- $\widetilde{Kh}^{*,*}_{\text{odd}}(\text{alternating}) \cong \widetilde{Kh}^{**}(\text{alternating})$ but in general $\widetilde{Kh}$ neither determines or is determined by $\widetilde{Kh}_{\text{odd}}$.

Remark 6. 1. The is a spectral sequence with E_2-page $Kh_{\mathbb{F}_2}(L^!)$ converging to the
0309170 Heegaard-Floer homology of the double branched cover branched along L . To
lift this integrally the correct theory to put at E_2 is (conjecturally) odd integral
Khovanov homology. Indeed this was one of the motivations for the invention
of that theory.

2. There are other interesting spectral sequences featuring odd Khovanov homol- 1205.2256
ogy at the E_2-page. Beir has one starting with odd Khovanov homology and
converging to an integral version of a theory made by Szabo. Scaduto has one
starting with odd Khovanov homology and converging to the framed instanton 1401.2093
homology of the double cover.
3. Kris Putyra extends the Bar-Natan picture to 2-categories which gives a general 1310.1895
framework into which odd Khovanov homology fits.
A somewhat similar approach is taken in Beliakova-Wagner where 2-cateogries
consisting of links, cobordisms and diffeomorphisms play a role. 0910.5050
4. On seeing a typical resolution in the construction of odd Khovanov homology
it is tempting to convert it into a graph whose vertices are the circles and whose 0903.3746
directed edges are the dotted arrows. There is a description of of odd Khovanov
homology in terms of these *arrow graphs* due to Bloom.

6. Generalisations

There is a general procedure, due to Witten and Reshetikhin-Turaev, for the construction of (quantum) link invariants using the representation theory of quantum groups as input. Starting with a simple Lie algebra $\mathfrak{g}$, link components are labelled with irreducible representations of the quantum group $U_q(\mathfrak{g})$ to produce a link invariant. From this point of view the Jones polynomial arises from the two dimensional representation when $\mathfrak{g} = sl(2)$.

An important and natural question in Khovanov homology is: are there link homology theories associated to other Lie algebras which generalise Khovanov homology in some appropriate sense.

6.1. Khovanov-Rozansky $sl(N)$-homology

An obvious place to start is $\mathfrak{g} = sl(N)$; the case $N = 2$ is already done and the analogue of the Jones polynomial, the $sl(N)$-polynomial has been extensively studied.

A very nice summary and possibly the best place to start is Rasmussen's paper:

- *Khovanov-Rozansky homology of two-bridge knots and links* 0508510

Details are contained in the original paper by Khovanov and Rozansky:

- *Matrix factorizations and link homology* 0401268

Theorem 6. *(Existence of $sl(N)$-homology) There exists a (covariant) projective functor*

$$KR_N^{*,*}\colon \mathcal{L}inks \to \mathcal{V}ect_{\mathbb{Q}}$$

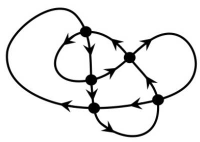
satisfying

1. *If $\Sigma\colon L_1 \to L_2$ is an isotopy then Σ_* is an isomorphism.*
2. $KR_N^{*,*}(L_1 \sqcup L_2) \cong KR_N^{*,*}(L_1) \otimes KR_N^{*,*}(L_2)$.
3. $KR_N^{i,j}(unknot) = \begin{cases} \mathbb{Q} & i = 0 \text{ and } j = 2k - N - 1 (k = 1, \ldots, N) \\ 0 & else \end{cases}$
4. *There are long exact sequences:*

$\xrightarrow{\delta} KR_N^{i-1,j+N}$  $\to KR_N^{i,j}$  $\to KR_N^{i,j+N-1}$  $\xrightarrow{\delta}$

$\xrightarrow{\delta} KR_N^{i,j-N+1}$  $\to KR_N^{i,j}$  $\to KR_N^{i+1,j-N}$  $\xrightarrow{\delta}$

