemma 4.12, $SFC_P(\mathcal{H})$ is a subcomplex—in fact, a direct summand—of $SFC(\mathcal{H})$.

By Lemma 4.13, an equivalent formulation of Theorem 4.1 is:

*There is an isomorphism* $SFH_P(\mathcal{H}) \cong SFH(\mathcal{H}')$.

This statement has two advantages: it is more concrete (so we can prove it), and it does not make reference to spin$^c$ structures (which we have not discussed much). It has the disadvantage that it is not intrinsic—it talks about diagrams, not sutured manifolds.

Notice that $\boldsymbol{\alpha}' \cap \boldsymbol{\beta}'$ corresponds (via the projection $\pi$) to $\boldsymbol{\alpha} \cap \boldsymbol{\beta} \cap (\Sigma \setminus P)$. This induces an identification of generators between $SFC_P(\mathcal{H})$ and $SFC(\mathcal{H}')$. We will show that for an appropriate choice of complex structure, this identification intertwines the differentials. (As usual, we are suppressing transversality issues and assuming we can work with split almost complex structures.)

Working in the cylindrical formulation (see Proposition 4.9), suppose $\mathbf{x}$ and $\mathbf{y}$ are generators of $SFC_P(\mathcal{H})$ and that $\mathbf{y}$ occurs in $\partial(\mathbf{x})$. Then there is a diagram $\mathbb{D} \xleftarrow{u_{\mathbb{D}}} S \xrightarrow{u_\Sigma} \Sigma$ as in Formula (4.11). We want to produce a similar diagram, but in $\mathcal{H}'$.

The idea is to insert long necks in $\Sigma$ along $A$ and $B$, or equivalently, to pinch $A$ and $B$, decomposing $\Sigma$ into two parts: $P/\partial P$ and $\Sigma/P$. (The argument is similar to the first part of the argument in [LOT08, Chapter 9].) Consider a sequence of curves $u_i = (u_{\mathbb{D},i}, u_{\Sigma,i})$ as above, with respect to a sequence of neck lengths converging to $\infty$.

*Claim 1.* As $A$ and $B$ collapse, one can find a subsequence of the $u_i$ so that:

- The surfaces $S_i$ converge to a nodal Riemann surface $S_\infty$.
- $S_\infty$ has two components, $S_\infty^P$ and $S_\infty^\Sigma$, attached at a collection of boundary points (nodes).
- The maps $u_i$ converge to holomorphic maps

$$u_{\mathbb{D},\infty}^\Sigma\colon S_\infty^\Sigma \to \mathbb{D}^2 \qquad u_{\mathbb{D},\infty}^P\colon S_\infty^P \to \mathbb{D}^2$$
$$u_{\Sigma,\infty}^\Sigma\colon S_\infty^\Sigma \to \Sigma \qquad u_{\Sigma,\infty}^P\colon S_\infty^P \to P.$$

- The maps $u_{\mathbb{D},\infty}^P$ and $u_{\mathbb{D},\infty}^\Sigma$ send each side of each node to the same point in $\partial\mathbb{D}^2$; that is, $u_{\mathbb{D},\infty}$ extends continuously over the nodes.
- At each node, $u_{\Sigma,\infty}^P$ and $u_{\Sigma,\infty}^\Sigma$ map to an arc between two $\alpha$- or $\beta$-circles and, further, both sides of the node map to the same such arc.

See Figure 13 for a schematic example.

Claim 1 is a version of Gromov's compactness theorem [Gro85] (see also [BEH$^+$03]), though the fact that we are considering maps between surfaces make it considerably easier than the general case.

*Claim 2.* Near any limiting surface as in Claim 1 there is a sequence of holomorphic curves converging to it.

