On stable commutator length in the mapping class groups of punctured spheres

(joint work with Danny Calegari and Naoyuki Monden)

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November 8, 2012

1. Definition and Properties of stable commutator length.

Stable commutator length (scl)

G: group,
$[a, b] = aba^{-1}b^{-1}$: commutator ($a, b \in G$),
$[G, G]$: the commutator subgroup of G,

Definition

The commutator length of $x \in [G, G]$ is defined by

$$\mathrm{cl}_G(x) = \mathrm{cl}(x) = \min\{l \in \mathbb{Z} \mid x = [a_1, b_1][a_2, b_2] \cdots [a_l, b_l], a_i, b_i \in G\}.$$

The stable commutator length of $x \in [G, G]$ is defined by

$$\mathrm{scl}_G(x) = \mathrm{scl}(x) = \lim_{n \to \infty} \frac{\mathrm{cl}(x^n)}{n}.$$

Let $x \in G$ such that $x^k \in [G,G]$ for some $k \in \mathbb{Z}$.
For such $x \in G$, we can also define the stable commutator length by

$$\mathrm{scl}(x) = \frac{\mathrm{scl}(x^k)}{k}.$$

If $x^k \notin [G,G]$ for all $k \in \mathbb{Z}$, we define $\mathrm{scl}(x) = \infty$.

Example (Culler)

$F_2 = \langle a, b \rangle$: free group of rank 2,

$$\mathrm{cl}_{F_2}([a,b]) = 1, \quad \mathrm{cl}_{F_2}([a,b]^2) = 2,$$

$$\mathrm{cl}_{F_2}([a,b]^3) = 2,$$

$$([a,b]^3 = [aba^{-1}, b^{-1}aba^{-2}][b^{-2}ab, b^{-2}]).$$

Generally, $\mathrm{cl}_{F_2}([a,b]^n) = \left[\frac{n}{2}\right] + 1$. Hence,

$$\mathrm{scl}_{F_2}([a,b]) = \lim_{n\to\infty} \frac{\mathrm{cl}_{F_2}([a,b]^n)}{n} = \frac{1}{2}.$$

Some Properties of scl

Lemma

1. Let G, H be groups, and $f : G \to H$ a homomorphism. Then,

$$\mathrm{scl}_G(a) \geq \mathrm{scl}_H(f(a)).$$

2. Let $1 \to K \to G \to H \to 1$ be an exact sequence between groups. If K is a finite group,

$$\mathrm{scl}_G(a) = \mathrm{scl}_H(f(a)).$$

2. Known Results on scl in Mapping Class Groups of Surfaces and Our Results.

Mapping Class Groups of Closed Surfaces

Σ_g: a closed oriented surface of genus g,
$\mathrm{Diff}_+ \Sigma_g$: the diffeomorphism group of Σ_g,

$$\mathcal{M}_g = \mathrm{Diff}_+ \Sigma_g/\text{isotopy}\,(= \pi_0\,\mathrm{Diff}_+ \Sigma_g).$$

It is generated by Dehn twists t_c along nonseparating SCCs.

Dehn twists

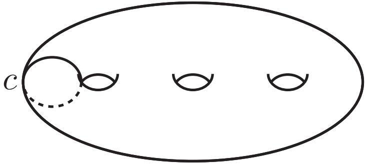

Figure: a nonseparating curve C

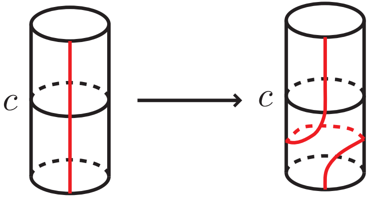

Figure: Dehn twist t_C

Known results

Theorem (Endo-Kotschick 2000, Korkmaz 2004)

When $g \geq 2$,

$$\frac{1}{18g-6} \leq \mathrm{scl}_{\mathcal{M}_g}(t_c).$$

Theorem (Korkmaz 2004)

When $g \geq 2$,

$$\mathrm{scl}_{\mathcal{M}_g}(t_c) \leq \frac{3}{20}.$$

Theorem (Kotschick 2008)

$$\mathrm{scl}_{\mathcal{M}_g}(t_c) = O(1/g).$$

mapping class groups of pointed spheres

$q_1, q_2, \cdots, q_m \in S^2$: m-points on a 2-sphere.
$\mathrm{Diff}_+(S^2, \{q_i\}_{i=1}^m)$: the diffemorphism group which preserves $\{q_i\}_{i=1}^m$ setwise.