Immediately we see there is something fishy with this: the long exact sequences feature an as yet undefined object. In fact $KR_N^{*,*}$ assigns a bigraded vector space to each *singular link diagram* (where crossing of the form ,  and  are allowed). Up to isomorphism this assignment is invariant on deforming the diagram by Reidemeister moves away from singularities. The situation is not quite as good as for Khovanov homology because even with perfect information about the long exact sequences, the basic normalising set of object consists not of one single simple object (the unknot) but an infinity of 4-valent planar graphs such as the one shown here.



Remark 7. If Σ is a cobordism then Σ_* has bi-degree $(0, (1-N)\chi(\Sigma))$.

Exercise 8. Attempt a computation of $KR_N^{*,*}$(Hopf link) from the existence theorem, carefully observing why this is much harder than when attempting the same computation for Khovanov homology.

Proposition 9. *Let $P_N(D) = \Sigma_{i,j}(-1)^i q^j dim(KR_N^{i,j}(L))$. We have*

$q^{-N}P_N($  $) - q^N P_N($  $) + (q - q^{-1})P_N($  $) = 0$

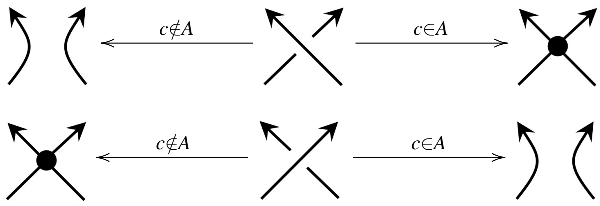
and

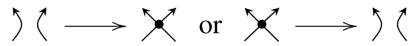
$$P_N(unknot) = \frac{q^N - q^{-N}}{q - q^{-1}} = \sum_{k=1}^{N} q^{2k-N-1}$$

We recognise these two properties as the ones characterising the $sl(N)$-polynomial showing that P_N *is* the $sl(N)$-polynomial.

Exercise 9. Prove this proposition using the long exact sequences and the fact that the alternating sum of dimensions in a long exact sequence is always zero.

The construction of $sl(N)$-homology (and the proof of the existence theorem above) proceeds by resolving each crossing into either the singular resolution or the oriented resolution giving 2^n choices which may herded, as before, into a cube of resolutions, this time each resolution being a planar singular graph. Given a subset of crossings $A \subset X$ the rules for making a resolution are:



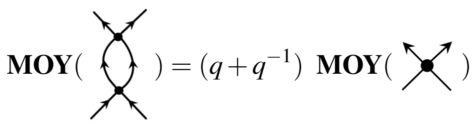
Along cube edges we see local changes: 

and what is needed is a way of associating algebraic data to resolutions and maps to cube edges. This would give a functor $\mathbb{B}(X) \to \mathcal{V}ect_{\mathbb{Q}}$ from which a complex and its homology can be extracted as before. For this to be worth anything it must result in a link invariant and therein, of course, lies the difficulty.

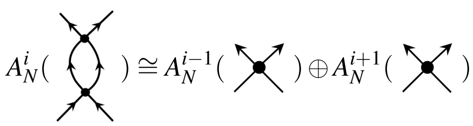
Khovanov and Rozansky employ *matrix factorizations* in order to carry this out. There are some guiding principles coming from the description of the $sl(n)$-polynomial given by H. Murakami, Ohtsuki and Yamada who describe it in terms of certain graphs which suitably interpreted are the ones considered here. One may think of their construction as associating a Laurent polynomial $\mathbf{MOY}(\gamma)$ to each planar singular graph γ and the $sl(N)$-polynomial is then expressed as a sum (over resolutions) of such polynomials. Matrix factorizations can be used to make an assignment

