A hitchhiker's guide to Khovanov homology
– preliminary version 19 June 2014 –

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I dedicate these notes to the memory of Ruty Ben-Zion

Abstract. These notes from the summer school Quantum Topology at the CIRM in Luminy attempt to provide a rough guide to a selection of developments in Khovanov homology over the last fifteen years.

Foreword

There are already too many introductory articles on Khovanov homology and certainly another is not needed. On the other hand by now - 15 years after the invention of subject - it is quite easy to get lost after having taken those first few steps. What could be useful is a rough guide to some of the developments over that time and the summer school *Quantum Topology* at the CIRM in Luminy has provided the ideal opportunity for thinking about what such a guide should look like.

It is quite a risky undertaking because it is all too easy to offend by omission, misrepresentation or other. I have not attempted a complete literature survey and inevitably these notes reflects my personal view, jaundiced as it may often be. My apologies for any offence caused. [1].

I would like to express my warm thanks to Lukas Lewark, Alex Shumakovitch, Liam Watson and Ben Webster.

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[1] You are reading a preliminary version - there is still time to correct any errors, add references and so on - please let me know if you have comments

1. A beginning

There are a number of introductions to Khovanov homology. A good place to start is Dror Bar-Natan's paper

0201043 – *On Khovanov's categorification of the Jones polynomial*

perhaps followed by Alex Shumakovitch review

1101.5614 – *Khovanov homology theories and their applications*

The original paper by Khovanov should be studied

9908171 – *A categorification of the Jones polynomial*

and while a little dated there are always the notes from my last trip to Marseilles

0606464 – *Five lectures on Khovanov homology*

1.1. There is a link homology theory called Khovanov homology

What are the minimal requirements of something deserving of the name *link homology theory* ? We should expect a functor

$$H\colon \mathcal{L}inks \to \mathcal{A}$$

where $\mathcal{L}inks$ is some category of links in which isotopies are morphisms and $\mathcal{A}$ another category, probably abelian, where we have in mind a category of finite dimensional vector spaces or of modules over a fixed ring. This functor should satisfy a number of properties.

- *Invariance.* If $L_1 \to L_2$ is an isotopy then the induced map $H(L_1) \to H(L_2)$ should be an isomorphism.
- *Disjoint unions.* Given two disjoint links L_1 and L_2 we want the union expressed in terms of the parts
$$H(L_1 \sqcup L_2) \cong H(L_1) \Box H(L_2)$$
where $\Box$ is some monoidal operation in $\mathcal{A}$ such as $\oplus$ or $\otimes$.
- *Normalisation.* The value of $H(\text{unknot})$ should be specified. (Possibly also the value of the empty knot)
- *Computational tool.* We want something like a long exact sequence which relates homology of a given link with associated "simpler" ones - something like the Meyer-Vietoris sequence in ordinary homology.

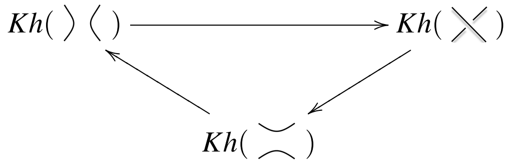
If these are our expectations then Khovanov homology is bound to please. Let us take $\mathcal{L}inks$ to be the category whose objects are oriented links in S^3 and whose morphisms are link cobordisms, that is to say compact oriented surfaces-with-boundary in $S^3 \times I$ defined up to isotopy. All manifolds are assumed to be smooth.