$\mathcal{M}_0^m = \mathrm{Diff}_+(S^2, \{q_i\}_{i=1}^m)/$isotopy fixing $\{q_i\}_{i=1}^m = \pi_0 \mathrm{Diff}_+(S^2, \{q_i\}_{i=1}^m)$.

It is generated by half twists $\{\sigma_i\}_{i=1}^{m-1}$.

Half twists

We denote by $\sigma_i \in \mathcal{M}_0^m$ the mapping class which twist the disk D_i counter-qlockwise, and permute q_i and q_{i+1} as in the figure.

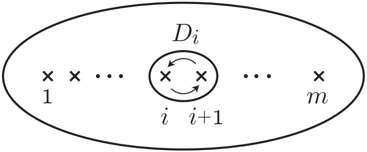

Figure: half twist in D_i

Theorem (Monden 2012)

When $m \geq 6$ and even,

$$\frac{1}{4(m-1)} \leq \mathrm{scl}_{\mathcal{M}_0^m}(\sigma_1).$$

Main Theorem 1 (Calegari-Monden-S)

Let $m \geq 4$. Then,

$$\mathrm{scl}_{\mathcal{M}_0^m}(\sigma_1) \leq \frac{1}{2m + 2 + \frac{4}{m-2}}.$$

When $m = 4$,

this upper bound coincides with the exact value $\mathrm{scl}(\sigma_1) = \frac{1}{12}$.

Corollary

Let $g \geq 1$. Then,

$$\mathrm{scl}_{\mathcal{M}_g}(t_c) \leq \frac{1}{4g + 6 + \frac{2}{g}}.$$

When $g = 1$,

this upper bound coincides with the exact value $\mathrm{scl}(t_c) = \frac{1}{12}$.

homogeneous quasimorphisms

Definition

A map $\phi : G \to \mathbb{R}$ is called a quasimorphism if

$$D(\phi) := \sup_{x,y \in G} |\phi(x) + \phi(y) - \phi(xy)| < \infty.$$

We call $D(\phi)$ the defect of the quasimorphism ϕ.

Definition

A quasimorphism $\phi : G \to \mathbb{R}$ is called homogeneous if it satisfies

$$\phi(x^n) = n\phi(x)$$

for any $x \in G$ and $n \in \mathbb{Z}$.

Remark

If $\phi : G \to \mathbb{R}$ is a quasimorphism, $\bar{\phi} : G \to \mathbb{R}$ defined by

$$\bar{\phi}(x) = \lim_{n \to \infty} \frac{\phi(x^n)}{n}$$

is a homogeneous quasimorphism.

We denote by $Q(G)$ the set of homogeneous quasimorphisms.

It is a vector space.

Theorem (Bavard's duality theorem)

Let $x \in G$. Then,

$$\mathrm{scl}(x) = \sup_{\phi \in Q(G), D(\phi) \neq 0} \frac{|\phi(x)|}{2D(\phi)}.$$

Theorem (Bestvina-Fujiwara 2007)

When g and m are nonnegative integers satisfying $3g+m-4>0$, $Q(\mathcal{M}_g^m)$ is infinite dimensional.
Moreover, for any subgroup $G \subset \mathcal{M}_g^m$ which is not virtually abelian, $Q(G)$ is infinite dimensional.