$$A_N^*(-): \text{Planar singular graphs} \to \text{Graded vector spaces}$$

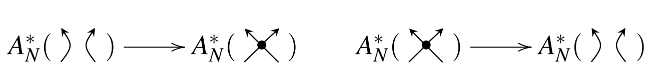
such that $\sum_i q^i \dim A_N^i(\Gamma) = \mathbf{MOY}(\Gamma)$. The local relations satisfied by $\mathbf{MOY}(-)$ are lifted to $A_N^*(-)$; for example



becomes



Moreover, there are maps



and applying $A_N^*(-)$ to the cube of resolutions gives a functor $\mathbb{B}(X) \to \mathcal{V}ect_{\mathbb{Q}}$, giving a complex whose homology defines $sl(N)$-homology.

Remark 8. (on gradings for which we follow the conventions given in Rasmussen's paper cited above). Let $A \subset X$ and let Γ be the associated resolution. If $x \in A_N^k(\Gamma)$ then the corresponding element of the (bi-graded) complex has bi-degree

$$(|A| - n_+, k - i + (N-1)(n_+ - n_-))$$

Calculations with $sl(N)$-homology are much harder than for Khovanov homology. An understanding of why this is so can be obtained by attempting to compute the $sl(N)$-homology of Hopf link and comparing it to the Khovanov homology calculation. It simply gets worse from there on in. Despite this there are programs to carry out computations which give a wealth of data and some interesting conjectures and calculations. Here are some of these.

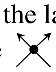
- Jake Rasmussen has completely determined the $sl(N)$-homology of two bridge
0508510 knots in terms of the HOMFLYPT polynomial and signature.

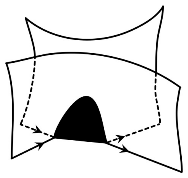
  Proposition 10. *Let K be a two bridge knot and $T(K)$ its HOMFLYPT polynomial. Then for $N > 4$ (working over $\mathbb{C}$)*

  $$\sum_{i,j} t^i q^j dim(\widetilde{KR}_N^{i,j}(K)) = (-t)^{\frac{\sigma}{2}}[T(K)]_{a \mapsto q^N t^{-1}, b \mapsto iqt^{\frac{1}{2}}}$$

1310.3100 Notice that this result computes the *reduced* $sl(N)$-homology. The reduced and un-reduced theories are related by a spectral sequence studied by Lewark.
- A special case of the above is that of torus knots $T(2,n)$ and the result in this
0505662 case was first conjectured by Dunfield-Gukov-Rasmussen.
- There are some very interesting conjectures concerning other torus knots:
  - 1404.0623 Gorsky and Lewark give explicit conjectures for the $sl(N)$-homology of 3 stranded torus knots,
  - As $n \to \infty$ the homology $KR_N^{*,*}(T(k,n))$ stabilises in bounded degree and it
1206.2226 makes sense to consider the $sl(N)$-homology of $T(k,\infty)$. Gorsky-Oblomov-Rasmussen give explicit conjectures for what this should be.

Here are some further remarks and comments.

1302.0331
1. KR_2^{**} should be isomorphic to Kh^{**} and indeed it is. Mark Hughes has written this isomorphism down very explicitly.
2. 1105.3985 One might hope that taken collectively the family of $sl(N)$-homologies are a complete invariant. Not so: Andrew Lobb has found an infinite family of distinct knots undistinguishable by $KR_N^{*,*}$ (all N).
3. What about Bar-Natan's geometric approach? For simplicity we have been using the language of singular link diagrams which are allowed crossings looking like  . In fact the notation used by Khovanov and Rozansky is to elongate the vertex into a *thick* edge (or *double edge)*  This depiction more accurately



reflects the viewpoint of Murakami, Ohtsuki and Yamada. With this in mind Bar-Natan's picture can be generalised, but the cobordism featuring their must be replaced by *foams* which take into account the existence of thick edges. This program has been carried out by Mackaay-Stosic-Vaz. 0708.2228