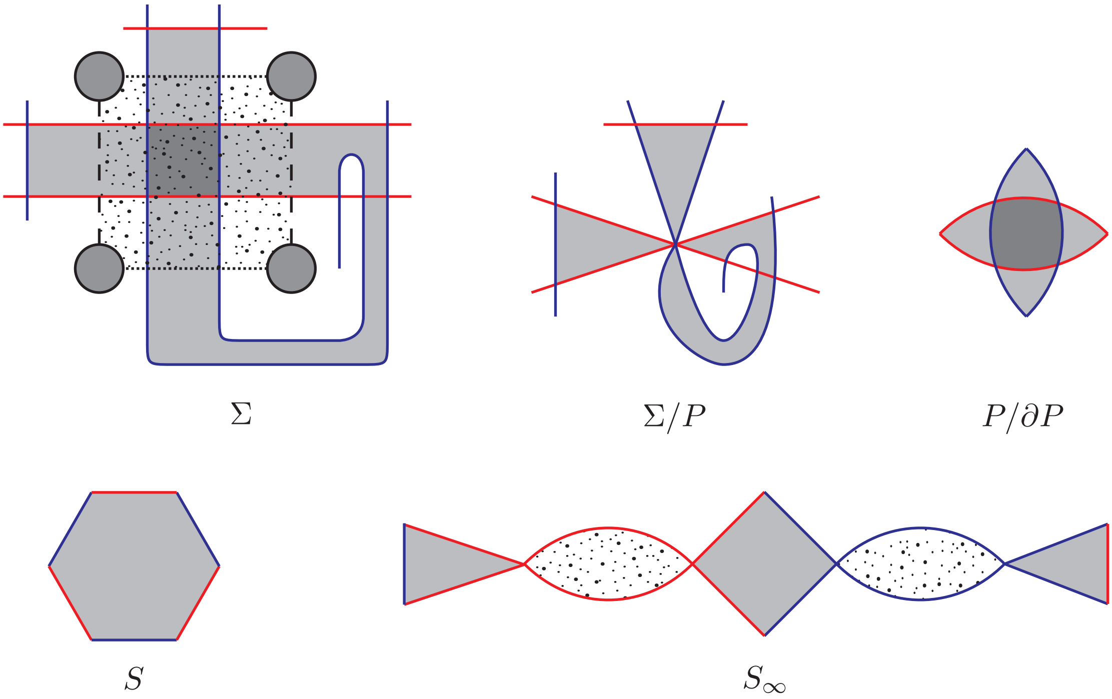



FIGURE 13. **A holomorphic curve after degeneration.** The domain of $u$ is shaded, and $P$ is speckled. In the domain of $u$, the darkly-shaded part is covered twice. In $P/\partial P$, the four corners are identified. The conformal structure of $S$ is not (usually) the one indicated. In $S_\infty$, $S_\infty^\Sigma$ is shaded and $S_\infty^P$ is speckled.

Claim 2 is called a gluing theorem. (Again, the fact that we are looking at maps between surfaces means this is a reasonably simple case.)

Together, Claims 1 and 2 mean that we can use this degenerated surface to compute the differential on $SFC_P$.

*Claim 3.* The surface $S_\infty^P$ consists of a disjoint union of bigons (disks with two boundary nodes). The map $u_\Sigma^P$ sends each bigon to a strip in $P$, with boundary either two $\alpha$-circles or two $\beta$-circles. The map $u_\mathbb{D}^P$ is constant on each bigon.

Notice that Claim 3 implies that $u_\Sigma^P$ and $u_\mathbb{D}^P$ can be reconstructed from $u_\Sigma^\Sigma$ and $u_\mathbb{D}^\Sigma$, and that $u_\Sigma^P$ and $u_\mathbb{D}^P$ exist if and only if $u_\Sigma^\Sigma$ and $u_\mathbb{D}^\Sigma$ satisfy certain easy-to-state properties (Exercise 8).

Similar results hold for holomorphic curves in $\Sigma'$, after collapsing the arcs $A$ and $B$ there. The difference is that we now have three components: $\Sigma/P$, $P_A/A$ and $P_B/B$. The analogue of Claim 3 says:

*Claim 3′.* Each of the surfaces $S_\infty^{P_A}$ and $S_\infty^{P_B}$ consists of a disjoint union of bigons. The map $u_\Sigma^{P_A}$ sends each bigon to a strip in $P_A$, with boundary on two $\alpha$-circles. The map $u_\Sigma^{P_B}$ sends each bigon to a strip in $P_B$, with boundary on two $\beta$-circles. The map $u_\mathbb{D}^P$ is constant on each bigon.

Again, Claim 3′ implies that the curves $u_\Sigma^{P_A}$, $u_\mathbb{D}^{P_A}$, $u_\Sigma^{P_B}$ and $u_\mathbb{D}^{P_B}$ can be reconstructed from $u_\Sigma^\Sigma$ and $u_\mathbb{D}^\Sigma$. In particular, there is an identification between the curves in Claim 3 and the curves in Claim 3′. Since we can use these degenerated curves to compute the differentials on $SFC_P(\mathcal{H})$ and $SFC(\mathcal{H}')$, this completes the proof. □

4.3. **Some open questions.** Here are some questions about sutured Floer homology which I think are open, and which I would find interesting to have answered. (Whether or not anyone else would find them interesting I cannot say.)

(1) Can one give a proof of Theorem 4.5 using sutured Floer homology? In this context [KM10, Section 7.8] seems relevant.
(2) Further explore the constructions in [AE11], or other "minus" variants of sutured Floer homology.
(3) What can be said about the next-to-outer spin$^c$ structures? What topological information do they contain? (Perhaps the pairing theorem in [Zar09] is relevant.)

