Heegaard-Floer homology, ordered groups and exceptional surgeries

Dale Rolfsen

A survey and recent work with Adam Clay

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Khovanov homology . . .

'categorification' of the Jones polynomial.

A bigraded homology theory $\widetilde{Kh}(K)$ associated to a knot K whose 'Euler characteristic' is the Jones polynomial:

$$\chi_{\widetilde{Kh}}(K) = V_K(t)$$

Today we'll discuss a similar theory – Knot Floer Homology

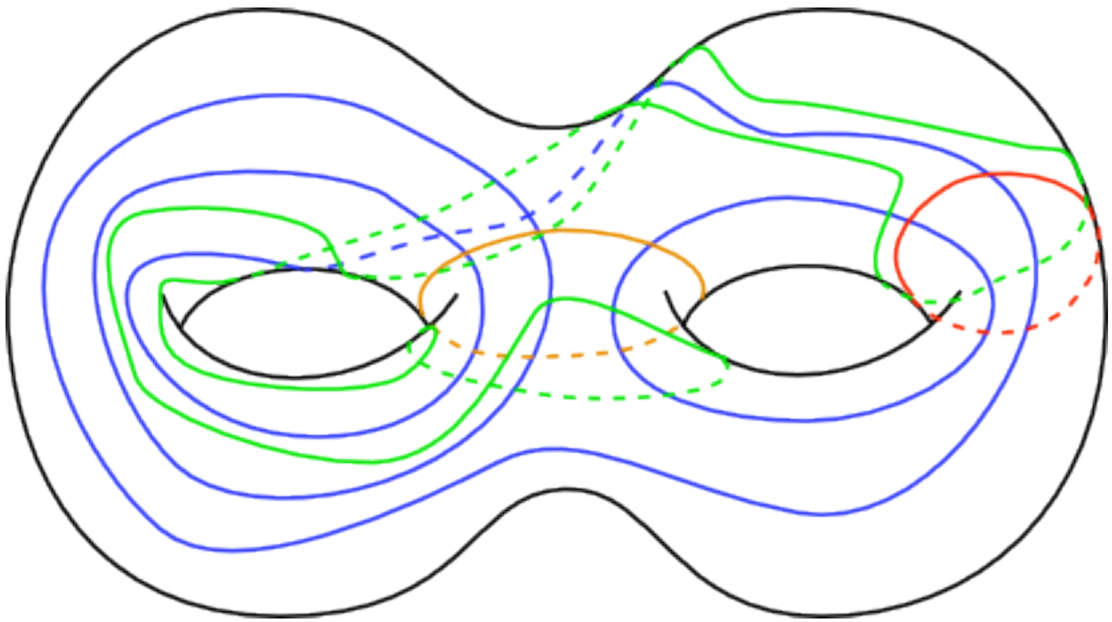
$$\widehat{HFK}_*(K, i)$$

and its cousin Heegaard Floer Homology

$$\widehat{HF}_*(M)$$

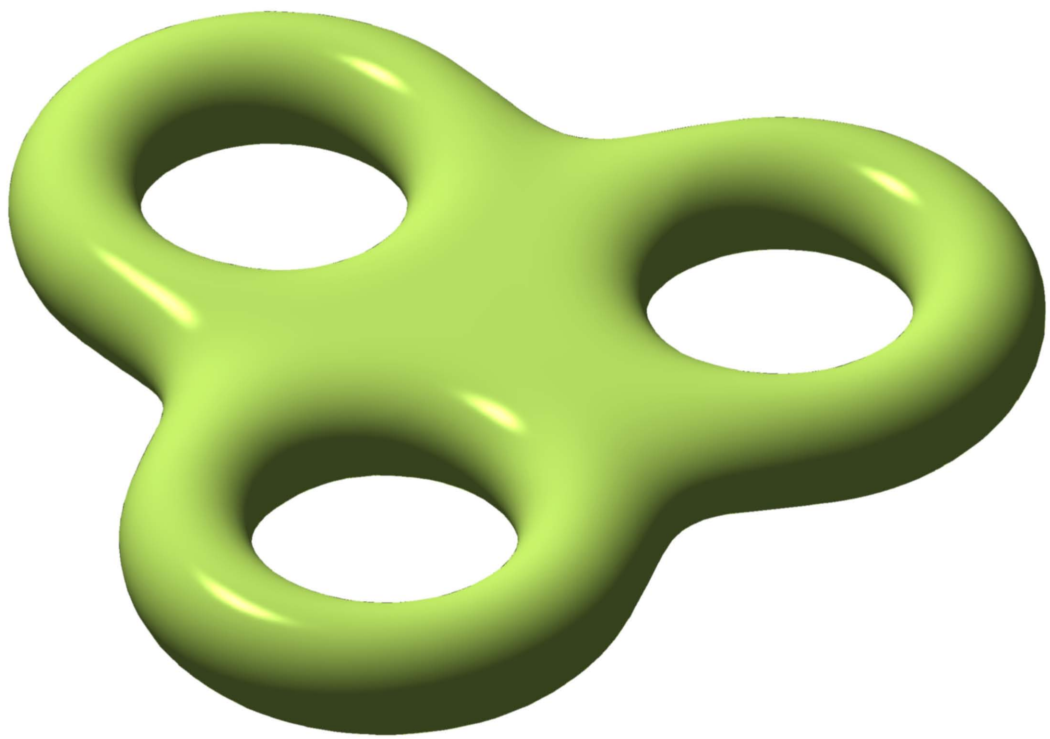
Their definition depends on a classical description of a 3-manifold M via . . .

Heegaard diagrams



Every closed orientable 3-manifold M is the union of two handlebodies:

$$M^3 \cong H_1 \cup_{\Sigma_g} H_2.$$



The "handles" of a handlebody H of genus g may be regarded as g disjoint disks in H, with their boundaries in $\partial H \cong \Sigma_g$, and so that the complement in H of the union of regular neighbourhoods of the disks is a 3-ball:

$$H \setminus \cup_{i=1}^{g} N(D_i) \cong B^3$$

The associated Heegaard diagram is

$$(\Sigma_g, \vec{a}, \vec{b})$$

where the curves $\vec{\alpha} = \alpha_1 \sqcup \cdots \sqcup \alpha_g$ bound disks in H_1 and likewise the curves $\vec{\beta} = \beta_1 \sqcup \cdots \sqcup \beta_g$ bound disjoint disks in H_2.