4. The main new algebraic ingredient in the construction of $sl(N)$-homology is the notion of a matrix factorization and these used locally: a matrix factorization is associated to a small neighbourhood of a resolved diagram. Locality of diagrams was best patched together (when incorporating link cobordisms too) using the formalism of *canopolis*. 0610650 Webster has shown that certain categories of matrix factorizations can be given the structure of a canopolis and he describes $sl(N)$-homology in these terms.
5. The construction of Khovanov homology can be modified to produced Lee theory. In a similar way there are "degenerate" variants of $sl(N)$ homology defined 0402266 by Gornik. Like Lee theory these are filtered theories and can be completely computed in terms of linking numbers. There is also an analogue of the Lee-Rasmussen spectral sequence for these theories which has been studied in depth by Wu. 0612406
6. Throughout this section we have been assuming that we are working over $\mathbb{Q}$ (or $\mathbb{C}$). It is interesting to ask if there are also integral theories. These have been studied 0910.1790 by Krasner.

6.2. Khovanov-Rozansky HOMFLYPT-homology

For each N the (normalised) $sl(N)$-polynomial $\widetilde{P}_N$ is a specialisation of the following version of the HOMFLYPT polynomial $\widetilde{P}$

$$a\widetilde{P}(\nwarrow\!\!\!\!\nearrow) - a\widetilde{P}(\nwarrow\!\!\!\!\nearrow) + (q-q^{-1})\widetilde{P}(\uparrow\;\uparrow) = 0$$

and

$$\widetilde{P}(\text{unknot}) = 1$$

Note that when $a = 1$ this also gives the (Conway)-Alexander polynomial. A very natural question in the context of link homology theories is: is there a link homology theory $\widetilde{H}$ whose graded dimensions combine to give the polynomial $\widetilde{P}$ and which specialize to $\widetilde{KR}_N^{*,*}$? (In asking this question it is not immediately clear what "specialise" should mean.) There is such a theory constructed once again by Khovanov and Rozansky. The main reference is their paper:

- *Matrix factorizations and link homology II* 0505056

A good place to start might be the expository sections of Rasmussen's

- *Some differentials on Khovanov-Rozansky homology* 0607544

Khovanov and Rozansky's *HOMFLYPT homology* (reduced version) assigns to each braid closure diagram D a *triply* graded $\mathbb{Q}$-vector space $\widetilde{H}^{*,*,*}(D)$ such that

1. $D_1 \sim D_2 \Longrightarrow \widetilde{H}(D_1) \cong \widetilde{H}(D_2)$
2. The HOMFLYPT polynomial is recovered
$$\sum_{r,s,t} (-1)^{\frac{t-s}{2}} a^s q^r \dim(\widetilde{H}^{r,s,t}(D)) = \widetilde{P}(D)$$
3. $\widetilde{H}(\text{unknot}) \cong \mathbb{Q}$ in grading $(0,0,0)$
4. $\widetilde{H}(L_1 \sqcup L_2) \cong \widetilde{H}(L_1) \otimes \widetilde{H}(L_2) \otimes \mathbb{Q}[x]$
5. There are Skein long exact sequences.

Remark 9. 1. The notation and grading conventions we are following are Rasmussen's. In fact Khovanov and Rozansky work with the unreduced theory H which is related to reduced by $H \cong \widetilde{H} \otimes \mathbb{Q}[x]$. The reduced theory has the nice property that for any connected sum we have $\widetilde{H}(L_1 \sharp L_2) \cong \widetilde{H}(L_1) \otimes \widetilde{H}(L_2)$.
2. The construction of HOMFLYPT homology uses matrix factorizations (though see section 6.3 below). They are *graded* matrix factorizations which is the grading which pushes through to ultimately give three gradings.
3. The theory is not as well behaved as previous theories. For one thing it is restricted to braid closures (a priori this could be removed but is required for the proof of Reidemeister invariance). Another drawback is that there is no functoriality.

