3-manifolds and their groups

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3-manifolds and their groups

A 3-manifold M is a space which is locally modelled on $\mathbb{R}^3$, or $\mathbb{R}^3_+$ for boundary points.
M is closed if it is compact and has no boundary points.
What does $\pi_1(M)$ tell us about M itself?

That question is the theme of these talks.

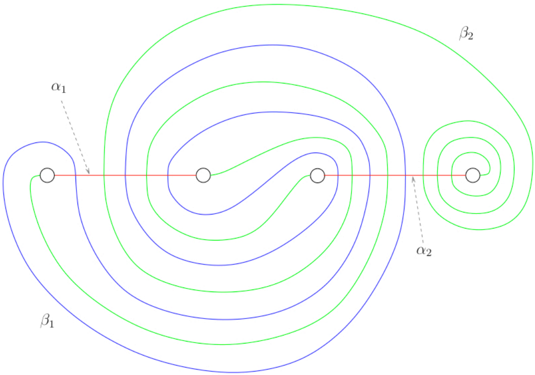
How is a 3-manifold "given?"

One can describe a 3-manifold by:

- triangulation
- Heegaard splitting
- surgery on a link in S^3
- orbit space of group of isometries of a geometry
- special constructions such as Seifert fibre spaces
- and fibre bundles over S^1

In each case, calculation of a presentation for $\pi_1(M)$ is straightforward.

A Heegaard diagram of Poincaré's dodecahedral space.



The Poincaré conjecture

Recently proved by Perelman a century after conjecture by Poincaré

Theorem

If M is a closed 3-manifold, then $\pi_1(M) = 1$ *iff* $M \cong S^3$

An equivalent form of this theorem is

Theorem

If M is a compact contractible 3-manifold with boundary, then M is a 3-dimensional ball.

History: Kneser's conjecture

Theorem

If $\pi_1(M) \cong G_1 * G_2$ is a free product of nontrivial groups, then M is a connected sum $M \cong M_1 \sharp M_2$ with $\pi_1(M_i) \cong G_i$.

In other words, there is a 2-sphere in M separating it into two submanifolds which realizes the splitting of the group.
By a theorem of Milnor, a closed 3-manifold has a unique splitting

$$M \cong M_1 \sharp M_2 \sharp \cdots \sharp M_n$$

into prime factors (up to order of the factors).

Now that the PC is proved we have

Theorem

$\pi_1(M)$ is a free product if and only if M is a connected sum.

Or, put another way ...

Theorem

M is prime if and only if $\pi_1(M)$ does not split as a free product.

History: Sphere theorem

Speaking of 2-spheres, there is a nice algebraic criterion for the existence of an essential S^2 in a 3-manifold.

Theorem

If M is an orientable 3-manifold, then M contains an embedded homotopically nontrivial 2-sphere if and only if $\pi_2(M) = 0$

From the same era (about 50 years ago) came Dehn's lemma and the loop theorem – useful technical tools for detecting essential surfaces, doing surgery, etc. We won't discuss them today, though they are part of the story of algebra reflecting topology.

Does $\pi_1(M)$ determine M?

Certainly not! For example certain lens spaces have isomorphic (cyclic) fundamental groups, but may not be homeomorphic, or even homotopy equivalent.

As another example, the complements of the reef knot and granny knot in S^3 have isomorphic groups, but they are not topologically equivalent.

Does $\pi_1(M)$ determine M?

An important class of 3-manifolds is the Haken manifold. M is Haken if it is compact, orientable and irreducible and "sufficiently large" in the sense that it contains an incompressible surface.
Irreducible means that every (tame) 2-sphere bounds a 3-ball in M. A prime manifold is irreducible, or else homeomorphic to $S^1 \times S^2$.
The surface S in M is incompressible if it is not a sphere and the inclusion-induced homomorphism $\pi_1(S) \to \pi_1(M)$ is injective.
Being Haken enables one to prove properties of M by induction, using a heirarchy obtained by cutting M open along S, showing the result contains another impressible surface, then cut along that surface, etc. After a finite number of steps M is decomposed into a disjoint union of 3-balls.

Does $\pi_1(M)$ determine M?

Positive results in this direction are due to Waldhausen (1968).

Theorem

Suppose M_1 and M_2 are compact orientable manifolds with nonempty boundary, and $\phi : \pi_1(M_1) \to \pi_1(M_2)$ an isomorphism. Then ϕ is induced by a homeomorphism $M_1 \to M_2$ provided the M_i are irreducible and the boundary components are incompressible and ϕ preserves the peripheral structure.