Theorem 1. *(Existence of Khovanov homology) There exists a (covariant) functor*

$$Kh\colon \mathcal{L}inks \to \mathcal{V}ect_{\mathbb{F}_2}$$

satisfying

1. *If $\Sigma\colon L_1 \to L_2$ is an isotopy then $Kh(\Sigma)\colon Kh(L_1) \to Kh(L_2)$ is an isomorphism.*
2. *$Kh(L_1 \sqcup L_2) \cong Kh(L_1) \otimes Kh(L_2)$.*
3. *$Kh(unknot) = \mathbb{F}_2 \oplus \mathbb{F}_2$ and $Kh(\emptyset) = \mathbb{F}_2$.*
4. *If L is presented by a link diagram part of which is*  *then there is an exact triangle*

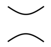


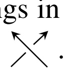
In fact Kh carries a bigrading

$$Kh^{*,*}(L) = \bigoplus_{i,j\in\mathbb{Z}} Kh^{i,j}(L)$$

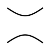
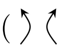
and with respect to this

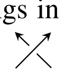
- a link cobordism $\Sigma\colon L_1 \to L_2$ induces a map $Kh(\Sigma)$ of bidegree $(0, \chi(\Sigma))$,
- the generators of the unknot have bidegree $(0,1)$ and $(0,-1)$ (and for the empty knot bidegree $(0,0)$),
- the exact triangle unravels as follows:
  Case I:  For each j there is a long exact sequence

$\xrightarrow{\delta} Kh^{i,j+1}($  $) \to Kh^{i,j}($  $) \to Kh^{i-\omega,j-1-3\omega}($  $) \xrightarrow{\delta} Kh^{i+1,j+1}($  $) \to$

where ω is the number of negative crossings in the chosen orientation of  minus the number of negative crossings in .

Case II:  For each j there is a long exact sequence

$\to Kh^{i-1,j-1}($  $) \xrightarrow{\delta} Kh^{i-1-c,j-2-3c}($  $) \to Kh^{i,j}($  $) \to Kh^{i,j-1}($  $) \xrightarrow{\delta}$

where c is the number of negative crossings in the chosen orientation of  minus the number of negative crossings in .

To prove the theorem one must construct such a functor, but first let's see a few consequences of the existence theorem.

Proposition 1. *If a link L has an odd (resp. even) number of components then $Kh^{*,even}(L)$ (resp. $Kh^{*,odd}(L)$) is trivial.*

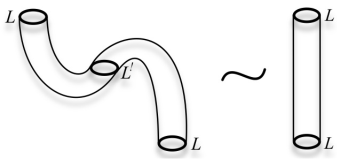
Proof. The proof is by induction on the number of crossing and uses the following elementary result.

> **Lemma 1.** *In the discussion of the long exact sequences above (i) if the strands featured at the crossing are from the same component then ω is odd and c is even, and (ii) if they are from different components then ω is even and c odd.*

For the inductive step we use this and, depending on the case, one of the long exact sequence shown above, observing that in each case two of the three groups shown are trivial. $\square$

Proposition 2. *Letting $L^!$ denote the mirror image of the link L we have $Kh^{i,j}(L^!) \cong Kh^{-i,-j}(L)$.*

Proof. There is a link cobordism $\Sigma : L^! \sqcup L \to \emptyset$ with $\chi(\Sigma) = 0$ obtained by bending the identity cobordism (a cylinder) $L \to L$. Since Kh is a functor there is an induced map of bidegree $(0, \chi(\Sigma)) = (0,0)$



$$\Sigma_* : Kh^{*,*}(L^!) \otimes Kh^{*,*}(L) \to Kh^{*,*}(\emptyset) = \mathbb{F}_2.$$

By a standard "cylinder straightening isotopy" argument the bilinear form is non-degenerate, and the result follows recalling that we are in a bigraded setting so

$$(Kh^{*,*}(L^!) \otimes Kh^{*,*}(L))^{0,0} = \bigoplus_{i,j} Kh^{i,j}(L^!) \otimes Kh^{-i,-j}(L).$$

$\square$

Proposition 3. *For any oriented link L,*

$$\frac{1}{t^{\frac{1}{2}} + t^{-\frac{1}{2}}} \sum_{i,j} (-1)^{i+j+1} t^{\frac{j}{2}} dim(Kh^{i,j}(L))$$

is the Jones polynomial of L.