4.4. **Suggested exercises.**

(1) Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, P)$ be a sutured Heegaard diagram adapted to a decomposing surface $S$. How does one compute the genus of $S = S(P)$?
(2) Deduce Proposition 3.23 from Theorem 4.1.
(3) Convince yourself that Figure 12 represents the figure 8 knot, and that $S(P)$ is a minimal genus Seifert surface. (Hint: the figure 8 knot is fibered with monodromy $ab^{-1}$.) Give a sutured Heegaard diagram for the trefoil complement, adapted to a minimal genus Seifert surface.
(4) What is the relationship between the proof of the "easy direction" of Theorem 1.1, i.e., Proposition 4.7, that we gave and the proof in [OSz04b]? (That is, are the diagrams used in [OSz04b] examples of diagrams adapted to the Seifert surface $F$ in the sense of Definition 4.2?)
(5) Prove that if $K$ is a fibered knot then $\mathit{HFK}_*(K, -g(K)) \cong \mathbb{F}_2$ (i.e., the easy direction of Theorem 4.6).
(6) Fill in the details in the proof of Proposition 4.9.
(7) Prove Lemma 4.12.
(8) In the proof of Theorem 4.1, say precisely what properties $S^\Sigma_\infty$, $u^\Sigma_\Sigma$ and $u^\Sigma_\mathbb{D}$ must satisfy for the surface $S^P_\infty$ and the maps $u^P_\Sigma$ and $u^P_\mathbb{D}$ to exist. (See the discussion immediately after Claim 3.)

## 5. Miscellaneous further remarks

The main goal of this lecture is to draw some connections with the lecture series on Khovanov homology. As a side benefit, I will mention another nice applications of Heegaard Floer homology (mostly without proof, unfortunately).

For most of this talk we will focus on the invariant $\widehat{\mathit{HF}}(Y) = \mathit{SFH}(Y \setminus \mathbb{D}^3, \Gamma)$ associated to a closed 3-manifold $Y$, as in Section 3.4.2.

5.1. **Surgery exact triangle.** A *framed knot* in a 3-manifold $Y$ is a knot $K \subset Y$ together with a slope $n$ (isotopy class of essential simple closed curves) on $\partial\,\mathrm{nbd}(K)$. Given a framed knot $(K, n)$ we can do *surgery* on $(K, n)$ by gluing a thickened disk (3-dimensional 2-handle) to $Y \setminus \mathrm{nbd}(K)$ along $n$, and then capping the resulting $S^2$ boundary component with a $\mathbb{D}^3$. Let $Y_n(K)$ denote the result of doing surgery to $Y$ along $(K, n)$.

**Theorem 5.1.** *[OSz04c] Let $n$, $n'$ and $n''$ be slopes in $\partial\,\mathrm{nbd}(K)$ whose intersection numbers satisfy*

$$n \cdot n' = n' \cdot n'' = n'' \cdot n = 1.$$

*Then there is an exact triangle*

$$\begin{array}{ccc} & \widehat{HF}(Y_n(K)) & \\ \nearrow & & \searrow \\ \widehat{HF}(Y_{n''}(K)) & \longleftarrow & \widehat{HF}(Y_{n'}(K)). \end{array}$$

In fact, the same result holds for $\widehat{HF}(Y)$ replaced by $SFH(Y,\Gamma)$ for any sutured 3-manifold $(Y,\Gamma)$. The original proof (from [OSz04c]) extends to this case. This also follows immediately from the exact triangle for bordered solid tori [LOT08, Section 11.2] together with Zarev's bordered-sutured theory—particularly [Zar10, Theorem 3.10].

There is always a distinguished slope for $K$, the *meridian*, which bounds a disk in $\mathrm{nbd}(K)$. If $K \subset S^3$ then $K$ also has a well-defined *longitude*, a slope which is nullhomologous in $S^3 \setminus \mathrm{nbd}(K)$. For knots in $S^3$, therefore, we can identify slopes with rational numbers, by declaring that $p/q$ corresponds to $p$ times the meridian plus $q$ times the longitude.

5.2. **Lens space surgery.** Theorem 5.1 has many applications. It is a central tool in computations of 3-manifold invariants; see for instance [JM08] for an intricate example.

In a different direction, let us consider 3-manifolds $Y$ for which $\widehat{HF}(Y)$ is trivial. First, we must decide what we mean by "trivial". To start, we have:

**Theorem 5.2.** *[OSz04c, Proposition 5.1] Given a torsion* $\mathrm{spin}^c$ *structure* $\mathfrak{s}$ *on* $Y$, $\chi(\widehat{HF}(Y,\mathfrak{s})) = \pm 1$. *In particular, for* $Y$ *a rational homology sphere (meaning* $H_1(Y;\mathbb{Q}) = 0$*), there is a choice of absolute* $\mathbb{Z}/2\mathbb{Z}$ *grading so that* $\chi(\widehat{HF}(Y)) = |H_1(Y)|$*, the number of elements of* $H_1(Y)$.

