The diagonal slice of $\mathrm{SL}(2, \mathbb{C})$-character variety of free group of rank two[1]

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Aspects of representation theory in low-dimensional topology and 3-dimensional invariants

[1]Part of this is a joint work with Caroline Series and Ser Peow Tan

Character variety

- Σ : a one-holed torus
- $\pi = \pi_1(\Sigma) = F_2 = \langle X, Y \rangle$: free group of rank 2
- $G = \mathrm{SL}(2, \mathbb{C})$
- G acts on $\mathrm{Hom}(\pi, G)$ by $g : \rho \mapsto \iota_g \circ \rho$. ($\rho \in \mathrm{Hom}(\pi, G)$, $\iota_g : x \mapsto gxg^{-1}$)
- $\mathcal{X} = \mathrm{Hom}(\pi, G)//G$: character variety
- $\mathcal{X}$ is identified with $\mathbb{C}^3$. For $[\rho] \in \mathcal{X}$,

$$[\rho] \mapsto (x, y, z) := (\mathrm{tr}\,\rho(X), \mathrm{tr}\,\rho(Y), \mathrm{tr}\,\rho(XY))$$

- κ-relative character variety ($\kappa \in \mathbb{C}$):

$$\begin{aligned}\mathcal{X}_k &:= \left\{[\rho] \in \mathcal{X} \,\middle|\, \mathrm{tr}\,\rho(XYX^{-1}Y^{-1}) = \kappa\right\} \\ &\cong \left\{(x, y, z) \,\middle|\, x^2 + y^2 + z^2 - xyz - 2 = \kappa\right\}\end{aligned}$$

Aim

Conditions for the characters

- discrete and faithful
- Bowditch's "Q-condision"
- primitive stability

We want to "see" and compare these conditions by computer experiments.

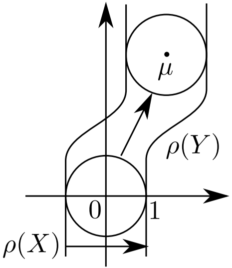
A 1-dimensional slice (Maskit)

- A 1-dimensional slice of $\mathcal{X}_{-2}$ where the generator X is pinched to a parabolic.

$$\left\{(x, y, z) \in \mathcal{X}_{-2} \,\middle|\, x = 2\right\}$$

- Each ρ can be normalized as follows. ($-i\mu = y$)

$$\rho(X) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \rho(Y) = \begin{pmatrix} -i\mu & -i \\ -i & 0 \end{pmatrix}.$$



Maskit slice

Maskit slice:

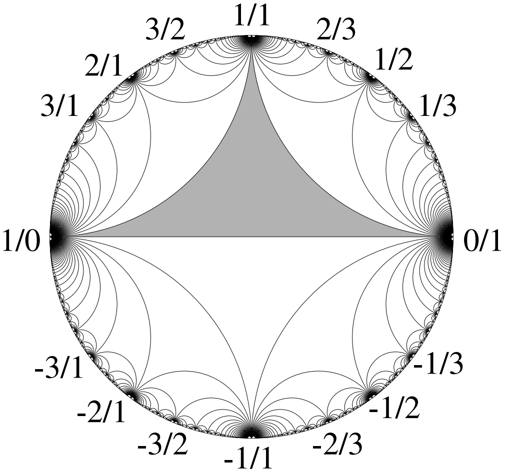
$$\mathcal{M} := \left\{ (x, y, z) \in \mathcal{X}_{-2} \,\middle|\, x = 2,\ \rho_{x,y,z} \text{ is discrete and faithful} \right\}$$

David Wright's method to plot Maskit boundary:

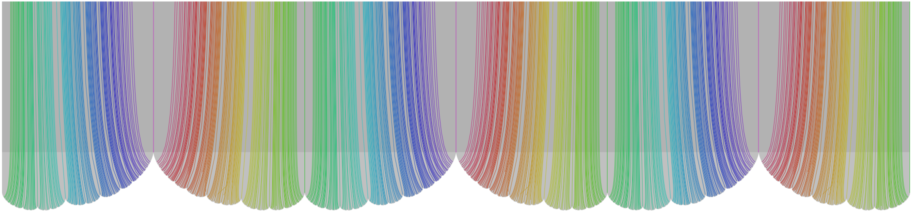
1. Enumerate homotopy classes of simple closed curves on the once punctured torus. ($\longleftrightarrow\ \mathbb{Q} \cup \{\infty\}$)
2. Find representatives of these curves as elements in F_2 and compute their traces as functions of μ. ($\operatorname{tr} W_{p/q} = \varphi_{p/q}(\mu) \ \in\ \mathbb{Z}[\mu]$)
3. Find points where the traces are ± 2.