Proof. Let $P(L) = \Sigma_{i,j}(-1)^i q^j \dim(Kh^{i,j}(L))$ and suppose L is represented by a diagram D. The alternating sum of dimensions in a long exact sequence of vector spaces is always zero, so from the long exact sequence for a negative crossing we have that for each $j \in \mathbb{Z}$ the sum

$$\sum_i (-1)^i \dim(Kh^{i,j+1}(\uparrow\,\uparrow))$$
$$- \sum_i (-1)^i \dim(Kh^{i,j}(\times)) + \sum_i (-1)^i \dim(Kh^{i-\omega, j-1-3\omega}(\asymp))$$

is zero. Written in terms of the polynomial P this becomes

$$q^{-1} P(\uparrow\,\uparrow) - P(\times) + (-1)^{\omega} q^{1+3\omega} P(\asymp) = 0.$$

Similarly, using the long exact sequence for a positive crossing (noting that $c = \omega + 1$) we get

$$(-1)^{\omega} q^{5+3\omega} P(\asymp) - P(\times) + q P(\uparrow\,\uparrow) = 0.$$

Combining these gives

$$q^{-2} P(\times) - q^2 P(\times) + (q - q^{-1}) P(\uparrow\,\uparrow) = 0$$

which becomes the skein relation of the Jones polynomial when $q = -t^{\frac{1}{2}}$. Since $P(\text{unknot}) = q + q^{-1} = -(t^{\frac{1}{2}} + t^{-\frac{1}{2}})$, the uniqueness of the Jones polynomial gives

$$P(D)\,|_{q=-t^{\frac{1}{2}}} = -(t^{\frac{1}{2}} + t^{-\frac{1}{2}})J(D)$$

whence the result. □

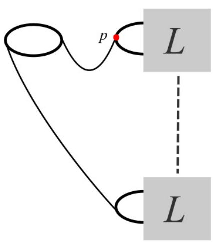
Remark 1. Another expository paper to look at is Rasmussen's

– *Knot polynomials and knot homologies* 0504045

in which both Khovanov homology and Heegaard-Floer knot homology are discussed. A tentative definition of what a knot homology theory should encompass (somewhat different from the expectations given above) is also presented.

1.2. Reduced Khovanov homology

There is a further piece of structure induced on Khovanov homology defined in the following way. The Khovanov homology of the unknot is a ring with unit courtesy of the cobordisms  and  which induce multiplication and unit respectively. The Khovanov homology of a link L together with a chosen point p is a module over this ring, using the link cobordism indicated. A priori this module structure depends on the point p and in particular on the component of L to which p belongs. Although it does not follow from the existence theorem directly, for the version of Khovanov homology presented above (namely over $\mathbb{F}_2$), this structure does not depend on these choices. In fact more is true (again not immediate from the existence theorem) and the structure of $Kh^{*,*}(L)$ over $U^{**} = Kh^{*,*}(\text{unknot})$ can be described as follows: there exists a bigraded vector space $\widetilde{Kh}^{*,*}(L)$ with the property that



$$Kh^{*,*}(L) \cong \widetilde{Kh}^{*,*}(L) \otimes U^{*,*}$$

Theorem 2. *(Existence of reduced Khovanov homology) There exists a (covariant) functor*

$$\widetilde{Kh}^{*,*} : \mathcal{L}inks \to \mathcal{V}ect_{\mathbb{F}_2}$$

satisfying

1. *If $\Sigma : L_1 \to L_2$ is an isotopy then $\widetilde{Kh}(\Sigma)$ is an isomorphism.*
2. $\widetilde{Kh}^{*,*}(L_1 \sqcup L_2) \cong \widetilde{Kh}^{*,*}(L_1) \otimes \widetilde{Kh}^{*,*}(L_2) \otimes U^{*,*}$.
3. $\widetilde{Kh}^{*,*}(\textit{unknot}) = \mathbb{F}_2$ *in bidegree* $(0,0)$.
4. $\widetilde{Kh}^{*,*}$ *satisfies the same long exact sequence (with the same bigradings) written down previously for the unreduced case.*

There is a question about which category of links we should be using here. A natural one would be links with a marked point and cobordisms with a marked line. Similarly to before if L has an odd (resp. even) number of components then $\widetilde{Kh}^{*,\text{odd}}$ (resp. $\widetilde{Kh}^{*,\text{even}}$) is trivial and

$$\sum_{i,j}(-1)^{i+j}t^{\frac{j}{2}}\dim(\widetilde{Kh}^{i,j}(L)) = J(L)$$

In order to compute Khovanov homology we should use our tools for that purpose which are the two long exact sequences.