Suppose $Y$ is a rational homology sphere. Saying $\widehat{HF}(Y)$ is trivial, then, should mean $\dim(\widehat{HF}(Y)) = \chi(\widehat{HF}(Y)) = |H_1(Y)|$. In this case, we say that $Y$ is an *$L$-space*. The terminology comes from the fact that the lens spaces $L(p,q)$ are all $L$-spaces: $|H_1(L(p,q))| = p$ and $\widehat{HF}(L(p,q)) \cong (\mathbb{F}_2)^p$ (Exercise 12).

As a first application of Theorem 5.1, we have:

**Corollary 5.3.** *Let* $S^3_n(K)$ *denote $n$-surgery on the knot $K$. If* $S^3_n(K)$ *is an $L$-space then so is* $S^3_m(K)$ *for any* $m > n$.

*Proof.* By induction, it suffices to prove that $S^3_{n+1}(K)$ is an $L$-space. Applying Theorem 5.1 to the slopes $n$, $n+1$ and $\infty$, we see that $\dim \widehat{HF}(S^3_{n+1}(K)) \leq n+1$. Theorem 5.2 gives the opposite inequality, proving the result. □

$L$-spaces are fairly rare, though many examples are known. (For example, the branched double cover of any alternating link is an $L$-space [OSz05]; this follows from the techniques in Section 5.3.)

Via Theorem 5.1 and its refinements (like the surgery formulas from [OS08, OS11]), one can give restrictions on which surgeries can yield $L$ spaces and, in particular, lens spaces. Perhaps the most dramatic example (so far) is a theorem of Kronheimer-Mrowka-Ozsváth-Szabó, originally proved using monopole (Seiberg-Witten) Floer homology:

**Theorem 5.4.** *[KMOSz07] Suppose that for some* $p/q \in \mathbb{Q}$, $S^3_{p/q}(K)$ *is orientation-preserving diffeomorphic to the lens space* $L(p,q)$*. Then* $K$ *is the unknot.*

Many cases of Theorem 5.4 were already known; see the introduction to [KMOSz07] for a discussion of the history.

Note that there are nontrivial knots $K$ in $S^3$ admitting lens space surgeries. For instance, $S^3_{pq+1}(T_{p,q}) = L(pq+1, q^2)$. A number of other knots (called *Berge knots*) are known to have lens space surgeries, and many others have $L$-space surgeries (see, for instance, [HLV14], and its references). So, the following is *false*: if $\widehat{HF}(S^3_{p/q}(K)) \cong \widehat{HF}(S^3_{p/q}(U))$ then $K = U$. In particular, the proof of Theorem 5.4 needs (at least) one more ingredient.

In Theorem 5.4, the case of 0-surgeries is called the *property R conjecture*, and was proved by Gabai [Gab87]. So, to prove Theorem 5.4 for $q = 1$, say, it suffices to show that if $S^3_p(K) = L(p, 1)$ then $S^3_{p-1}(K) = L(p-1, 1)$. (This is the opposite direction of induction from Corollary 5.3.)

To accomplish this downward induction, one can either use the *absolute* $\mathbb{Q}$-*grading* on $\widehat{HF}(Y)$, as in the original proof of Theorem 5.4 or, for a quicker proof, the surgery formula from [OS08]. (In fact, the latter proof is sufficiently simple that Theorem 5.4 makes a good exercise when learning the surgery formula.)

Theorem 5.4 is one of everyone's favorite application of low-dimensional Floer theories (though it does have some stiff competition), and so gets mentioned a lot. For some other striking applications to lens space surgeries, which are fairly accessible from the discussion in these lectures, see for instance [OS05b].

5.3. **The spectral sequence for the branched double cover.** As another application of the surgery exact triangle (and related techniques), we discuss a relationship between Heegaard Floer homology and Khovanov homology:

**Theorem 5.5.** *[OSz05, Theorem 1.1] For any link $L$ in $S^3$ there is a spectral sequence $\widetilde{Kh}(m(L)) \Rightarrow \widehat{HF}(\Sigma(L))$.*

Here, $\widetilde{Kh}(m(L))$ denotes the reduced Khovanov homology of the mirror of $L$. The manifold $\Sigma(L)$ is the double cover of $S^3$ branched along $L$. That is, the meridians of $L$ define a canonical isomorphism $H_1(S^3 \setminus \mathrm{nbd}(L)) \cong \mathbb{Z}^{|L|}$. The composition

$$\pi_1(S^3 \setminus \mathrm{nbd}(L)) \to H_1(S^3 \setminus \mathrm{nbd}(L)) = \mathbb{Z}^{|L|} \to \mathbb{Z}/2\mathbb{Z},$$

where the last map sends each basis vector (i.e., meridian) to 1, defines a connected double cover $\tilde{Y}$ of $S^3 \setminus \mathrm{nbd}(L)$. The boundary of $\tilde{Y}$ is a union of tori. Each of these tori has a distinguished meridian—the total preimage of a meridian of the corresponding component of $L$. Filling in these meridians with thickened disks and the resulting $S^2$ boundary components with $\mathbb{D}^3$'s gives the double cover of $S^3$ branched along $L$.