Main Theorem 2 (Calegari-Monden-S)

Let $m \geq 4$. There exist homogeneous quasimorphisms

$$\bar{\phi}_{m,j} : \mathcal{M}_0^m \to \mathbb{R}$$

parametrized by j, where $1 \leq j \leq \left[\frac{m}{2}\right]$.
For $2 \leq r \leq m-1$, their values are as follows.

$$\begin{aligned}&\bar{\phi}_{m,j}(\sigma_1 \cdots \sigma_{r-1}) \\ &= -\frac{2}{r}\left\{\frac{jr(m-j)(m-r)}{m^2(m-1)} + \left(\frac{rj}{m} - \left[\frac{rj}{m}\right] - \frac{1}{2}\right)^2 - \frac{1}{4}\right\}.\end{aligned}$$

Corollary

If m is not too large ($m < 30$),
the set $\{\bar{\phi}_{m,j}\}_{j=2}^{[\frac{m}{2}]}$ is linearly independent.

Proposition 1

$$D(\bar{\phi}_{m,j}) \leq m - 2.$$

When m is even and $j = m/2$, $D(\bar{\phi}_{m,m/2}) = m - 2$.

Corollary

Let c be a non-separating SCC in Σ_2,
and d an essential separating SCC in Σ_2, s.t. $c \cap d = \emptyset$. Then.

$$\mathrm{scl}_{\mathcal{M}_2}(t_c^{12} t_d^{-1}) = \frac{1}{2}.$$

The value of $-\bar{\phi}_{m,j}(\sigma_1)$ ($\sigma_1 \in \mathcal{M}_0^m$, $1 \leq j \leq [\frac{m}{2}]$)

$$-\bar{\phi}_{m,j}(\sigma_1) = \frac{2j(j-1)}{m(m-1)}.$$

	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$	$m=10$
$j=1$	0	0	0	0	0	0	0
$j=2$	$\frac{4}{12}$	$\frac{4}{20}$	$\frac{4}{30}$	$\frac{4}{42}$	$\frac{4}{56}$	$\frac{4}{72}$	$\frac{4}{90}$
$j=3$			$\frac{12}{30}$	$\frac{12}{42}$	$\frac{12}{56}$	$\frac{12}{72}$	$\frac{12}{90}$
$j=4$					$\frac{24}{56}$	$\frac{24}{72}$	$\frac{24}{90}$
$j=5$							$\frac{40}{90}$

Obtained lower bounds of $\mathrm{scl}_{\mathcal{M}_0^m}(\sigma_1)$.

$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$	$m=10$
$\frac{1}{12}$	$\frac{1}{30}$	$\frac{1}{20}$	$\frac{1}{35}$	$\frac{1}{28}$	$\frac{1}{42}$	$\frac{1}{36}$

Lower bounds and Upper bounds.

	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$	$m=10$
Lower	$\frac{1}{12}$	$\frac{1}{30}$	$\frac{1}{20}$	$\frac{1}{35}$	$\frac{1}{28}$	$\frac{1}{42}$	$\frac{1}{36}$
Upper	$\frac{1}{12}$	$\frac{3}{40}$	$\frac{1}{15}$	$\frac{5}{84}$	$\frac{3}{56}$	$\frac{7}{144}$	$\frac{2}{45}$

3. Relation between $\mathrm{scl}_{\mathcal{M}_g}$ and $\mathrm{scl}_{\mathcal{M}_0^m}$.

Symmetric mapping class groups

There are groups which relate the mapping class groups $\mathcal{M}_g$ and $\mathcal{M}_0^m$.
Let d be a positive integer such that $d|m$.
If we choose a surjective homomorphism $H_1(S^2 - \{q_i\}_{i=1}^m; \mathbb{Z}) \to \mathbb{Z}/d\mathbb{Z}$, we obtain a d-cyclic branched covering space $p : \Sigma_g \to S^2$.
Let us denote

$$t \in \mathrm{Diff}_+ \Sigma_g\text{: a generator of the deck transformation group,}$$