- There are many points where an element is parabolic, but we cannot conclude that the group is on the boundary.
- Keen and Series proposed plotting the branches of $\varphi_{p/q} > 2$, $\varphi_{p/q} \in \mathbb{R}$ moving away from the cusp.

Farey diagram



Maskit slice



Maskit slice

Observation:

1. they are pairwise disjoint
2. they end in "cusps"
3. they contain no critical points
4. they are asymptotic to a fixed direction at ∞
5. they appear to be dense in the presumed parameter space $\mathcal{M}$.

- To understand the picture, the key was to study the action of $\rho(\pi)$ on hyperbolic 3-space $\mathbb{H}^3$, in particular, on the boundary of the convex hull.
- The real trace curves in the picture are exactly the "pleating rays" $P_{p/q}$ for $q \neq 0$.

Riley slice

- Groups generated by two parabolic transformations.

$$G_w = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \right\rangle$$

Def (Riley slice)

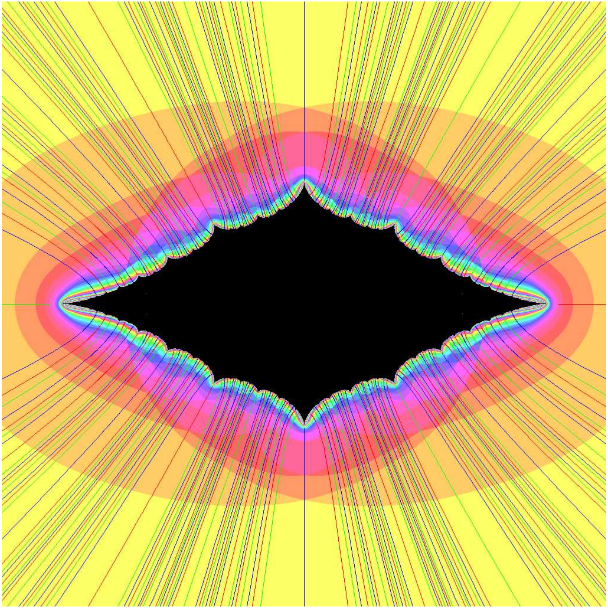
The *Riley slice of the Schottky space* is the subspace of $\mathbb{C}$ consisting of those complex numbers w such that G_w is discrete and free and that the quotient $\Omega(G_w)/G_w$ of the domain of discontinuity is homeomorphic to the 4-times punctured sphere S.

- We will consider:

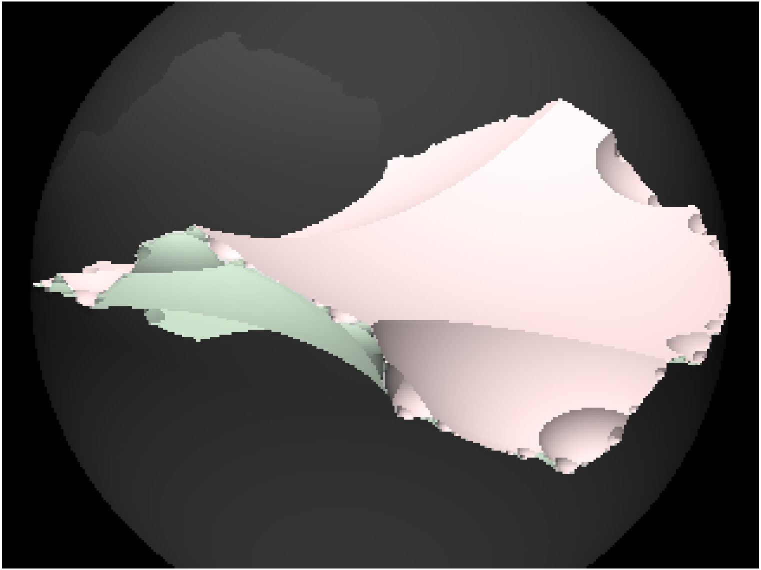
$$\mathcal{R} = \left\{ (x, y, z) \in \mathcal{X}_{-2} \;\middle|\; x = 0,\ \rho_{x,y,z} \text{ satisfies the above condition} \right\}$$

- G_w corresponds to $(0, y, yi)$, where $w = -y^2$.