Theorem 5.5 has received a lot of attention. See [LOT10, Section 1.2] for references to related work. In particular, Kronheimer-Mrowka later used similar ideas to prove that Khovanov homology detects the unknot [KM11].

*Sketch of Proof of Theorem 5.5.* The relationship between Theorem 5.5 and Theorem 5.1 comes from the following observation: let $L$ be a link diagram, $c$ a crossing in $L$, and $L_0$ and $L_1$ the two resolutions of $L$ at $c$, as in Figure 14. Let $\gamma$ be the (vertical) arc in $\mathbb{R}^3$ with boundary on $L$ lying above $c$. The total preimage $\widetilde{\gamma}$ of $\gamma$ in $\Sigma(L)$ is a circle $K$. There are surgery slopes $n$ and $n'$ on $\partial\,\mathrm{nbd}(K)$ so that $n$ (respectively $n'$) surgery on $K$ gives $L_1$ (respectively $L_0$), and $(\infty, n, n')$ satisfy the conditions of Theorem 5.1. Thus, Theorem 5.1 gives a long exact sequence

$$\cdots \to \widehat{HF}(\Sigma(L)) \to \widehat{HF}(\Sigma(L_1)) \to \widehat{HF}(\Sigma(L_0)) \to \widehat{HF}(\Sigma(L)) \to \cdots.$$

There is an analogous skein sequence for Khovanov homology. This skein relation does not characterize Khovanov homology but, as we will see, it almost implies the existence of a spectral sequence of the desired form.

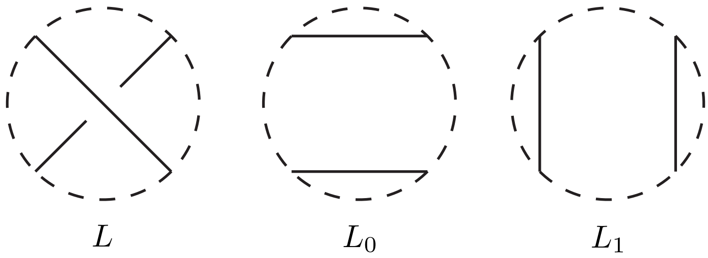



FIGURE 14. **Resolutions of a crossing.** The conventions agree with [Kho00], not [OSz05].

To proceed, we actually need a slight strengthening of Theorem 5.1: with notation as in that theorem, there is a short exact sequence of chain complexes

$$0 \to \widehat{CF}(Y_{n'}(K)) \to \widehat{CF}(Y_{n''}(K)) \to \widehat{CF}(Y_n(K)) \to 0 \tag{5.6}$$

(for appropriately chosen Heegaard diagrams). Further, this surgery triangle is local in the following sense. Fix two disjoint knots $K$ and $L$ in $Y$, and framings $m, m', m''$ for $K$ and $n, n', n''$ for $K'$ as in Theorem 5.1. Let $Y_{m,n}(K \cup L)$ be the result of performing $m$ surgery on $K$ and $n$ surgery on $L$. Then there is a homotopy-commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \widehat{CF}(Y_{m',n'}(K\cup L)) & \xrightarrow{f_{00,10}} & \widehat{CF}(Y_{m'',n'}(K\cup L)) & \to & \widehat{CF}(Y_{m,n'}(K\cup L)) & \to & 0 \\
 & & \downarrow{\scriptstyle f_{00,01}} & & \downarrow{\scriptstyle f_{10,11}} & & \downarrow & & \\
0 & \to & \widehat{CF}(Y_{m',n''}(K\cup L)) & \xrightarrow{f_{01,11}} & \widehat{CF}(Y_{m'',n''}(K\cup L)) & \to & \widehat{CF}(Y_{m,n''}(K\cup L)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \widehat{CF}(Y_{m',n}(K\cup L)) & \to & \widehat{CF}(Y_{m'',n}(K\cup L)) & \to & \widehat{CF}(Y_{m,n}(K\cup L)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{5.7}$$

where the rows and columns are the exact sequences of Equation 5.6. Here, homotopy-commutative means, for instance, that there is a map $f_{00,11} \colon \widehat{CF}(Y_{m',n'}(K \cup L)) \to \widehat{CF}(Y_{m'',n''}(K \cup L))$ so that

$$\partial \circ f_{00,11} + f_{00,11} \circ \partial = f_{01,11} \circ f_{00,01} + f_{10,11} \circ f_{00,10}.$$

The natural analogue holds for surgeries on a link of $n > 2$ components, as well.