$$C(t) := \{f \in \mathrm{Diff}_+ \Sigma_g \,|\, ft = tf\},$$

$$\mathcal{M}_g(p) := \pi_0 C(t).$$

The inclusion homomorphism $C(t) \to \mathrm{Diff}_+ \Sigma_g$ induces $\iota : \mathcal{M}_g(p) \to \mathcal{M}_g$.
Thus, for $\varphi \in \mathcal{M}_g(p)$, we have

$$\mathrm{scl}_{\mathcal{M}_g(p)}(\varphi) \geq \mathrm{scl}_{\mathcal{M}_g}(\iota(\varphi)).$$

For some technical reason, let $d = m$, and choose a homomorphism

$$H_1(S^2 - \{q_i\}_{i=1}^m) \to \mathbb{Z}/m\mathbb{Z}$$

which maps each loop which rounds q_i once clockwise to $1 \in \mathbb{Z}/m\mathbb{Z}$. Then, the symmetric mapping class group $\mathcal{M}_g(p)$ satisfies an exact sequence

$$0 \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathcal{M}_g(p) \xrightarrow{\mathcal{P}} \mathcal{M}_0^m \longrightarrow 1.$$

The homomorphism $\mathcal{P}$ is defined by $[f] \mapsto [\bar{f}]$,
where $f \in C(t)$ and $\bar{f} \in \mathrm{Diff}_+(S^2, \{q_i\}_{i=1}^m)$ satisfy the commutative diagram

$$\begin{array}{ccc} \Sigma_g & \xrightarrow{f} & \Sigma_g \\ \downarrow & & \downarrow \\ S^2 & \xrightarrow{\bar{f}} & S^2. \end{array}$$

Lemma (well-known)

Let K, G, H be groups. If

$$1 \longrightarrow K \longrightarrow G \xrightarrow{f} H \longrightarrow 1$$

is a (central) extension with K finite, then $Q(G) = Q(H)$.

Proof.

It suffices to construct a homomorphism from $Q(G)$ to $Q(H)$.
Let $\phi \in Q(G)$ and $h \in H$.
We will show that, for $a, a' \in f^{-1}(h)$, $\phi(a) = \phi(a')$.
If $a, a' \in \mathcal{P}^{-1}(h)$, we have $a = a'k$ ($k \in K$).
Since K is in the center of G, we have $a^n = (a')^n$ for some $n \in \mathbb{Z}$. Thus,

$$\phi(a) = \frac{1}{n}\phi(a^n) = \frac{1}{n}\phi((a')^n) = \phi(a').$$

Hence, we can define $\phi' : H \to \mathbb{R}$ by $\phi'(\bar{a}) = \phi(a)$. □

Therefore, for any $x \in \mathcal{M}_g(p)$, we have

$$\mathrm{scl}_{\mathcal{M}_g(p)}(x) = \mathrm{scl}_{\mathcal{M}_0^m}(\mathcal{P}(x)) \geq \mathrm{scl}_{\mathcal{M}_g}(\iota(x)).$$

4. Definition of $\bar{\phi}_{m,j} : \mathcal{M}_0^m \to \mathbb{R}$

Meyer's signature cocycle

Let $P = S^2 - \amalg_{i=1}^{3} \operatorname{Int} D^2$ as in Figure.

For $\varphi, \psi \in \mathcal{M}_g$, there exists a unique Σ_g-bundle $E_{\varphi,\psi}$ over P whose monodromies along α and β are φ and ψ.

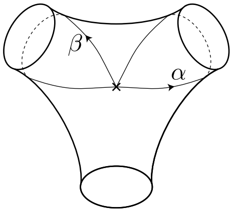

Figure: loops in a pair of pants

Then, we obtain a local system $\pi_1 P \to \mathrm{Aut}(H_1(\Sigma_g;\mathbb{Q}))$.