HOMFLYPT-homology reproduces the polynomial $\widetilde{P}$ but what about its re-
0607544 lationship to $sl(N)$-homology? Since $\widetilde{P}$ specialises to the $sl(N)$-polynomial one would hope there is a relationship. This question has been thoroughly investigated in a wonderful paper by Jake Rasmussen in which he analyses a family of spectral sequences relating these theories. His main result is:

Theorem 7. *1. For each* $N > 0$ *there exists a spectral sequence starting with* $\widetilde{H}$ *and converging to* $\widetilde{KR}_N^{*,*}$*. Moreover each page is a knot invariant.*
2. There is a spectral sequence starting with $\widetilde{H}$ *and converging to* $\mathbb{Q}$*.*

Remark 10. 1. The existence of these spectral sequences gives information about the form of HOMFLYPT-homology in much the same way as the Lee-Rasmussen spectral sequence gives information about the form of Khovanov homology. Here because there is a whole family of spectral sequences to be
0505662 considered there is a large amount of information given. Prior to Jake's paper a lot of this structure was predicted by Dunfield, Gukov, and Rasmussen where particular attention was given to torus knots.
2. Rasmussen himself carries out low crossing number computations. A nice
0812.1957 worked example of how to glean information from the existence of the spectral sequences is given by Mackaay and Vaz who calculate $\widetilde{H}$(Conway knot) and $\widetilde{H}$(Kinoshita-Terasaka knot), showing the two to be isomorphic.
3. One can construct other theories similar to $\widetilde{KR}_N^{*,*}$ and there are Rasmussen-type
0612406 spectral sequences starting with $\widetilde{H}$ and converging to these. Some of these have been studied in depth by Wu.
4. Manolescu proposes a spectral sequence converging to knot Floer homology
1108.0032 which has E_1-page HOMFLPT-homology.

6.3. Khovanov-Rozansky HOMFLYPT-homology using Hochschild homology

There is another construction of triply-graded HOMFLYPT-homology which uses Hochschild homology and is due to Khovanov. 0510265

There is a construction due to Rouquier that associates a cochain complex $F^*(\sigma)$ to a word σ representing an element in the braid group. This assignment is such that if two words represent the same group element then the associated complexes are isomorphic. Khovanov uses this construction in the following way. Suppose we have a link presented as the closure of an m-braid diagram D and let σ be the corresponding braid word and $F^*(\sigma)$ its Rouquier complex. Now apply Hochschild homology $HH(R,-)$ to this to get a complex 0409593

$$\cdots \longrightarrow HH(R,F^i(\sigma)) \longrightarrow HH(R,F^{i+1}(\sigma)) \longrightarrow HH(R,F^{i+2}(\sigma)) \longrightarrow \cdots$$

where R is a certain ring ($R = \mathbb{Q}[x_1-x_2,\dots,x_{m-1}-x_m] \subset \mathbb{Q}[x_1,\dots,x_m]$). There are internal gradings and each term in the sequence is in fact bigraded.

Theorem 8. *The homology of this complex, denoted $\mathcal{H}^{*,*,*}$, is independent (up to isomorphism) of the choices made and (module juggling grading conventions) is isomorphic to Khovanov and Rozansky's HOMFLYPT-homology.*

6.4. Coloured link homologies

Coloured Jones polynomials arise from $sl(2)$ using higher dimensional representations rather than the fundamental two dimensional one. Link homology theories related to these have been studied. There is a similar story for $sl(N)$.