This has been generalized in several directions since then.

Geometric structures

Thurston showed that their are eight possible 3-dimensional geometric structures on 3-manifolds:

Hyperbolic ($\mathbb{H}^3$), spherical (S^3) and euclidean ($\mathbb{E}^3$)

and $S^2 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^1$, Nil, Sol and $\widetilde{PSL}(2, \mathbb{R})$

A 3-manifold is geometric if it can be expressed as the orbit space of a discrete group of isometries acting freely on the model geometry.

Geometric structures

Thurston also showed:

Theorem

If the compact 3-manifold M is geometric, then $\pi_1(M)$ determines its geometry.

JSJ decomposition

Jaco and Shalen, and (independently) Johannson proved

Theorem

Irreducible orientable closed 3-manifolds have a unique (up to isotopy) minimal collection of disjointly embedded incompressible tori such that each component of the 3-manifold obtained by cutting along the tori is either atoroidal or Seifert-fibered.

The tori correspond to $\mathbb{Z} \oplus \mathbb{Z}$ subgroups of the fundamental group. "Atoroidal" means there is no essential torus, or in algebraic terms, the fundamental group contains no subgroup isomorphic with $\mathbb{Z} \oplus \mathbb{Z}$.

Thurston's geometrization conjecture

A Seifert-fibred 3-manifold (roughly speaking) is one which is the disjoint union of topological circles. Such manifolds, with the exception of $\mathbb{R}P^3 \sharp \mathbb{R}P^3$, are irreducible.

It is known that irreducible Seifert-fibred 3-manifolds are geometric.

Thurston's geometrization conjecture asserts that each component of the JSJ decomposition is geometric.

Thurston's geometrization conjecture

TGC was proved for Haken manifolds by Thurston and others.
It was recently proved in general by Perelman.

Theorem

Each component of the JSJ decomposition is geometric.

It has many consequences.

Consequences of TGC

The spherical space form conjecture ...

Theorem

If $\pi_1(M)$ is finite, then M has a metric of constant positive curvature.

Consequences of TGC

Theorem

If the prime 3-manifold M is non-Haken and has infinite fundamental group, then M is Seifert fibred or hyperbolic.

Consequences of TGC

Combining with other recent results, it solves the homeomorphism problem.

Theorem

There is an algorithm to decide if two given compact 3-manifolds are homeomorphic.

This is in contrast with dimensions greater than 3, in which the homeomorphism problem cannot be solved algorithmically.

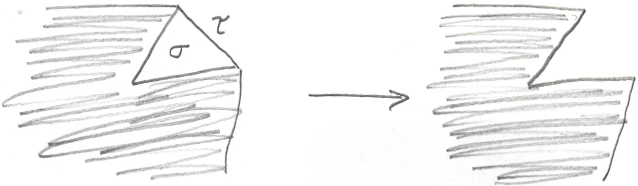
Further consequences of PC

Collapsing and simple-homotopy:

Suppose the finite polyhedron K has a simplex σ^n which has a free face τ^{n-1} (meaning $int(\tau)$ does not intersect any other part of K). Then the transition:

$$K \longrightarrow K \setminus \{int(\sigma) \cup int(\tau)\}$$

is called an elementary collapse. The inverse of this operation is an elementary expansion.



Further consequences of PC

J. H. C. Whitehead defined simple homotopy to be the equivalence relation among polyhedra which is generated by elementary collapse and expansion. Subdivision is also allowed.

If two polyhedra have the same simple homotopy type, then they are homotopy equivalent, but the converse is not true. Whitehead torsion is an obstruction to going in the other direction.

A sequence of expansions and collapses involving simplices of dimension at most n is called an n-deformation.

Further consequences of PC

Theorem

(Whitehead-Wall): If $n \neq 2$, and the polyhedra K^n and L^n are simple-homotopy equivalent, then there exists an $n+1$-deformation from K to L.

The case $n = 2$ is still open. It is related to a problem which is equivalent to a group-theoretic conjecture made by Andrews and Curtis – that a balanced presentation of the trivial group can be reduced to the trivial presentation by certain specific moves.

Further consequences of PC

The A-C conjecture concerns presentations of the trivial group which are "balanced" in the sense of having the same number of generators and relations. Examples

$\langle x, y | \, x, y \rangle$

$\langle x, y | \, x^p y^q, x^r y^s \rangle, \quad ps - rq = \pm 1$

$\langle x, y | \, x^{-1} y^2 x = y^3, y^{-1} x^2 y = x^3 \rangle$

$\langle x, y | \, x^4 y^3 = y^2 x^2, x^6 y^4 = y^3 x^3 \rangle$

$\langle x, y, z | \, y^{-1} xy = x^2, z^{-1} yz = y^2, x^{-1} zx = z^2 \rangle$

$\langle , \rangle$

all present the trivial group.