We also have an intersection form defined by

$$\begin{aligned} H_1(P; H_1(\Sigma_g;\mathbb{Q}))^{\otimes 2} &\cong H^1(P,\partial P; H_1(\Sigma_g;\mathbb{Q}))^{\otimes 2} \\ &\xrightarrow{\cup} H^2(P,\partial P; H_1(\Sigma_g;\mathbb{Q})^{\otimes 2}) \\ &\xrightarrow{\cdot} H^2(P,\partial P;\mathbb{Q}) \\ &\xrightarrow{[P]} \mathbb{Q}. \end{aligned}$$

Theorem (Meyer)

The signature of $E_{\varphi,\psi}$ coincides with that of this intersection form. The map

$$\tau_g : \mathcal{M}_g \times \mathcal{M}_g \to \mathbb{Z}$$

defined by $\tau_g(\varphi,\psi) = \mathrm{Sign}(H_1(P; H_1(\Sigma_g;\mathbb{Q}))$ is a bounded 2-cocycle on $\mathcal{M}_g$.

When $g = 1, 2$, $H^2(\mathcal{M}_g; \mathbb{Q}) = 0$. Hence, we obtain a quasimorphism

$$\phi_g : \mathcal{M}_g \to \mathbb{Q}$$

which satisfies

$$\delta\phi_g(\varphi, \psi) := \phi_g(\varphi) + \phi_g(\psi) - \phi_g(\varphi\psi) = \tau_g(\varphi, \psi).$$

Gambaudo-Ghys' ω-signature

Let us consider another intersection form when $\varphi, \psi \in \mathcal{M}_g(p)$.

Let $p : \Sigma_g \to S^2$ be the cyclic branched covering as before.

Let $\omega = \exp(2\pi\sqrt{-1}/m)$.

Since, φ, ψ commutes with the deck transfomation t,

$\mathbb{Z}/m\mathbb{Z}$ acts on the Σ_g-bundle $E_{\varphi,\psi}$ preserving each fiber. Hence, we obtain a local system $\pi_1 P \to \mathrm{Aut}(V^{\omega^j})$,

where $V^{\omega^j} \subset H_1(\Sigma_g; \mathbb{C})$ is the eigenspace with eigenvalue ω^j.

We also have an intersection form defined by

$$\begin{aligned} H_1(P; V^{\omega^j})^{\otimes 2} &\cong H^1(P, \partial P; V^{\omega^j})^{\otimes 2} \\ &\xrightarrow{\cup} H^2(P, \partial P; (V^{\omega^j})^{\otimes 2}) \\ &\xrightarrow{\cdot} H^2(P, \partial P; \mathbb{C}) \\ &\xrightarrow{[P]} \mathbb{C}. \end{aligned}$$

The map $\tau_{m,j}(\varphi, \psi) = \mathrm{Sign}(H_1(P; V^{\omega^j}))$ is a bounded 2-cocycle on $\mathcal{M}_g(p)$.

Since $H^2(\mathcal{M}_g(p); \mathbb{Q}) = 0$, we also obtain a quasimorphism $\phi_{m,j} : \mathcal{M}_g(p) \to \mathbb{Q}$ which satisfies

$$\phi_{m,j}(\varphi) + \phi_{m,j}(\psi) - \phi_{m,j}(\varphi\psi) = \tau_{m,j}(\varphi, \psi),$$

for all $m \geq 4$.

By Meyer's results, this cocycle can be calculated explicitly as follows.

$$H_1(P; V^{\omega^j}) = \{(x, y) \in V^{\omega^j} \oplus V^{\omega^j} \mid (\varphi_*^{-1} - 1)x + (\psi_* - 1)y = 0\},$$

and the hermitian form is written as

$$\langle (x_1, y_1), (x_2, y_2) \rangle = (x_1 + x_2) \cdot (1 - \psi_*) y_2.$$

We can see that $\dim(H_1(P; V^{\omega^j})/\text{Annihilator}) \leq m - 2$, and we have

$$D(\phi_{m,j}) = \sup_{\varphi, \psi \in \mathcal{M}_g(p)} \tau_{m,j}(\varphi, \psi) \leq m - 2.$$

Cochran-Harvey-Horn generalized this construction to infinite coverings using von Neumann signature.
They considered it when p is the universal abelian covering of Σ_g, and constructed infinite number of quasiphisms on the Johnson kernel which are linearly independent.