As an algebraic corollary, we have the fact that

$$\widehat{CF}(Y_{m,n}(K \cup L)) \simeq \mathrm{Cone}\left(\begin{array}{ccc}
\widehat{CF}(Y_{m',n'}(K\cup L)) & \xrightarrow{f_{00,10}} & \widehat{CF}(Y_{m'',n'}(K\cup L)) \\
\downarrow{\scriptstyle f_{00,01}} & \searrow^{f_{00,11}} & \downarrow{\scriptstyle f_{10,11}} \\
\widehat{CF}(Y_{m',n''}(K\cup L)) & \xrightarrow{f_{01,11}} & \widehat{CF}(Y_{m'',n''}(K\cup L))
\end{array}\right); \tag{5.8}$$

see Exercise 2. Here, the Cone means that we take the whole diagram and view it as a complex. That is, take the direct sum of the complexes at the four vertices, and use the differentials on the complexes and maps between them to define a differential. For instance, if $x \in \widehat{CF}(Y_{m',n'}(K \cup L))$ then the differential of $x$ is given by $\partial_{CF}(x) + f_{00,10}(x) + f_{00,01}(x) + f_{00,11}(x)$. (If we were not working over $\mathbb{F}_2$, there would be some signs.) Again, the analogous results hold for a link with more than two components.

Notice that the complex in Formula 5.8 has an obvious filtration. The terms in the associated graded complex are $\widehat{CF}(Y_{m',n'}(K \cup L))$, $\widehat{CF}(Y_{m'',n'}(K \cup L)) \oplus \widehat{CF}(Y_{m',n''}(K \cup L))$, and $\widehat{CF}(Y_{m'',n''}(K \cup L))$. Thus, there is a spectral sequence with $E^1$ term given by

$$\widehat{HF}(Y_{m',n'}(K \cup L)) \oplus \widehat{HF}(Y_{m'',n'}(K \cup L)) \oplus \widehat{HF}(Y_{m',n''}(K \cup L)) \oplus \widehat{HF}(Y_{m'',n''}(K \cup L))$$

converging to $\widehat{HF}(Y_{m,n}(K \cup L))$. This is called the *link surgery spectral sequence.*

Returning to the branched double cover, suppose that $L$ has $k$ crossings. Consider the link $K$ in $\Sigma(L)$ corresponding to the $k$ crossings, as in the first paragraph of the proof. The surgery spectral sequence corresponding to this link has $E^\infty$-page $\widehat{HF}(\Sigma(L))$. It remains to identify the $E^2$-page with Khovanov homology. In fact, the $E^1$-page is identified with the reduced Khovanov complex. At the level of vertices, notice that the Floer group corresponding to each vertex is the branched double cover of an unlink. Hence, by Exercise 3, if the unlink has $\ell$ circles in it then this branched double cover has $\widehat{HF}$ given by $(\mathbb{F}_2 \oplus \mathbb{F}_2)^{\otimes(n-1)}$, in agreement with the corresponding term in the reduced Khovanov complex. Identifying the differential on the $E^1$-page is then a fairly short computation; see [OSz05]. □

5.4. **Suggested exercises.**

(1) Corollary 5.3 holds for rational surgeries, as well. Prove it.
(2) Prove Formula (5.8) (assuming Formula (5.7)).
(3) Show that the branched double cover of an $n$-component unlink in $S^3$ is the connected sum of $(n-1)$ copies of $S^2 \times S^1$. Deduce that the branched double cover has $\widehat{HF}$ given by $(\mathbb{F}_2 \oplus \mathbb{F}_2)^{\otimes(n-1)}$.

## References

[AE11] Akram S. Alishahi and Eaman Eftekhary, *A refinement of sutured Floer homology*, 2011, arXiv:1112.3540.

[BEH+03] Frdric Bourgeois, Yakov Eliashberg, Helmut Hofer, Kris Wysocki, and Eduard Zehnder, *Compactness results in symplectic field theory*, Geom. Topol. **7** (2003), 799–888, arXiv: math.SG/0308183.

[Eft05] Eaman Eftekhary, *Longitude Floer homology and the Whitehead double*, Algebr. Geom. Topol. **5** (2005), 1389–1418, arXiv:math.GT/0407211.

[Gab83] David Gabai, *Foliations and the topology of* 3*-manifolds*, J. Differential Geom. **18** (1983), no. 3, 445–503.

