matrix element whenever B covers A. It can easily be checked that $d^2 = 0$ (remember we are working over $\mathbb{F}_2$). If F takes values graded objects we should require the matrix elements to have grading zero and the resulting complex will be bigraded.

Definition 1. *Let L be an oriented link and let D be a diagram representing L having n crossings of which n_- are negative. Let $F_D\colon \mathbb{B}(X_D) \to \mathcal{V}ect_{\mathbb{F}_2}$ be the Khovanov cube. Apply the above construction to give a bigraded cochain complex $\mathcal{C}^{*,*}(D)$ and define the* Khovanov homology *of L to be*

$$Kh^{i,*}(D) = H^{i+n_-}((\mathcal{C}^{*,*}(D), d)$$

Remark 2. Gradings: for $A \subset X_D$, the element $v \in V_A$ of degree k defines a cochain in bidegree

$$(|A| - n_-, k + |A| - 2n_- + n_+)$$

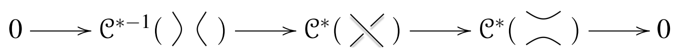
This construction appears to depend on the diagram, but Khovanov's result is that it doesn't.

Theorem 3. *Up to isomorphism the definition above does not depend on the choice of diagram representing the link.*

If we are not working over $\mathbb{F}_2$ the construction above requires one modification but otherwise everything remains the same. As it stands d^2 is zero only mod 2 but this can be rectified by introducing a *signage* function $\varepsilon\colon \{\text{edges}\} \to \mathbb{Z}/2$ which satisfies $\varepsilon(e_1) + \varepsilon(e_2) + \varepsilon(e_3) + \varepsilon(e_4) = 1 \bmod 2$ whenever $e_1, \cdots, e_4$ are the four edges of a square. The required modification is to the matrix elements: we must now take $(-1)^{\varepsilon} d_{A,B}$ which will give $d^2 = 0$.

Exercise 4. Show that $Kh(L_1 \sqcup L_2) \cong Kh(L_1) \otimes Kh(L_2)$.

Exercise 5. Show that if L is presented by a diagram part of which is  then there is a short exact sequence of complexes



In fact one need not go all the way to taking homology: the cochain complex itself is an invariant up to homotopy equivalence of complexes.

2.3. Functoriality

The existence theorem asserts that there is a *functor* and the domain category has link cobordisms as morphism. We therefore need to know how to define linear maps associated to such cobordisms.

Exercise 6. Find out how link cobordisms are represented by *movies* and how to associate maps in Khovanov homology to such things (by looking in the papers given below).

Because of the dependence on diagrams there are things to check. One can show that up to an over all factor of ± 1 there is no dependence of the maps on the diagrams chosen. This is enough to give a functor over $\mathbb{F}_2$. The papers showing functoriality up to ± 1 are Jacobsson's paper

- *An invariant of link cobordisms from Khovanov homology* 0206303

and Khovanov's paper

0207264 – *An invariant of tangle cobordisms*

and then reproved by Bar-Natan

0410495 – *Khovanov's homology for tangles and cobordisms*

It is hard work to remove the innocent looking "up to ± 1" and something additional is needed to make it work. Using Bar-Natan's local point of view (see below) it is addressed in the paper by Clark,Morrison and Walker

0701339 – *Fixing the functoriality of Khovanov homology*

which requires working over $\mathbb{Z}[i]$. A somewhat similar picture is given by Caprau

0707.3051 – *An $sl(2)$ tangle homology and seamed cobordisms*

A different construction working over $\mathbb{Z}$ is given in Blanchet's

1405.7246 – *An oriented model for Khovanov homology*

2.4. Another extraction technique

There is another, more abstract, way of extracting information from the cube. To motivate this kind of approach think about the definitions of group cohomology where one can either define an explicit cochain complex using the bar resolution or use derived functors. Each approach has its uses and if the definition is taken to be the explicit complex then the derived functors approach becomes an "interpretation", but if the definition is in terms of derived functors then the explicit complex becomes a "calculation".