Exercise 1. (for beginners) Using the long exact sequences calculate the reduced Khovanov homology of the Hopf link, left and right trefoils, and the figure eight knot.

Exercise 2. (for experts) Find the first knot in the tables for which the reduced Khovanov homology can not be calculated using only the long exact sequences and calculations of the reduced Khovanov homology of knots and links occurring previously in the tables.

0201105 Alternating links have particularly simple Khovanov homology. The following is a result of E.S. Lee.

Proposition 4. *For a non-split alternating link L the vector space* $\widetilde{Kh}^{i,j}(L)$ *is trivial unless* $j-2i$ *is the signature of L.*

0409328 This result can also be proved using an approach to Khovanov homology using spanning trees due to Wehrli.

As a corollary we note that for an alternating link $\widetilde{Kh}^{*,*}(L)$ is completely determined by the Jones polynomial and signature.

1.3. Integral Khovanov homology

One can also define an integral version of the above which satisfies the same long exact sequences, but some changes are necessary.

1. functoriality is much trickier
   - up to sign ± 1 everything works okay
   - strict functoriality requires work
2. there is a reduced version but
   - it is dependent on the component of the marked point
   - the relationship to the unreduced theory is more complicated and is expressed via a long exact sequence

$$\xrightarrow{\delta} \widetilde{Kh}_{\mathbb{Z}}^{i,j+1}(L,L_\alpha) \longrightarrow Kh_{\mathbb{Z}}^{i,j}(L) \longrightarrow \widetilde{Kh}_{\mathbb{Z}}^{i,j-1}(L,L_\alpha) \xrightarrow{\delta}$$

     where L_α is a chosen component of L.
   - in this exact sequence the coboundary map δ is zero modulo 2

The integral theory is related to the $\mathbb{F}_2$-version by a universal coefficient theorem:

$$0 \longrightarrow Kh^{i,j}_{\mathbb{Z}}(L) \otimes \mathbb{F}_2 \longrightarrow Kh^{i,j}_{\mathbb{F}_2}(L) \longrightarrow \mathrm{Tor}(Kh^{i+1,j}_{\mathbb{Z}}(L), \mathbb{F}_2) \longrightarrow 0$$

There is a similar universal coefficient theorem relating the two reduced theories.

Any theory defined over the integers has a chance of revealing interesting torsion phenomena. Unreduced integral Khovanov homology has a lot of 2-torsion and much of this arises in the passage from reduced to unreduced coming from that fact that in the long exact sequence relating the two the coboundary map is zero mod 2. Correspondingly the reduced theory has much less 2-torsion.

Proposition 5. *The reduced integral Khovanov homology of alternating links has no 2-torsion.*

Proof. Suppose that L is non-split. Any 2-torsion in $\widetilde{Kh}^{i,j}_{\mathbb{Z}}(L)$ would contribute non-trivial homology in $\widetilde{Kh}^{i,j}_{\mathbb{F}_2}(L)$ via the leftmost group in the universal coefficient theorem for reduced theory and also in $\widetilde{Kh}^{i-1,j}_{\mathbb{F}_2}(L)$ via the Tor group. This contradicts the conclusion of Proposition 4, namely that there is only non-trivial homology when $j - 2i = \text{signature}(L)$. □

In general torsion is not very well understood. Calculations (by Alex Shumakovitch) show

- the simplest knot having 2-torsion in reduced homology has 13 crossings, for example 13n3663
- the simplest knot having odd torsion in unreduced homology is $T(5,6)$ which has a copy of $\mathbb{Z}/3$ and a copy of $\mathbb{Z}/5$.
- the simplest knot having odd torsion in reduced homology is also $T(5,6)$ which has a copy of $\mathbb{Z}/3$
- some knots, e.g. $T(5,6)$, have odd torsion in unreduced homology which is not seen in the reduced theory, but the other way around is also possible: $T(7,8)$ has an odd torsion group in reduced that is not seen in unreduced.