[Gab84] ———, *Foliations and genera of links*, Topology **23** (1984), no. 4, 381–394.

[Gab86] ———, *Genera of the arborescent links*, Mem. Amer. Math. Soc. **59** (1986), no. 339, i–viii and 1–98.

[Gab87] ———, *Foliations and the topology of* 3*-manifolds. III*, J. Differential Geom. **26** (1987), no. 3, 479–536.

[Ghi08] Paolo Ghiggini, *Knot Floer homology detects genus-one fibred knots*, Amer. J. Math. **130** (2008), no. 5, 1151–1169.

[Gro85] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), no. 2, 307–347.

[GW10] J. Elisenda Grigsby and Stephan M. Wehrli, *On the naturality of the spectral sequence from Khovanov homology to Heegaard Floer homology*, Int. Math. Res. Not. IMRN (2010), no. 21, 4159–4210.

[HKM09] Ko Honda, William H. Kazez, and Gordana Matić, *On the contact class in Heegaard Floer homology*, J. Differential Geom. **83** (2009), no. 2, 289–311.

[HLV14] Jennifer Hom, Tye Lidman, and Faramarz Vafaee, *Berge-Gabai knots and L-space satellite operations*, 2014, arXiv:1406.1597.

[HN10] Matthew Hedden and Yi Ni, *Manifolds with small Heegaard Floer ranks*, Geom. Topol. **14** (2010), no. 3, 1479–1501.

[JM08] Stanislav Jabuka and Thomas E. Mark, *On the Heegaard Floer homology of a surface times a circle*, Adv. Math. **218** (2008), no. 3, 728–761.

[JT12] András Juhász and Dylan P. Thurston, *Naturality and mapping class groups in Heegaard Floer homology*, 2012, arXiv:1210.4996.

[Juh06] András Juhász, *Holomorphic discs and sutured manifolds*, Algebr. Geom. Topol. **6** (2006), 1429–1457, arXiv:math/0601443.

[Juh08] Andrs Juhsz, *Floer homology and surface decompositions*, Geom. Topol. **12** (2008), no. 1, 299–350, arXiv:math/0609779.

[Juh09] András Juhász, *Cobordisms of sutured manifolds*, 2009, arXiv:0910.4382.

[Juh13] ———, *A survey of Heegaard Floer homology*, 2013, arXiv:1310.3418.

[Kho00] Mikhail Khovanov, *A categorification of the Jones polynomial*, Duke Math. J. **101** (2000), no. 3, 359–426.

[KM10] Peter Kronheimer and Tomasz Mrowka, *Knots, sutures, and excision*, J. Differential Geom. **84** (2010), no. 2, 301–364, arXiv:0807.4891.

[KM11] P. B. Kronheimer and T. S. Mrowka, *Khovanov homology is an unknot-detector*, Publ. Math. Inst. Hautes Études Sci. (2011), no. 113, 97–208, arXiv:1005.4346.

[KMOSz07] Peter Kronheimer, Tomasz Mrowka, Peter S. Ozsváth, and Zoltán Szabó, *Monopoles and lens space surgeries*, Ann. of Math. (2) **165** (2007), no. 2, 457–546, arXiv:math.GT/0310164.

[Lip06] Robert Lipshitz, *A cylindrical reformulation of Heegaard Floer homology*, Geom. Topol. **10** (2006), 955–1097, arXiv:math.SG/0502404.

[LOT08] Robert Lipshitz, Peter S. Ozsváth, and Dylan P. Thurston, *Bordered Heegaard Floer homology: Invariance and pairing*, 2008, arXiv:0810.0687v4.

[LOT10] ———, *Bordered Floer homology and the spectral sequence of a branched double cover I*, 2010, arXiv:1011.0499.

[LOT11] ———, *Tour of bordered Floer theory*, PNAS **108** (2011), no. 20, 8085–8092.

[LOT12] ———, *Notes on bordered Floer homology*, 2012, arXiv:1211.6791.

[Man14] Ciprian Manolescu, *An introduction to knot floer homology*, 2014, arXiv:1401.7107.

[Mil63] J. Milnor, *Morse theory*, Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51, Princeton University Press, Princeton, N.J., 1963.

[Mil65] John Milnor, *Lectures on the h-cobordism theorem*, Notes by L. Siebenmann and J. Sondow, Princeton University Press, Princeton, N.J., 1965.

[MOS09] Ciprian Manolescu, Peter Ozsváth, and Sucharit Sarkar, *A combinatorial description of knot Floer homology*, Ann. of Math. (2) **169** (2009), no. 2, 633–660, arXiv:math.GT/0607691.