3. Proof of Proposition 1.

Proposition 1 (Calegari-Monden-S.)

$$D(\bar{\phi}_{m,j}) \leq m - 2.$$

We have already seen that

$$D(\phi_{m,j}) \leq m - 2.$$

It suffices to show that

$$D(\bar{\phi}_{m,j}) \leq D(\phi_{m,j}).$$

Proposition (cf. scl)

Let G be a group, and $\phi : G \to \mathbb{R}$ a quasimorphism. Then,

$$D(\bar{\phi}) \leq 2D(\phi).$$

Actually, $\phi_{m,j}$ satisfies

$$\phi_{m,j}(xyx^{-1}) = \phi_{m,j}(y), \quad \phi_{m,j}(x^{-1}) = -\phi(x).$$

In this setting, we can show:

Proposition 2 (Calegari-Monden-S.)

Let G be a group, and $\phi : G \to \mathbb{R}$ a quasimorphism satisfying

$$\phi(xyx^{-1}) = \phi(y), \quad \phi(x^{-1}) = -\phi(x).$$

Then,

$$D(\bar{\phi}) \leq D(\phi).$$

Lemma 1

$$|\phi((ab)^3x) - \phi(xba^3b^2)| \leq 2D(\phi).$$

Proof.

$$\begin{aligned}
&\delta\phi([b,a],(ab)^3x) + \delta\phi([a,b], babxba^2) \\
&= \phi((ab)^3x) + \phi([b,a]) - \phi(ba(ab)^2x) \\
&\quad + \phi(babxba^2) + \phi([a,b]) - \phi(ab^2xba^2).
\end{aligned}$$

By the assumption on ϕ, we have

$\phi(ba(ab)^2x) = \phi(babxba^2), \phi([b,a]) = -\phi([a,b]), \phi(ab^2xba^2) = \phi(xba^3b^2).$

Thus, we have

$$\delta\phi([b,a],(ab)^3x) + \delta\phi([a,b], babxba^2) = \phi((ab)^3x) - \phi(xba^3b^2).$$

□

Lemma 2

$$|\phi((ab)^{3^n}) - \phi(a^{3^n} b^{3^n})| \leq (3^n - 1)D(\phi).$$

Proof.

By Lemma 1, $|\phi((ab)^3 x) - \phi(xba^3b^2)| \leq 2D(\phi)$.

If we substitute

$x = (ab)^{3^n - 3}, (ab)^{3^n-6}(ba^3b^2), (ab)^{3^n-9}(ba^3b^2)^3, \cdots, (ba^3b^2)^{3^{n-1}}$,

we have

$$|\phi((ab)^{3^n}) - \phi((a^3b^3)^{3^{n-1}})| \leq 3^{n-1}(2D(\phi)).$$

In the same way, we have

$$|\phi((a^{3^{n-k}} b^{3^{n-k}})^{3^k}) - \phi((a^{3^{n-k+1}} b^{3^{n-k+1}})^{3^{k-1}})| \leq 3^{k-1}(2D(\phi)).$$

for $k = 1, 2, \cdots, n$.