Riley slice



Limit set and convex core



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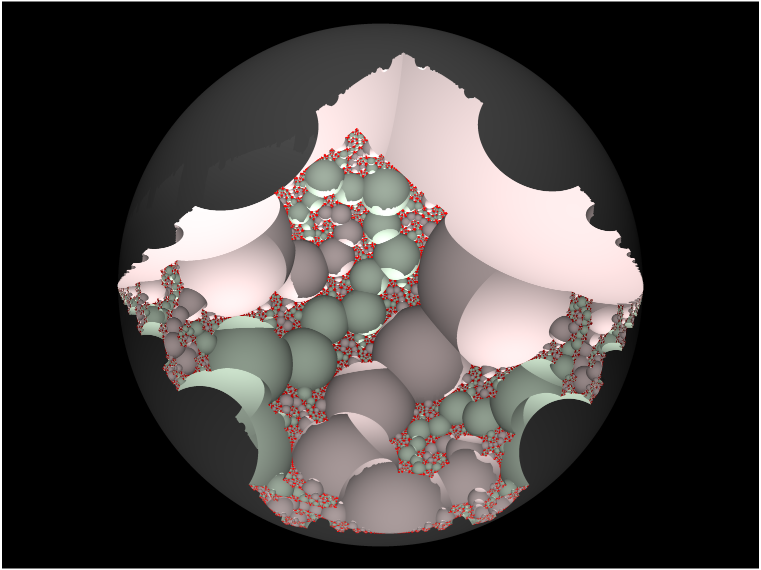


Bowditch's Q-condition

Bowditch defined the following condition on $[\rho] \in \mathcal{X}$, which Tan-Wong-Zhang called condition BQ:

1. $\rho(x)$ is loxodromic for all primitive $x \in F_2$.
2. The number of conjugacy classes of primitive elements x such that $|\operatorname{tr}(\rho(x))| \leq 2$ is finite.

Conjecture (Bowditch)

$BQ \cap \mathcal{X}_{-2}$ = the set of quasi-fuchsian groups (punctured torus groups).

1. One of the motivations was to study McShane's identity.
2. Bowditch, Tan–Wong–Zhang gave an algorithm for showing that given representation is BQ.
3. Ng and Tan gave an algorithm for showing that given representation in $\mathcal{X}_{-2}$ is not BQ.

We want to consider the “diagonal slice”:

$$\mathcal{D} := \{(x, x, x) | x \in \mathbb{C}\} \subset \mathcal{X}$$

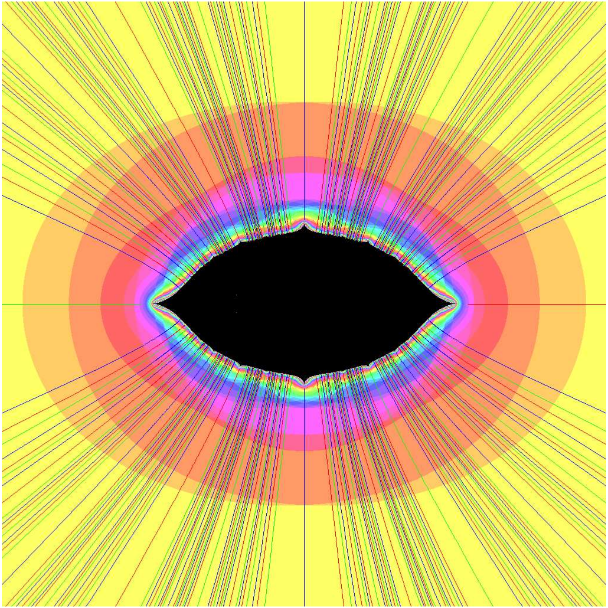
Set

- $\mathcal{D}' = \left\{(1, 1, x) \,\middle|\, x \in \mathbb{C}\right\}$
- $\mathcal{R}' = \left\{(\sqrt{-x+2},\ 0,\ \sqrt{x+1}) \,\middle|\, x \in \mathbb{C}\right\} \subset \mathcal{X}_1$:

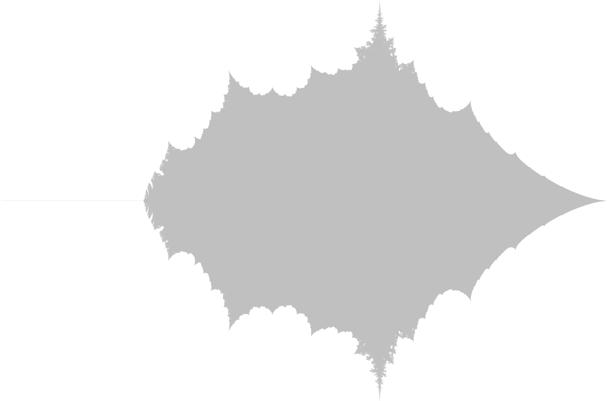
Then

- $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{R}'$ are ”commensurable”.
- $\mathcal{R}'$ corresponds to (elliptic version of) ”generalized Riley slice” and contained in a relative character variety.
- We are able to compute the Keen-Series pleating rays and thus fully determine the discreteness locus.

"generalized" Riley slice



Diagonal slice (BQ condition)

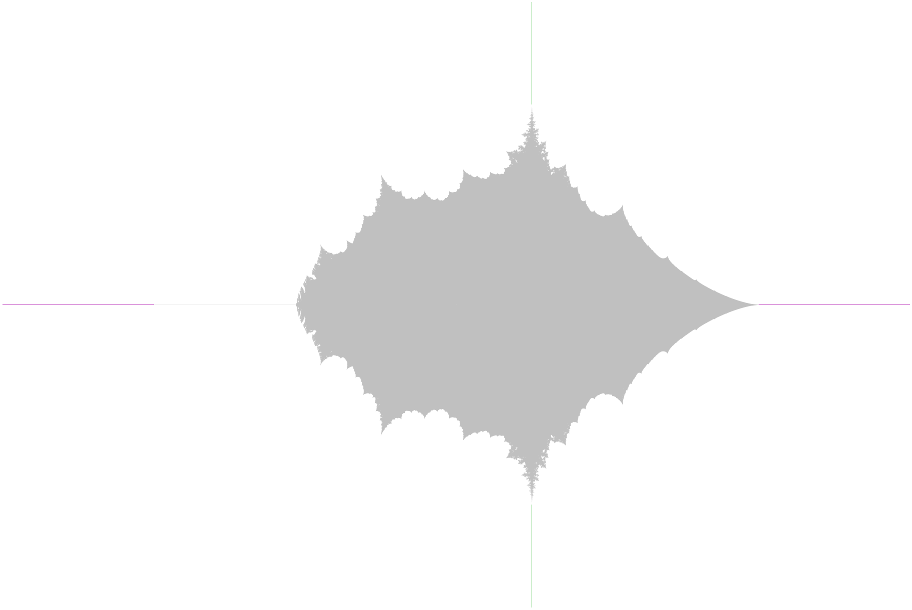


Diagonal slice

Can we describe the boundary by the rays?

1. Enumerate homotopy classes of simple closed curves on the once punctured torus. ($\longleftrightarrow \ \mathbb{Q} \cup \{\infty\}$)
2. Find representatives of these curves as elements in F_2 and compute their traces as functions of μ. ($\mathrm{tr}\, W_{p/q} = \varphi_{p/q}(\mu) \ \in \ \mathbb{Z}[\mu]$)
3. Plotting the branches of $\varphi_{p/q} > 2$, $\varphi_{p/q} \in \mathbb{R}$ moving away from the cusp.