The data of a Heegaard diagram, together with the notion of 'pseudo-holomorphic' discs, produces a chain complex whose homology is $\widehat{HF}(M)$, in the case that M is a rational homology sphere: $b_1(M) = 0$.

More generally, one needs a $spin^c$ structure, which may be regarded as a cohomology class $s \in H^2(M;\mathbb{Z})$, to define $\widehat{HF}(M, s)$

Relation between HF and Khovanov homology:

There's a spectral sequence with $E_2 \cong \widetilde{Kh}(L^*)$ converging to $E_\infty \cong \widehat{HF}(\Sigma(S^3, L))$

Here $\Sigma(S^3, L)$ is the 2-fold branched cover of S^3 branched along the link L.

Knot Floer homology can be considered with various gradings:

$$\widehat{HFK}_*(K,i)$$

$i =$Alexander grading

$* =$ Maslov (homological) grading

Knot Floer homology categorifies the Alexander polynomial:

$$\chi(HFK) = \Delta_K$$

More precisely

$$\sum_{i \in \mathbb{Z}} [\sum_{* \in \mathbb{Z}} (-1)^* rk \widehat{HFK}_*(K,i)] \cdot T^i = \Delta_K(T)$$

and $\Delta_K(T)$ is the symmetrized Alexander polynomial.

Applications:

Theorem. *(O-S)* $\max\{i \in \mathbb{Z} | \widehat{HFK}(K,i) \neq 0\} = g(K)$, *where* $g(K)$ *is the (minimal) Seifert genus of* K.

In particular $\widehat{HFK}$ detects the unknot.

Theorem. *(Kronheimer-Mrowka)* $\widetilde{Kh}$ *detects the unknot.*

$\widehat{HFK}$ also detects fibred knots . . .

A knot $K \subset S^3$ is fibred if there is a locally trivial fibre bundle

$$S^3 \setminus K \to S^1$$

in which the fibres are surfaces whose closures have boundary K.