Some papers:

Khovanov

- *Categorifications of the colored Jones polynomial* 0302060

Beliakova-Wehrli

- *Categorification of the colored Jones polynomial and Rasmussen invariant of links* 0510382

Mackaay-Stosić-Vaz

- *The 1,2-colored HOMFLY-PT link homology* 0809.0193

Webster-Williams

- *A geometric construction of colored HOMFLYPT homology* 0905.0486

Wu

- *A colored $sl(N)$ homology for links in S^3* 0907.0695

- *Colored $sl(N)$ link homology via matrix factorizations* 1110.2076

6.5. Higher representation theory

This is now a vast and important subject providing the most comprehensive answers to the question of how one should generalise Khovanov homology to other Lie algebras. To get an idea you could read the introductory sections of:

Lauda-Queffelec-Rose

1212.6076 – *Khovanov homology is a skew Howe 2-representation of categorified quantum* sl_m

Lauda

1106.2128 – *An introduction to diagrammatic algebra and categorified quantum* sl_2

Webster

1309.3796 – *Knot invariants and higher representation theory*

The $sl(2)$ case is treated separately in his:

1312.7357 – *Tensor product algebras, Grassmannians and Khovanov homology*

The paper by Lauda-Queffelec-Rose has a follow-up:

1405.5920 – *The* sl_n *foam 2-category: a combinatorial formulation of Khovanov-Rozansky homology via categorical skew How duality*

The work of Rouquier has been of fundamental importance in this area. One can get an idea of the vision from the introduction of his paper:

0812.5023 – *2-Kac-Moody algebras*

7. Applications of Khovanov homology

7.1. Concordance invariants

The first paper to read on this subject is definitely Rasmussen's

0402131 – *Khovanov homology and the slice genus*

Recall that for a knot the Lee-Rasmussen spectral sequence leaves only two generators on the E_∞-page. If we use the grading conventions which impose the differentials on the usual picture for Khovanov homology, then denoting the E_∞ page by $\mathbb{K}_\infty^{*,*}$, the statement that the spectral sequence *converges* to Lee theory means that

$$\mathbb{K}_\infty^{i,j} = \frac{F^j Lee^i}{F^{j+1} Lee^i}$$

where $F^* Lee^i$ is the induced filtration on Lee theory. Since we are working over $\mathbb{Q}$ this means that $Lee^i \cong \bigoplus_j \mathbb{K}_\infty^{i,j}$.

Usually the filtration grading (here the grading denoted by j) of the E_∞-page of a spectral sequence is not meaningful, but in this case the entire spectral sequence from the second page onwards is a knot invariant and thus the filtration gradings of the generators (two of them) surviving to the E_∞-page are too.

In fact their difference is always 2:

Proposition 11. *For a knot K there exists an even integer $s(K)$ such that the two surviving generators in the Lee-Rasmussen spectral sequence have filtration degree $s(K)+1$ and $s(K)-1$.*

Definition 2. *The integer $s(K)$ is called the* Rasmussen *s-invariant of the knot K.*

Exercise 10. Show that for an alternating knot, Rasmussen's invariant agrees with the signature.

By digging down a bit into the filtration, Rasmussen shows that his invariant has the following properties:

1. $s(\text{unknot}) = 0$,
2. $s(K_1 \sharp K_2) = s(K_1) + s(K_2)$,
3. $s(K^!) = -s(K)$.

Rasmussen's invariant provides a lower bound for the slice genus, but we begin with the simpler:

Theorem 9. *If K is a smoothly slice knot then $s(K) = 0$*

Proof. Let Σ be a slice disc with another small disc removed. This can be viewed as (connected) link cobordism $\Sigma\colon K \to U$ (the unknot). Since the Euler characteristic of Σ is zero, this cobordism induces a filtered isomorphism of filtered degree zero $\Sigma_*\colon Lee^0(K) \to Lee^0(U)$. Thus for any $\alpha \in Lee^0(K)$ we have $s(\Sigma_*(\alpha)) \geq s(\alpha)$. But since $Lee^0(U)$ has two generators in filtration degrees ± 1 (and since Σ_* is an isomorphism) we have $-1 \leq s(\Sigma(\alpha)) \leq 1$ proving that $s(\alpha) \leq 1$. Now $s(K)$ is equal to $s(\alpha) - 1$ for *some* α so $s(K) \leq 0$. Finally, a similar argument applies to $K^!$ giving $s(K^!) \leq 0$ and so $s(K) = -s(K^!) \geq 0$.