There is a way of defining cohomology of posets equipped with coefficient systems which is "well known" and with a small modification to the cube, it gives an alternative way of getting Khovanov homology. Let Q be the poset formed from $\mathbb{B}(X)$ by the addition of a second minimal element. Extend the functor F_D to Q
1112.3460 by sending this new element to the trivial group. Everitt-Turner show that Khovanov homology can be interpreted as the right derived functors of the inverse limit functor:

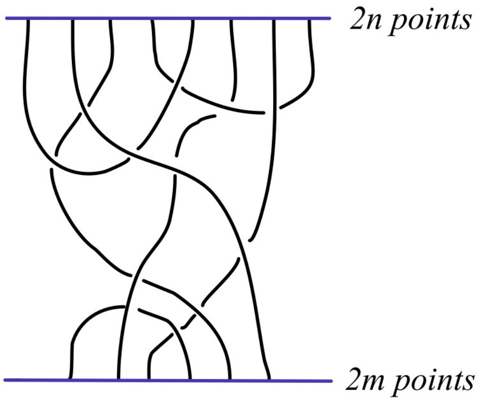
$$Kh(D) = R^* \varprojlim_{Q} F_D$$

3. Tangles

The topology of the resolutions of a link diagram require knowledge of the whole diagram and this is used in the construction of Khovanov homology (circles fuse or split depending on global information). None the less diagrams are made up of more basic pieces, namely tangles, and so it is natural to ask if Khovanov homology may be defined more locally. The difficulty is that while piecing together geometric data is easy, doing the same with algebraic data is never so simple.

3.1. Khovanov's approach

The first approach is due to Khovanov who studies (m,n)-tangles. 0103190



For such a diagram T there is a cube of resolutions as before. To $A \subset X_T$ one associates

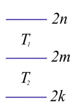
$$M_A = \bigoplus P[x_\gamma \mid \gamma \text{ a circle }]/(x_\gamma^2 = 0)$$

where the direct sum is over all tangle closures of the type shown here.

This is an (H^m, H^n)-bimodule where $\{H^i\}$ is a certain family of rings. By the usual extraction of a complex from a cube this yields a complex of bi-modules $\mathcal{C}(T)$. When $m = n = 0$ one recovers the usual Khovanov complex. Isotopic tangles produce complexes that are homotopy equivalent and the construction is functorial (up to ± 1) with respect to tangle cobordisms. 0267264

The key new property is that by using the bi-module structure tangle composition can be captured algebraically.

Proposition 6. *Let T_1 be a (m,n)-tangle and T_2 a (k,m)-tangle. Then,*



$$\mathcal{C}(\;\cdot\;) \simeq \mathcal{C}(T_2) \otimes_{H^m} \mathcal{C}(T_1)$$

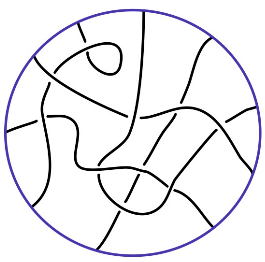
3.2. Bar-Natan's approach

0410495 A different approach to locality is due to Bar-Natan as his viewpoint has turned out to be very influential. The functor $F_D\colon \mathbb{B}(X_D) \to \mathcal{V}ect$ is a two step process: firstly make a cube of resolutions (which are geometric objects) and secondly associate to to these some algebraic data. The first step requires defining a functor from $\mathbb{B}(X_D)$ to a cobordism category and the second step consists of applying a 1+1-dimensional TQFT to the first step. Bar-Natan's central idea is to work with the "geometric" cube (the first step) as long as possible delaying the application of the TQFT until the last possible moment (or even not at all).