Diagonal slice (Series–Tan–Y)



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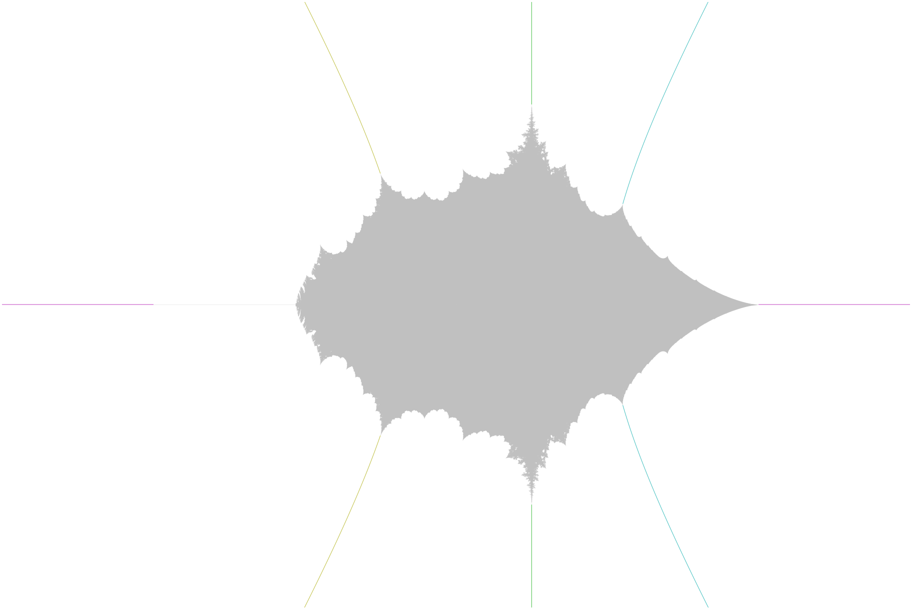

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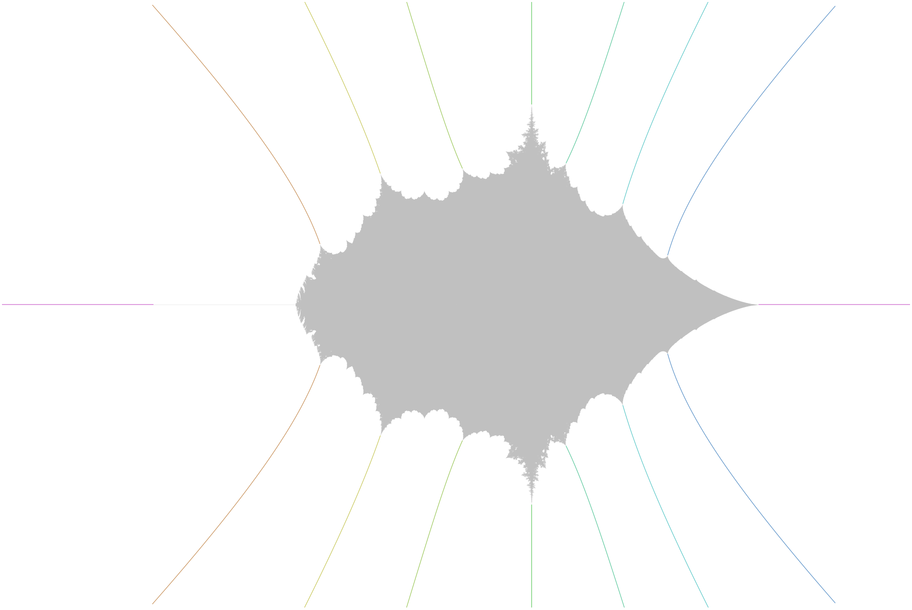

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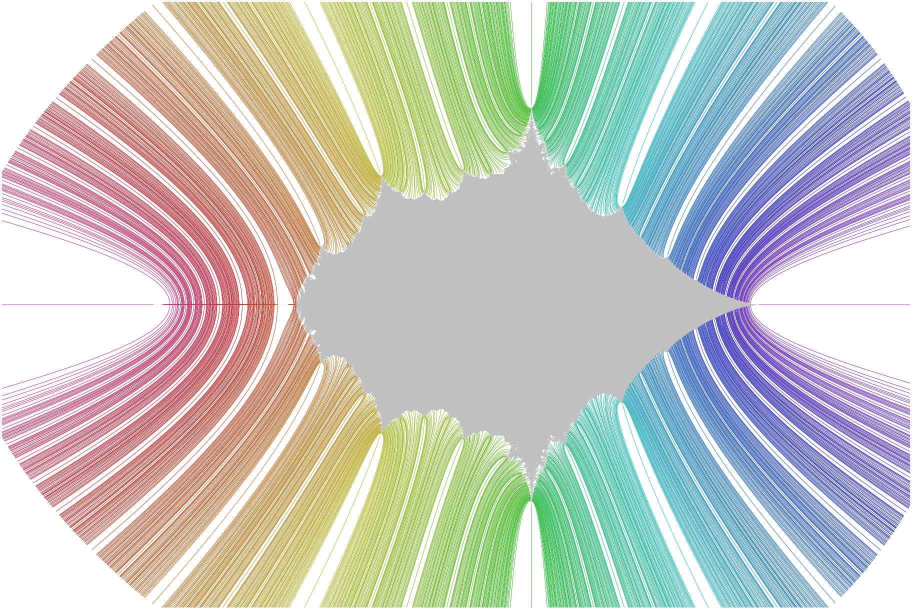