[MW95] Mario J. Micallef and Brian White, *The structure of branch points in minimal surfaces and in pseudoholomorphic curves*, Ann. of Math. (2) **141** (1995), no. 1, 35–85.

[Ni07] Yi Ni, *Knot Floer homology detects fibred knots*, Invent. Math. **170** (2007), no. 3, 577–608.

[Ni09] ———, *Heegaard Floer homology and fibred 3-manifolds*, Amer. J. Math. **131** (2009), no. 4, 1047–1063.

[OS05a] Peter Ozsváth and Zoltán Szabó, *On Heegaard diagrams and holomorphic disks*, European Congress of Mathematics, Eur. Math. Soc., Zürich, 2005, pp. 769–781.

[OS05b] ———, *On knot Floer homology and lens space surgeries*, Topology **44** (2005), no. 6, 1281–1300.

[OS06a] ———, *An introduction to Heegaard Floer homology*, Floer homology, gauge theory, and low-dimensional topology, Clay Math. Proc., vol. 5, Amer. Math. Soc., Providence, RI, 2006, pp. 3–27.

[OS06b] ———, *Lectures on Heegaard Floer homology*, Floer homology, gauge theory, and low-dimensional topology, Clay Math. Proc., vol. 5, Amer. Math. Soc., Providence, RI, 2006, pp. 29–70.

[OS08] Peter S. Ozsváth and Zoltán Szabó, *Knot Floer homology and integer surgeries*, Algebr. Geom. Topol. **8** (2008), no. 1, 101–153.

[OS09] Peter Ozsváth and Zoltán Szabó, *A cube of resolutions for knot Floer homology*, J. Topol. **2** (2009), no. 4, 865–910.

[OS11] Peter S. Ozsváth and Zoltán Szabó, *Knot Floer homology and rational surgeries*, Algebr. Geom. Topol. **11** (2011), no. 1, 1–68.

[OSS] Peter Ozsváth, András Stipsicz, and Zoltán Szabó, In preparation.

[OSS09] ———, *Floer homology and singular knots*, J. Topol. **2** (2009), no. 2, 380–404.

[OSz04a] Peter S. Ozsváth and Zoltán Szabó, *Holomorphic disks and genus bounds*, Geom. Topol. **8** (2004), 311–334, arXiv:math.GT/0311496.

[OSz04b] ———, *Holomorphic disks and knot invariants*, Adv. Math. **186** (2004), no. 1, 58–116, arXiv:math.GT/0209056.

[OSz04c] ———, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. (2) **159** (2004), no. 3, 1159–1245, arXiv:math.SG/0105202.

[OSz04d] ———, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2) **159** (2004), no. 3, 1027–1158, arXiv:math.SG/0101206.

[OSz05] ———, *On the Heegaard Floer homology of branched double-covers*, Adv. Math. **194** (2005), no. 1, 1–33, arXiv:math.GT/0309170.

[OSz06] ———, *Holomorphic triangles and invariants for smooth four-manifolds*, Adv. Math. **202** (2006), no. 2, 326–400, arXiv:math.SG/0110169.

[OSz08] ———, *Holomorphic disks, link invariants and the multi-variable Alexander polynomial*, Algebr. Geom. Topol. **8** (2008), no. 2, 615–692, arXiv:math/0512286.

[Ras03] Jacob Rasmussen, *Floer homology and knot complements*, Ph.D. thesis, Harvard University, Cambridge, MA, 2003, arXiv:math.GT/0306378.

[RH11] Vinicius Gripp Barros Ramos and Yang Huang, *An absolute grading on Heegaard Floer homology by homotopy classes of oriented 2-plane fields*, 2011, arXiv:1112.0290.

[RH12] ———, *A topological grading on bordered Heegaard Floer homology*, 2012, arXiv:1211.7367.

[Sch89] Martin Scharlemann, *Sutured manifolds and generalized Thurston norms*, J. Differential Geom. **29** (1989), no. 3, 557–614.

[SW10] Sucharit Sarkar and Jiajun Wang, *An algorithm for computing some Heegaard Floer homologies*, Ann. of Math. (2) **171** (2010), no. 2, 1213–1236, arXiv:math/0607777.

[Thu86] William P. Thurston, *A norm for the homology of* 3*-manifolds*, Mem. Amer. Math. Soc. **59** (1986), no. 339, i–vi and 99–130.

[Tur97] Vladimir Turaev, *Torsion invariants of* $\mathrm{Spin}^c$*-structures on* 3*-manifolds*, Math. Res. Lett. **4** (1997), no. 5, 679–695.

[Zar09] Rumen Zarev, *Bordered Floer homology for sutured manifolds*, 2009, arXiv:0908.1106.

[Zar10] ———, *Joining and gluing sutured Floer homology*, 2010, arXiv:1010.3496.

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