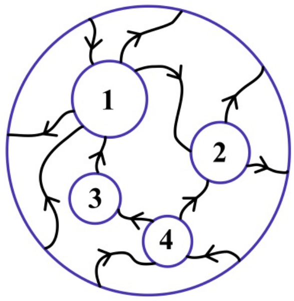
$$F_D\colon \mathbb{B}(X_D) \longrightarrow \mathrm{Cob}_{1+1} \xrightarrow{\quad \mathrm{TQFT} \quad} \mathcal{V}ect$$

Distilling the essential operations used to construct a cochain complex from the functor F_D one sees that we needed to 1) take direct sums of vector spaces (in the step often referred to as "flattening the cube"), and 2) assemble a linear map out of the matrix elements which involved taking linear combinations of maps between vector spaces. In order to delay the passage to the algebra and to build some notion of "complex" in the setting of a cobordism category we need some equivalent of these two operations. What is done is to replace direct sum by the operation of taking formal combinations of objects (closed 1-manifolds) and allowing linear combinations of cobordisms. A typical morphism will be a matrix of formal linear combinations of cobordisms. In this way it is possible to define a "formal" complex $[\![D]\!]$ associated to D. It is no longer possible to take homology of such formal complexes because we are working in a non-abelian category (the kernel of a linear combination of cobordisms makes no sense, for example) but one still has the notion of homotopy equivalence of formal complexes and indeed if $D \sim D'$ then $[\![D]\!] \simeq [\![D']\!]$.

This approach works perfectly well for tangles too. Given a tangle T of the type shown below a resolution will typically involved 1-manifolds with- and without-boundary and the cobordism category must be adapted appropriately but a formal complex $[\![T]\!]$ may be constructed as above. Things are as they should be because given isotopic tangles T_1 and T_2 then there is a equivalence of formal complexes $[\![T_1]\!] \simeq [\![T_2]\!]$. Moreover this construction is functorial (up to ± 1) with respect to tangle cobordisms.



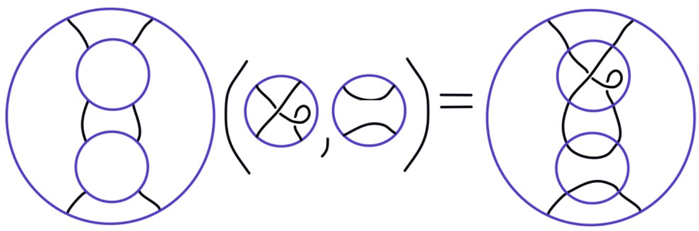
But now comes the beauty of this approach: by insisting on staying on the geometric side of the street for so long, the composition of tangles is accurately reflected at the level of formal complexes as well. The combinatorics of tangle composition is captured by the notion of a *planar algebra*: to each d-input arc diagram D like the one shown here



there is an operation

$$D\colon \mathcal{T}ang \times \cdots \times \mathcal{T}ang \to \mathcal{T}ang$$

defined by plugging the holes. For example



These operations are subject to various composition criteria that make up the structure of a planar algebra. The category $\mathcal{T}ang$ is very naturally a planar algebra, but other categories may admit the structure of a planar algebra too - all that is needed is operations of the type above. The category of formal complexes above (the one in which $[\![T]\!]$ lives) is an example - for the details of the construction read Bar-Natan's paper.

Proposition 7. *The construction sending a tangle T to the associated formal complex $[\![T]\!]$ respects the planar algebra structures defined on tangles and formal complexes. (In other words $[\![-]\!]$ is a morphism of planar algebras)*

Using the planar algebra structure all tangles can be built out of single crossings. What the proposition is telling us is that the same is true of the formal complex: it is enough to specify $[\![-]\!]$ on single crossings and the rest comes from the planar algebra structure.

The next question to ask about this approach is: how does the planar algebra structure interact with link cobordisms? What is needed is an extension of the notion of planar algebra to the situation where there are morphisms between the planar algebra constituents. The name given to the appropriate structure is a *canopolis*. Again read Bar-Natan's paper for details. Working with this local approach makes far more digestible the proofs of invariance and functoriality.