Theorem. *(O-S) If $K \subset S^3$ is a fibred knot, then $\widehat{HFK}(K, g(K))$ has rank equal to 1.*

Theorem. *(Ghiggini, Ni) The converse also holds.*

Other applications:

- 4-dimensional manifolds

- 4-dimensional genus of a knot – solution of Milnor's conjecture for torus knots.

- concordance group

- existence of taut foliations

- contact structures

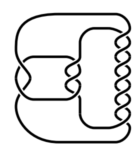
Surgery and L - spaces

If L is a link in S^3, one can construct a manifold by Dehn surgery along L, by specifying a surgery coefficient $p/q \in \mathbb{Q} \cup \infty$ for each curve of the link.

If K is a hyperbolic knot and a particular p/q surgery yields a non-hyperbolic 3-manifold, then this is said to be an exceptional surgery.

Thurston: there are only finitely many.

Example: the Fintushel-Stern knot, also known as the pretzel knot of type $(-2, 3, 7)$.



It admits SEVEN exceptional surgeries.

The Alexander polynomial of $P(-2, 3, 7)$ is $\Delta_K(t) = L(-t)$, where L is the Lehmer polynomial

$$L(x) = 1 + x - x^3 - x^4 - x^5 - x^6 - x^7 + x^9 + x^{10}$$

which is the polynomial (over $\mathbb{Z}$) of smallest known Mahler measure.

Upon symmetrizing we have

$$\Delta_K(x) = x^{-5} - x^{-4} + x^{-2} - x^{-1} + 1 - x + x^2 - x^4 + x^5$$

Exceptional surgeries can result in

- lens spaces
- the Poincaré homology sphere
- other manifolds with finite π_1
- Seifert-fibred manifolds
- connected sums

Def: A 3-manifold M is an L-space if

- M is a rational homology sphere, that is $b_1(M) = 0$
- $\widehat{HF}(M)$ has smallest possible rank, namely

$$rk\widehat{HF}(M) = |H_1(M;\mathbb{Z})|$$

Examples: lens spaces, other spaces with finite π_1, certain Seifert-fibred manifolds.

Theorem. *(Boyer-Watson) If M is a Seifert-fibred manifold, then M IS an L-space if and only if $\pi_1(M)$ is NOT left-orderable.*

Question: Is a rational homology sphere M an L-space $\Leftrightarrow$ $\pi_1(M)$ is not left-orderable?

More generally, what connection is there between $\widehat{HF}(M)$ and $\pi_1(M)$?

Theorem. *(O-S, 2005) Surgery on a hyperbolic alternating knot in* S^3 *never yields an* L*-space.*

Theorem. *(O-S) If* $M = \Sigma(S^3, K)$ *is the 2-fold branched cover of* S^3 *along the alternating knot* K*, then* M *is an L-space.*

Theorem. *(Boyer-Gordon-Watson) If* $M = \Sigma(S^3, K)$ *is the 2-fold branched cover of* S^3 *along the alternating knot* K*, then* $\pi_1(M)$ *is NOT left-orderable.*

Theorem. *(O-S) L-spaces do not admit taut foliations.*

Theorem. *(O-S) Let K be a knot in S^3 for which surgery on K yields an L-space. Then K is fibred and the Alexander polynomial of K has the form*

$$\Delta_K(t) = (-1)^k + \sum_{j=1}^{k} (-1)^{k-j}(t^{n_j} + t^{-n_j})$$

for some increasing sequence of positive integers

$0 < n_1 < n_2 < \cdots < n_k$.

For example the Fintushel-Stern knot polynomial has the form

$$\Delta_K(x) = 1 - (x + x^{-1}) + (x^2 + x^{-2}) - (x^4 + x^{-4}) + (x^5 + x^{-5})$$

Theorem. *(Clay-R) If surgery on a nontrivial knot K in S^3 results in an L-space, then the knot group $\pi_1(S^3 \setminus K)$ is not bi-orderable.*

We remark that all knot groups are left-orderable, and many are also bi-orderable

To prove this theorem, we assume for contradiction that K has bi-orderable knot group, and that surgery on K yields an L-space. It follows (Yi Ni) that K must be fibred. We now recall a recent result:

Theorem. *(C-R) If the group $\pi_1(S^3 \setminus K)$ of a nontrivial fibred knot K in S^3 is bi-orderable, then the Alexander polynomial of the knot must have a positive real root.*

Our goal is to show, then that a polynomial

$$\Delta_K(t) = (-1)^k + \sum_{j=1}^{k} (-1)^{k-j}(t^{n_j} + t^{-n_j})$$

cannot have a positive real root – the proof will be complete.