Torus knots are very interesting in the study of odd torsion and in fact this is the only place where odd torsion has been observed so far. In general torsion remains quite a mystery.

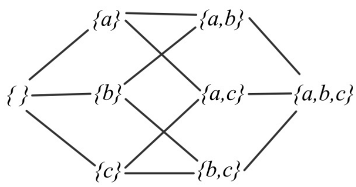
2. Constructing Khovanov homology

2.1. The Khovanov cube

The central combinatorial input in the construction of Khovanov homology is a hypercube decorated by vector spaces (or abelian groups) known variously as "the cube", "the cube of resolutions" and "the Khovanov cube". Such a thing is constructed from a diagram representing the link in question. This is by now considered to be "well known" and to know it well you should read the articles by Bar-Natan or Shumakovitch.

Exercise 3. (for beginners) Read sections 3.1 and 3.2 of Bar-Natan's *On Khovanov's categorification of the Jones polynomial* and/or sections 2.1 and 2.2 of Shumakovitch's *Khovanov homology theories and their applications* on the Khovanov cube. You need to know what a *resolution* (or *complete smoothing*) of a link diagram is; how to assemble these into a *cube*; how resolutions differ if they are found at each end of a cube edge and why *circles fuse* or *circles split*; how to interpret cube edges as cobordisms; how to attach a vector space to each resolution; how to *flatten* the cube and make a complex.

This cube of resolutions is actually an example of something more general: it is a Boolean lattice equipped with a local coefficient system. It can be a convenient point of view so we now explain it. Let D be a diagram and let X_D be the set of crossings of D. We can form the poset $\mathbb{B}(X_D)$ of subsets of X_D ordered by inclusion, which is to say the Boolean lattice on X_D.



If you want to see a cube make the Hasse diagram of this poset which is the graph having vertices the elements of the set X_D and an edge A •——• B if and only if $A < B$ and there is no C such that $A < C < B$ (in this case say B *covers* A). The convention adopted here will be that if $A < B$ then the Hasse diagram features A to the left of B.

The poset $\mathbb{B}(X_D)$ can be regarded as a category with a unique morphism $A \to B$ whenever $A \leq B$. The process of decorating the cube seen in the Khovanov cube (to vertices associate vector spaces; to edges associated linear maps) amounts to defining a (covariant) functor

$$F_D \colon \mathbb{B}(X_d) \to \mathcal{V}ect_{\mathbb{F}_2}$$

In fact the functor takes values in graded vector spaces: letting $||A||$ denote the number of connected components in the resolution of $A \subset X_D$, there is the associated graded vector space V_A defined by $V^{\otimes ||A||}$ and the graded vector space $F_D(A)$ is a shifted version of this, namely $F_D(A) = V_A\{||A|| + n_+ - 2n_-\}$ where n_+ and n_- are the number of positive and negative crossings in the diagram D.

2.2. *Extracting information from the cube*

Having defined this decorated hypercube the task is to extract some computable information from it. This comes in the form of a cochain complex. For a set S, a functor $F \colon \mathbb{B}(S) \to \mathcal{V}ect_{\mathbb{F}_2}$ gives rise to a cochain complex by defining cochain groups

$$\mathcal{C}^i = \bigoplus_{A \subset S, |A| = i} F(A)$$

with differential $d \colon \mathcal{C}^i \to \mathcal{C}^{i+1}$ given by the "matrix elements" $d_{A,B} \colon F(A) \to F(B)$ where A and B range over subsets of size i and $i+1$ respectively and there is such a