3.3. Further remarks on tangles

1. If one wishes to apply a TQFT to get something algebraic out of Bar-Natan's geometric complex one needs something slightly different capable of handling 0606331 manifolds with boundary. The appropriate thing is an open-closed TQFT, and these studied in the context of Khovanov homology by Lauda and Pfeiffer.
2. 0606318 Bar-Natan explains the technique of *de-looping* which helps simplify the complex at an early stage. His local point of view has sped up computer calculations considerably.
   0202199 Though very different in approach the de-looped complex is closely related to the one given by Viro.
3. 1304.0463 The question of algebraic gluing of tangle components has been carefully studied by Lawrence Roberts who draws inspiration from bordered Heegaard Floer homology. He builds (considerably) on the work of Asaeda-Przytycki-Sikora
   0410238 who study the skein module of tangles in a way very similar to Khovanov's original construction.

4. Variants

In Bar-Natan's approach the construction of Khovanov homology was very clearly divided into two steps: first construct the geometric cube, then apply a TQFT to obtain something algebraic. This means that any TQFT satisfying certain properties will give a link homology theory. Among these there is a universal example which in terms of circle variables has multiplication given by

$$x_\alpha x_\beta = t + hx_\gamma$$

where α and β fuse into γ. To ensure a bigraded theory the ground ring must also be graded and contain h and t of degree -2 and -4 respectively. For Khovanov homology we can take $t = h = 0$ and the bigrading of the ground ring can be concentrated in degree zero unproblematically. It should be noted that in Khovanov's original paper $t = 0$ but h is left in place as a formal variable in the ground ring.

4.1. Lee Theory

One special case is obtained by working over $\mathbb{Q}$ and setting $h = 0$ and $t = 1$. This
0210213 theory - the first variant of Khovanov homology to appear - is due to E.S. Lee. Since $\mathbb{Q}$ is ungraded here this means that Lee theory is a singly graded theory.

The two most important facts about Lee theory are:

- it can be completely calculated
- it is filtered

Lee proves:

Theorem 4. *Let K be a knot. Then*

$$Lee^i(K) \cong \begin{cases} \mathbb{Q} \oplus \mathbb{Q} & i = 0 \\ 0 & else \end{cases}$$

Let L be a two component link. Then

$$Lee^i(K) \cong \begin{cases} \mathbb{Q} \oplus \mathbb{Q} & i = 0, \ or\ lk(the\ two\ components) \\ 0 & else \end{cases}$$

In general, for a k component link $\sum dim(Lee^i(L)) = 2^k$ and there is a formula for the degrees of the generators in terms of linking numbers.

Bar-Natan and Morrision obtain a new proof this result in the context of Bar-Natan's local theory. 0606542

The filtration leads to the following (implicit in Lee's work, made explicit by Rasmussen):

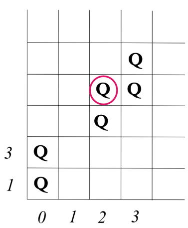
Theorem 5. *Let L be a link and γ its number of components modulo two. There exists a spectral sequence, the* Lee-Rasmussen spectral sequence*, which has the form* 0210213 0402131

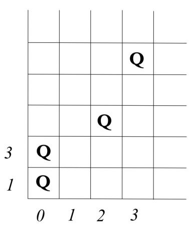
$$E_2^{p,q} = Kh_{\mathbb{Q}}^{p+q,2p+\gamma} \Longrightarrow Lee^*(L).$$

The differentials have the form $d_r \colon E_r^{p,q} \to E_r^{p+q,q-r-1}$. Moreover, each page of the spectral sequence is a link invariant.

Remark 3.
1. If one re-grades so that the differentials are expressed in terms of the gradings of Khovanov homology (rather than the pages of the spectral sequence) the differential d_r is zero for r odd and has bigrading $(1, 2r)$ when r is even.
2. In all known examples this spectral sequence (over $\mathbb{Q}$) collapses at the E_2-page. It is still an open question as to whether this is always the case or not.

The utility of this spectral sequence is that it puts considerable restrictions on the allowable shape of Khovanov homology. As an example consider an attempted calculation of the rational (unreduced) Khovanov homology of the right-handed trefoil only using the skein long exact sequences. At some point you will find that you need additional information (some boundary map may or may not be zero and you have no way of telling without some further input).