Further consequences of PC

Given a presentation $\langle x_1, \dots, x_n; r_1, \dots r_m \rangle$ of a group, consider the operations, which do not change the group presented:
(1) replace r_i by its inverse r_i^{-1},
(2) replace r_i by $r_i r_j, \quad i \neq j$,
(3) replace r_i by $g r_i g^{-1}$, where $g \in F(x_1, \dots, x_n)$.
(4) introduce a generator x_{n+1} and relator r_{n+1} which is just x_{n+1}.

Andrews-Curtis Conjecture: A balanced presentation of the trivial group can be reduced to the empty presentation by (1)-(3) above, and operation (4) and its inverse.

Further consequences of PC

The A-C conjecture is equivalent to

Geometric Andrews-Curtis conjecture: If K^2 is contractible then K 3-deforms to a point.

The A-C conjectures remains open in general, but the PC implies some progress in this....
If a 2-complex happens to embed in a 3-manifold, we will call it a spine. One sees easily that a regular neighbourhood of a spine collapses to the spine. There is an algorithm, due to Neuwirth, to decide if a given 2-complex is a spine.

Further consequences of PC

Theorem

The (geometric) A-C conjecture is true for spines.

proof: Let N^3 be a regular neighbourhood in a manifold containing the contractible K^2, so that N collapses to K. The PC implies N^3 is homeomorphic with the standard 3-ball, and hence collapsible to a point. This gives the 3-deformation asserted by the ACC:

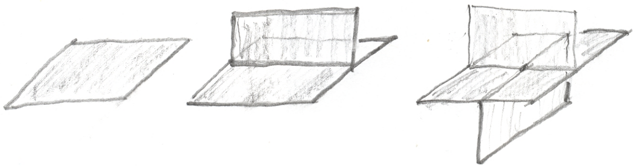
$$K^2 \swarrow N^3 \searrow pt$$

Further consequences of PC

Zeeman conjecture: If K^2 is a contractible complex, then $K \times I$ collapses to a point.
Clearly the ZC implies the ACC, because the transition $K \swarrow K \times I \searrow pt$ gives a 3-deformation.
The ZC also implies the PC, by the following argument: Suppose that Q^3 is a compact, contractible manifold. Q collapses to a "spine" K^2, also contractible. By ZC, $K \times I$ collapses to a point. Then $Q \times I$ collapses to $K \times I$, which then collapses to a point.
Being a collapsible 4-manifold, $Q \times I$ must be a 4-ball. Now Q clearly embeds in $\partial(Q \times I)$, which is a 3-sphere. Therefore Q is a 3-ball.

Further consequences of PC

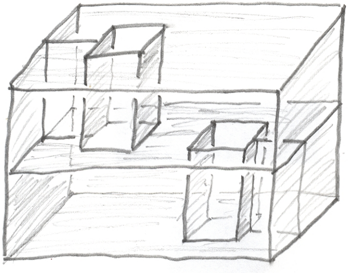
A converse....
A 2-complex is *standard* if it is modeled on the cone upon Δ_1^3, the 1-skeleton of a 3-simplex.
Every 3-manifold with nonempty boundary collapses to a standard spine and is determined by such a spine.



Local structure of a standard complex

Further consequences of PC

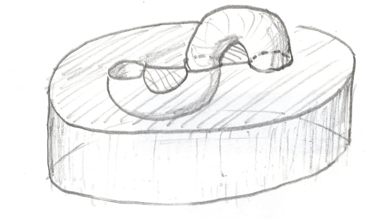
Bing's house with two rooms
A standard spine of the cube



It is contractible, but not collapsible

Further consequences of PC

The igloo



Another contractible, non-collapsible 2-polyhedron

Further consequences of PC

Theorem

(Gillman - R.) The ZC, restricted to standard spines, is equivalent to the PC.

Key idea of the proof: We've already seen that the ZC implies the PC. Since every 3-manifold with boundary collapses to a special spine, the same proof works for ZC, restricted to standard spines.
For the converse, if K^2 is a standard spine of M^3 and has trivial homology groups, then (by an explicit construction) $K \times I$ collapses to a subset homeomorphic to M. If K is contractible, so is M, and assuming PC, M is a 3-ball, and so $K \times I \searrow M \searrow *$ verifies the ZC for K.

Corollary

The ZC and ACC are true for standard spines.