If we sum up all these terms we obtain what we want. □

Proof of Proposition 2

For any $a, b \in G$,

$$|\bar{\phi}(a) + \bar{\phi}(b) - \bar{\phi}(ab)| = \lim_{n\to\infty} \frac{1}{3^n}|\phi(a^{3^n}) + \phi(b^{3^n}) - \phi((ab)^{3^n})|.$$

Since $|\phi(a^{3^n}) + \phi(b^{3^n}) - \phi(a^{3^n}b^{3^n})| \leq D(\phi)$, we have

$$|\bar{\phi}(a) + \bar{\phi}(b) - \bar{\phi}(ab)| = \lim_{n\to\infty} \frac{1}{3^n}|\phi(a^{3^n}b^{3^n}) - \phi((ab)^{3^n})|.$$

By Lemma 2, we obtain

$$\lim_{n\to\infty} \frac{1}{3^n}|\phi(a^{3^n}b^{3^n}) - \phi((ab)^{3^n})| \leq D(\phi).$$

3. Proof of Main Theorem 1.

Main Theorem 1

Let $m \geq 4$. Then,

$$\mathrm{scl}_{\mathcal{M}_0^m}(\sigma_1) \leq \frac{1}{2m+2+\frac{4}{m-2}}.$$

Lemma 3 (cf. scl)

Let $\phi : G \to \mathbb{R}$ be a homogeneous quasimorphism. Then, we have

1. $\phi(yxy^{-1}) = \phi(x)$,
2. If $xy = yx$, $\phi(xy) = \phi(x) + \phi(y)$.

Proof of Lemma 3 (2).

for any $n \in \mathbb{Z}$, we have

$$\begin{aligned} D(\phi) &\geq |\phi((x^n y^n) - \phi(x^n) - \phi(y^n)| \\ &= |\phi((xy)^n) - \phi(x^n) - \phi(y^n)| \\ &= n|\phi(xy) - \phi(x) - \phi(y)|. \end{aligned}$$

Thus,

$$|\phi(xy) - \phi(x) - \phi(y)| \leq \frac{D(\phi)}{n}.$$

Let $n \to \infty$, then we have $\phi(xy) = \phi(x) + \phi(y)$. □

Let G be a group, and $a \in G$.
If there exist $x, y \in G$ and $n_1, n_2, n_3 \in \mathbb{R}$ (which do not depend on ϕ) such that

$$\phi(x) = n_1\phi(a), \quad \phi(y) = n_2\phi(a), \quad \phi(xy) = n_3\phi(a),$$

for any $\phi \in Q(G)$, then

$$D(\phi) \geq |\phi(x) + \phi(y) - \phi(xy)| = |n_1 + n_2 - n_3||\phi(a)|.$$

$$\mathrm{scl}(a) = \sup_{\phi \in Q(G), D(\phi) \neq 0} \frac{\phi(a)}{2D(\phi)} \leq \frac{1}{2|n_1 + n_2 - n_3|}.$$

We will construct such $x, y \in \mathcal{M}_0^m$ in the next page.

For simplicity, we consider the case when m is even.

Let $x = \sigma_1^2 \sigma_3 \sigma_5 \cdots \sigma_{m-3} \sigma_{m-1}^2$ and $y = \sigma_2 \sigma_4 \sigma_6 \cdots \sigma_{m-2}$.

When $|i - j| \geq 2$ the supports of σ_i and σ_j are disjoint. Hence,

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ when } |i - j| \geq 2.$$

By Lemma 3 (2), we have

$$\begin{aligned}
\phi(x) &= \phi(\sigma_1^2) + \phi(\sigma_3) + \phi(\sigma_5) + \cdots + \phi(\sigma_{m-3}) + \phi(\sigma_{m-1}^2) \\
&= \frac{m+4}{2} \phi(\sigma_1), \\
\phi(y) &= \phi(\sigma_2) + \phi(\sigma_4) + \phi(\sigma_6) + \cdots + \phi(\sigma_{m-2}) \\
&= \frac{m-2}{2} \phi(\sigma_1),
\end{aligned}$$

for any $\phi \in Q(\mathcal{M}_0^m)$.Therefore, it suffices to show

$$\phi(xy) = -\frac{2}{m-2} \phi(\sigma_1).$$

We need some relations of $\mathcal{M}_0^m$ to show

$$\phi(xy) = -\frac{2}{m-2}\phi(\sigma_1).$$

When $a, b \in \mathcal{M}_0^m$ are conjugate, denote $a \sim b$.