You can conclude that the Khovanov homology must be one of the two possibilities shown here. The existence of the Lee-Rasmussen spectral sequence tells you that the correct answer is on the right: the two generators that survive to the E_∞-page of the spectral sequence are the two in homological degree zero and all the others must be killed by differentials; if the Khovanov homology were as given on the left, then a quick look at the degrees of the differentials shows that the generator in bi-degree (2,7) could never be killed, giving a contradiction.

Remark 4. Over other rings Lee theory behaves as follows:

1. over $\mathbb{F}_p$ for p odd it behaves as Lee theory over $\mathbb{Q}$ (i.e. it is "degenerate")
2. over $\mathbb{F}_2$ it is isomorphic (as ungraded theories) to $\mathbb{F}_2$ Khovanov homology
3. over $\mathbb{Z}$ it has a free part of rank $2^{\text{no. of components}}$, no odd torsion, but a considerable amount of 2-torsion.

4.2. Bar-Natan Theory

0410495 Another interesting case is to take $t = 0$ and $h = 1$. This theory, known as *Bar-Natan* theory, is quite similar to Lee theory: it is a "degenerate" theory requiring only linking numbers for a full calculation; it is filtered and there is a Lee-
0411275 Rasmussen type spectral sequence. It has been studied by Turner. There are some differences (which possibly make it a better theory than Lee theory): the integral version also degenerates; there is a reduced version with a reduced Lee-Rasmussen type spectral sequence.

5. Odd Khovanov homology

The construction of Khovanov homology makes no demands on the order of the circles appearing in a resolution. At the algebraic level this is reflected in the commutativity of the Frobenius algebra used. Put differently the vector space associated to a resolution has a variable associated to each circle and these variables commute among themselves. If one could impose a local ordering of strands near crossings then one might hope that this commutativity requirement could be removed. The subject of *odd* Khovanov homology is one approach to achieving this.

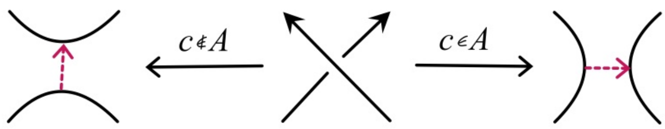
The defining paper is the one by Ozsváth, Rasmussen and Szabó

0710.4300 – *Odd Khovanov homology*

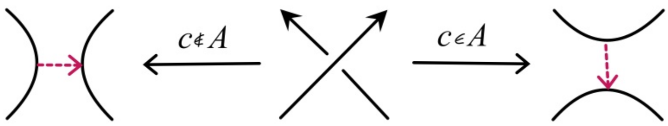
and there is also a nice expository article with many calculations by Alex Shumakovitch

1101.5607 – *Patterns in odd Khovanov homology*

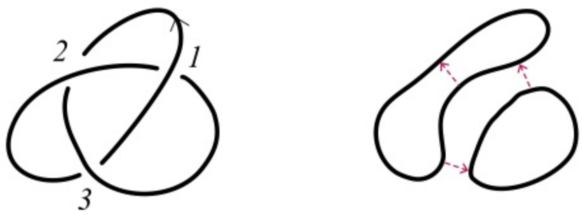
While constructing the cube of resolutions we can retain some additional information (the dotted arrow) given by the following two rules.
Negative crossing:



Positive crossing:



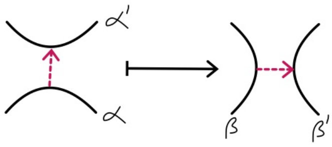
If presented with a local piece around a crossing for which the two strands are from different components we may now order these by the decree, tail before head. The right-handed trefoil has typical resolution of the form below (the one shown is for the subset $A = \{2\}$). Note that this does not give a global ordering on the circles and indeed in the example below it is clear that this would not be possible with the given rules.



The construction of a decorated cube now proceeds as follows. For $A \subset X$ resolve the crossings as above and associate to A the exterior algebra (over some fixed ring R)

$$\Lambda_A = \Lambda[x_\gamma \mid \gamma \text{ a component of the resolution}]$$

This tells us what to do on cube vertices. On edges, if $|B| = |A| + 1$ we must define a map $d_{A,B} \colon \Lambda_A \to \to_B$. In a neighbourhood of the additional crossing we see the following local change.