Remark 11. This proof uses the fact that Lee theory is a *functor*.

Exercise 11. (a) Use the properties of s and the theorem above to prove that s is a concordance invariant.
(b) Modify the proof of the theorem above to prove $|s(K)| \leq 2g_s(K)$, where $g_s(K)$ denotes the smooth slice genus of K.

Remark 12. 1. Gompf gives a way of constructing non-standard smooth structures on $\mathbb{R}^4$ from the data of a topologically slice but not smoothly slice knot. By work of Freedman if the Alexander polynomial Δ_K is 1 then K is topologically slice. Thus a non-standard smooth structure on $\mathbb{R}^4$ can be inferred from a knot K satisfying $\Delta_K = 1$ and $s(K) \neq 0$. Example of such knots are readily found, for example, the pretzel knot $P(-3,5,7)$.

2. By studying the s-invariant for positive knots, Rasmussen gives a proof of the Milnor conjecture: the slice genus of the torus knot $T(p,q)$ is $\frac{1}{2}(p-1)(q-1)$.
3. For a short time it looked like Rasmussen's invariant might help to find a counter-example to the smooth 4-dimensional Poincaré conjecture as explained by Freedman-Gompf-Morrsion-Walker. 0906.5177

This hope was short lived and Akbulut showed (by other means) that the po-
0907.0136 tential counter-examples are standard spheres.
In fact Kronheimer and Mrowka have related Rasmussen's invariant to a similar invariant from instanton homology and draw the consequence that Rasmussen's
1110.1297 invariant will never detect such counter-examples to the 4-dimensional Poincaré conjecture.

0512348 4. There is a similar invariant to Rasmussen's called τ coming from Heegaard-Floer knot homology. While in many cases $2\tau = s$, Hedden and Ording show in general $s \neq 2\tau$.
0602631 Livingston has found an example for which $\Delta_K = 1$ (so topologically slice) for which $s \neq 2\tau$.

1012.2802 Replacing Khovanov homology by $sl(N)$-homology and Lee theory by Gornik's theory G_N^* and there is again a spectral sequence starting with the latter and converging to the former which has dimension N concentrated in (homological) degree
0612406 0. From this one can get an invariant much like Rasmussen's which has been studied by Lobb and independently by Wu.

0703210 **Theorem 10.** *(1) Let K be a knot. There exists and integer $s_N(K)$ such that*

$$\sum q^j G_N^{0,j}(K) = q^{s_N(K)} \frac{q^N - q^{-N}}{q - q^{-1}}$$

where the second grading on Gornik theory is the filtration grading.
(2) $|s_N(K)| \leq 2(n-1)g_s(K)$.

1310.3100 *Remark 13.* The question of whether these invariants are related for various N has been studied by Lukas Lewark who conjectures that the invariants $\{s_N(K)\}_{N\geq 2}$ are linearly independent. Evidence for this is given by his results stating that $s_2(K)$ is not a linear combination of $\{s_N(K)\}_{N\geq 3}$ and a similar result for $s_3(K)$.

Another way of obtaining Rasmussen type invariant is by using the spectral
0509692 sequence to Bar-Natan theory giving rise to invariants $s_R^{BN}(K)$ for a variety of rings R. It was thought (incorrectly) that over $\mathbb{Z}$, $\mathbb{Q}$ and finite fields that these invariants always coincide with Rasmussen's invariant (this was claimed in Mackaay-Turner-Vaz) but Cotton Seed has done some calculations with $K = K14n19265$ showing that in this case $s(K) \neq s_{\mathbb{F}_2}^{BN}(K)$.

1206.3532 These invariants have been further refined by Lipshitz and Sarkar (where a discussion of $K14n19265$ can also be found).