Lemma. *(Calculus 100) Suppose that* $\alpha > 1$ *and* $s > t > 0$*. Then*

$$\alpha^s + \alpha^{-s} > \alpha^t + \alpha^{-t}.$$

Proof: For $\alpha > 1$, consider the function $f(x) = \alpha^x + \alpha^{-x}$. It is continuous and differentiable for all x, with derivative $f'(x) = \ln(\alpha)(\alpha^x - \alpha^{-x})$. Since $\alpha > 1$, both $\ln(\alpha)$ and $\alpha^x - \alpha^{-x}$ are positive whenever $x > 0$, hence $f'(x) > 0$ for all $x > 0$, and so f is an increasing function on $(0, \infty)$. Therefore, $s > t > 0$ implies $f(s) > f(t)$, in other words $\alpha^s + \alpha^{-s} > \alpha^t + \alpha^{-t}$.

If an Alexander polynomial has a positive real root α, because it is palindromic, it also has the root $1/\alpha$. Also, the equation $\Delta_K(1) = \pm 1$ rules out unity as a root.

So we may assume the equation

$$\Delta_K(t) = (-1)^k + \sum_{j=1}^{k} (-1)^{k-j}(t^{n_j} + t^{-n_j})$$

has a root $\alpha > 1$. We then use the lemma to derive a contradiction.

We illustrate by example, the Fintushel-Stern polynomial:

$$\Delta_K(x) = 1 - (x + x^{-1}) + (x^2 + x^{-2}) - (x^4 + x^{-4}) + (x^5 + x^{-5})$$

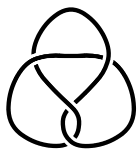
We see that each color contributes a positive quantity, by the calculus lemma, if $x = \alpha > 1$. Therefore we have $\Delta_K(\alpha) > 1$, so α cannot be a root.

The general case is proved similarly.

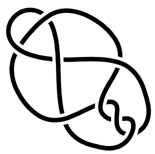
Summary: we have proved, in other words

Theorem. *(C-R) If a nontrivial knot K in S^3 has bi-orderable knot group, then surgery on K cannot produce an L-space.*

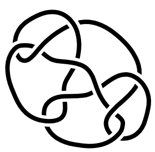
Some knots with biorderable group, and their polynomials.



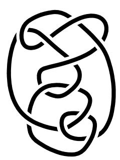
$1 - 3t + t^2$



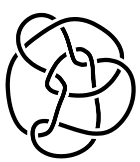
$1 - 7t + 13t^2 - 7t^3 + t^4$



$1 - 6t + 11t^2 - 6t^3 + t^4$

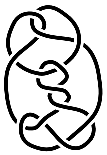


$1 - 9t + 30t^2 - 45t^3 + 30t^4 - 9t^5 + t^6$

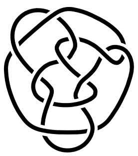


$1 - 8t + 15t^2 - 8t^3 + t^4$

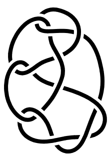
$1 - 12t + 44t^2 - 67t^3 + 44t^4 - 12t^5 + t^6$



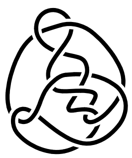
$$1 - 11t + 40t^2 - 61t^3 + 40t^4 - 11t^5 + t^6$$



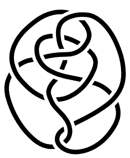
$$1 - 13t + 50t^2 - 77t^3 + 50t^4 - 13t^5 + t^6$$



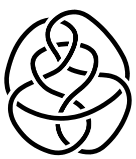
$$1 - 7t + 13t^2 - 7t^3 + t^4$$



$$1 - 6t + 11t^2 - 6t^3 + t^4$$



$$1 - 6t + 11t^2 - 6t^3 + t^4$$



$$1 - 6t + 11t^2 - 6t^3 + t^4$$