There are now two cases.

- if $\alpha \neq \alpha'$ (in which case we also have $\beta = \beta'$) we define $d_{A,B}$ to be the algebra map given by

$$1 \mapsto 1, \quad x_\alpha, x_{\alpha'} \mapsto x_\beta, \quad x_\gamma \mapsto x_\gamma \text{ for } \gamma \neq \alpha, \alpha'$$

- if $\alpha = \alpha'$ (in which case we also have $\beta \neq \beta'$) we define $d_{A,B}$ to be the *module* map given by

$$x_\alpha \wedge v \mapsto x_\beta \wedge x_{\beta'} \wedge v, \quad v \mapsto (x_\beta - x_{\beta'}) \wedge v$$

where x_α is assumed not to appear in v. Thus $1 \mapsto x_\beta - x_{\beta'}$ and $x_\alpha \mapsto x_\beta \wedge x_{\beta'}$.

The first of these makes no use of the local ordering, but in the second the asymmetry is very clear.

Exercise 7. Check that if the underlying ring is the field $\mathbb{F}_2$ then the exterior algebra is isomorphic to the truncated polynomial algebra and the maps $d_{A,B}$ agree with the ones used in construction of (ordinary) Khovanov homology.

For ordinary Khovanov homology the construction gives a functor $\mathbb{B}(X) \to \mathbf{Mod}_R$ without further trouble. Or, put differently, the square faces of the cube commute. (Immediately afterwards a sign assignment is made, but that is to turn commuting squares into anti-commuting ones which is only necessary because of the particular extraction technique used to obtain a complex out of the functor.) Here, for odd Khovanov homology things are not so simple and there is not obviously functor $\mathbb{B}(X) \to \mathbf{Mod}_R$; some squares commute, others anti-commute and others still produce maps which are zero. After a fair bit of digging into the possible cases Ozsváth-Rasmussen-Szabó prove:

Proposition 8. *There exists a signage making all squares commute.*

This gives a functor $F_D^{\text{odd}} \colon \mathbb{B}(X_D) \to \mathbf{Ab}$ and by one of the extraction techniques discussed previously this yields a complex whose homology defines *odd Khovanov homology*, denoted $Kh_{\text{odd}}^{*,*}(D;R)$.

Remark 5. The bigrading is as follows: for $A \subset X$, the element $v \in \Lambda^k \subset \Lambda_A$ defines a cochain with bigrading

$$(|A| - n, ||A|| + |A| + n_+ - 2n_- - 2k)$$

where $||A||$ is the number of circles in the resolution defined by A.

Here is a summary of some of the properties of odd Khovanov homology.

- there are skein long exact sequences precisely as for ordinary Khovanov homology (with the same indices).
- the Jones polynomial is obtained as

$$\sum_{i,j} (-1)^i q^j \dim(Kh_{\text{odd}}^{i,j}(L))\big|_{q=-t^{\frac{1}{2}}} = -(t^{\frac{1}{2}} + t^{-\frac{1}{2}})J(D)$$

- there is a reduced version, $\widetilde{Kh}_{\text{odd}}^{*,*}$, satisfying

$$Kh_{\text{odd}}^{i,j} \cong \widetilde{Kh}_{\text{odd}}^{i,j+1} \oplus \widetilde{Kh}_{\text{odd}}^{i,j-1}$$

  and which does not depend on the component of the base-point. So Kh_{odd} stands in the same relationship to $\widetilde{Kh}_{\text{odd}}$ as $Kh_{\mathbb{F}_2}$ to $\widetilde{Kh}_{\mathbb{F}_2}$ which is very different to the relationship between $Kh_{\mathbb{Z}}$ and $\widetilde{Kh}_{\mathbb{Z}}$.
- over $\mathbb{F}_2$ odd and ordinary Khovanov homology coincide (reduced and unreduced); this is courtesy of Exercise 7.