Lemma 4 (well-known)

1. $\sigma_1\sigma_2\cdots\sigma_{m-2}\sigma_{m-1}^2\sigma_{m-2}\cdots\sigma_2\sigma_1 = 1$,
2. $(\sigma_{m-2}\sigma_{m-3}\cdots\sigma_2)^{m-2} \sim \sigma_1^2$.

First, we will show

$$xy = (\sigma_1^2\sigma_3\cdots\sigma_{m-3}\sigma_{m-1}^2)(\sigma_2\sigma_4\cdots\sigma_{m-2}) \sim \sigma_1^2\sigma_2\sigma_3\cdots\sigma_{m-2}\sigma_{m-1}^2.$$

$$\begin{aligned}
xy &= (\sigma_1^2\sigma_3\cdots\sigma_{m-3}\sigma_{m-1}^2)(\sigma_2\sigma_4\cdots\sigma_{m-2}) \\
&= \sigma_{m-1}^2(\sigma_1^2\sigma_3\cdots\sigma_{m-5}\sigma_{m-3})(\sigma_2\sigma_4\cdots\sigma_{m-4}\sigma_{m-2}) \\
&\sim (\sigma_1^2\sigma_3\cdots\sigma_{m-3})(\sigma_2\sigma_4\cdots\sigma_{m-2})\sigma_{m-1}^2 \\
&= (\sigma_1^2\sigma_3\cdots\sigma_{m-5}\sigma_{m-3})(\sigma_2\sigma_4\cdots\sigma_{m-4})(\sigma_{m-2}\sigma_{m-1}^2) \\
&= \sigma_{m-3}(\sigma_1^2\sigma_3\cdots\sigma_{m-5})(\sigma_2\sigma_4\cdots\sigma_{m-4})(\sigma_{m-2}\sigma_{m-1}^2) \\
&\sim (\sigma_1^2\sigma_3\cdots\sigma_{m-5})(\sigma_2\sigma_4\cdots\sigma_{m-4})(\sigma_{m-2}\sigma_{m-1}^2)\sigma_{m-3} \\
&= (\sigma_{m-2}\sigma_{m-1}^2)(\sigma_1^2\sigma_3\cdots\sigma_{m-5})(\sigma_2\sigma_4\cdots\sigma_{m-4})\sigma_{m-3} \\
&\sim (\sigma_1^2\sigma_3\cdots\sigma_{m-5})(\sigma_2\sigma_4\cdots\sigma_{m-6}\sigma_{m-4})\sigma_{m-3}\sigma_{m-2}\sigma_{m-1}^2 \\
&= (\sigma_1^2\sigma_3\cdots\sigma_{m-5})(\sigma_2\sigma_4\cdots\sigma_{m-6})(\sigma_{m-4}\sigma_{m-3}\sigma_{m-2}\sigma_{m-1}^2) \\
&\sim \sigma_1^2\sigma_2\sigma_3\cdots\sigma_{m-2}\sigma_{m-1}^2.
\end{aligned}$$

By Lemma 4, we have

$$
\begin{aligned}
\phi(xy) &= \phi(\sigma_1^2\sigma_2\sigma_3\cdots\sigma_{m-2}\sigma_{m-1}^2) \\
&= \phi((\sigma_{m-2}\sigma_{m-3}\cdots\sigma_2)^{-1}) \\
&= -\frac{1}{m-2}\phi((\sigma_{m-2}\sigma_{m-3}\cdots\sigma_2)^{m-2}) \\
&= -\frac{1}{m-2}\phi(\sigma_1^2) \\
&= -\frac{2}{m-2}\phi(\sigma_1).
\end{aligned}
$$

Main Theorem 1

Let $m \geq 4$. Then,

$$\mathrm{scl}_{\mathcal{M}_0^m}(\sigma_1) \leq \frac{1}{2m + 2 + \frac{4}{m-2}}.$$