Remark 14. For links (rather than knots) there are various things one might try in
0510382 order to to obtain a Rasmussen-type invariant. The most direct analogue of Rasmussen's work is the proposal of Beliakova and Wehrli.

7.2. Unknot detection

Khovanov homology is a nice functorial invariant which is known not to be *complete* - it is not hard to find distinct knots with the same Khovanov homology. However, the question of whether or not Khovanov homology detects the unknot is an important one.

The following is a result of Hedden and Watson:

Theorem 11. *The dimension of the reduced Khovanov homology of the (2,1)-cable of a knot K is exactly 1 if and only if K is the unknot.* 0805.4423

See also the paper by Hedden and that of Grigsby and Wehrli. These papers all apply somewhat the same approach: make something else out of the knot and use a spectral sequence to Heegaard-Floer homology to make a conclusion about the minium size of the E_2-page; there are subtleties dues to the existence of L-space integer homology spheres. 0805.4418 0807.1432

It is now known that Khovanov homology itself detects the unknot by work of Kronheimer and Mrowka.

Theorem 12. *Khovanov homology detects the unknot.* 1005.4346

They also use a spectral sequence but this time using another theory defined (by them) using instantons. This is a deep result and the unsuspecting hitchhiker should be aware that their two papers have lengths 124 pages and 119 pages respectively. 0806.1053

7.3. Other applications

Legendrian knots, Thurston-Bennequin bounds, transvers knots:

Let L be a link and let $\overline{tb}(L)$ be the maximum Thurston-Bennequin number over all Legendrian representatives of L. A result of Ng is:

Theorem 13. 0508649

$$\overline{tb}(L) \leq min\{k \mid \oplus_{j-i=k} Kh^{i,j}(L) \neq 0\}$$

This bound is sharp for alternating links. Other, earlier related papers:

- Shumakovitch 0411643
- Plamenevskaya 0412184

Width of Khovanov homology and finite fillings.

- Watson 0807.1341
- Watson 1010.3051

In another paper Watson builds a subtle (vector space-valued) invariant of tangles using an natural inverse system of Khovanov homology groups and applies this to strongly invertible knots (his result: a strongly invertible knot is the trivial knot if and only if his invariant is trivial)

1311.1085

8. Geometrical interpretations and related theories

8.1. Symplectic geometry

Seidel-Smith

0405089

– *A link invariant from the symplectic geometry of nilpotent slices*

Manolescu

0601629

– *Link homology theories from symplectic geometry*

8.2. Instanton knot homology

Kronheimer-Mrowka

0806.1053

– *Knot homology groups from instantons*

1110.1290

– *Filtrations on instanton homology*

1110.1297

– *Gauge theory and Rasmussen's invariant*

8.3. Derived categories of coherent sheaves

Cautis-Kamnitzer

0701194

– *Knot homology via derived categories of coherent sheaves I, sl(2) case*

8.4. Knot groups and representation varieties

Jacobsson-Rubinsztein

0806.2902

– *Symplectic topology of SU(2)-representation varieties and link homology, I: symplectic braid action and the first Chern class*

8.5. Physics

Gukov-Schwarz-Vafa

0412243 – *Khovanov-Rozansky homology and topological strings*

Gukov-Iqbal-Kozcaz-Vafa

0705.1368 – *Link homologies and the refined topological vertex*

Witten

1101.3216 – *Fivebranes and knots*

1108.3103 – *Khovanov homology and gauge theory*

– *Two lectures on the Jones polynomial and Khovanov homology* 1401.6996

8.6. Factorization homology

Ayala-Francis-Tanaka

– *Structured singular manifolds and factorization homology* 1206.5164

8.7. Homotopy theory

Lipshitz-Sarkar

– *A Khovanov homotopy type* 1112.3932

Everitt-Turner

– *The homotopy theory of Khovanov homology* 1112.3460