Observation and questions

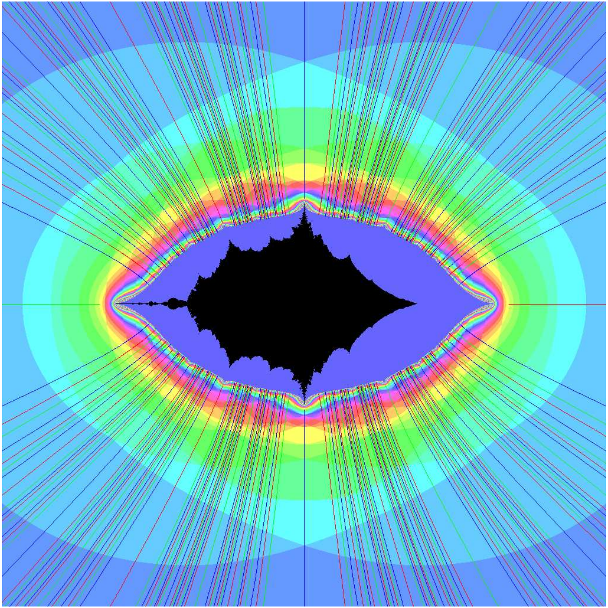
Observation:

1. they are pairwise disjoint
2. they end in “cusps”
3. they contain no critical points
4. they are asymptotic to a fixed direction at ∞
5. they appear to be dense in the presumed parameter space $\mathcal{M}$.

The relation to...

1. The group generated by two elliptic elements of order 3.

”generalized” Riley slice and diagonal slice (Series-Tan-Y)



Observations

Observation:

1. Our graphics indicate that the set BQ is both strictly larger than, and signicantly dierent from, the discreteness locus.

The action by the mapping class group

$MCG(\Sigma)$ is generated by two Dehn twist maps and an involution:

$$\begin{aligned} T_X &: X \mapsto X, Y \mapsto YX, \\ T_Y &: X \mapsto XY^{-1}, Y \mapsto Y, \\ \iota &: X \mapsto X^{-1}, Y \mapsto Y. \end{aligned}$$

This induces the polynomial automorphisms of $\mathbb{C}^3$.

$$\begin{aligned} \phi_X &: (x, y, z) \mapsto (x, z, zx - y) \\ \phi_Y &: (x, y, z) \mapsto (xy - z, y, x) \\ \phi_\iota &: (x, y, z) \mapsto (x, y, xy - z) \end{aligned}$$

- $Aut(\kappa)$: the group of polynomial automorphisms of $\mathbb{C}^3$ which leave invariant the fibers of $\kappa(x, y, z) = x^2 + y^2 + z^2 - xyz - 2 =$ constant.
- $MCG(\Sigma)$ is commensurable with $Aut(\kappa)$.

Dynamical decomposition of $\mathcal{X} = \mathrm{Hom}(\pi, G)//G$

Minsky and Lubotzky introduced a decomposition of $\mathcal{X}(F_n)$ by primitive stable ($\mathcal{PS}(F_n)$) and redundant ($\mathcal{R}(F_n)$) characters.

- $\mathcal{X}(F_n) \supset \mathcal{PS}(F_n) \cup \mathcal{R}(F_n)$, it is not known whether $\mathcal{X}(F_n) \setminus (\mathcal{PS}(F_n) \cup \mathcal{R}(F_n))$ has measure zero in $\mathcal{X}(F_n)$ or not.
- $\mathrm{Out}(F_n)$ acts ergodically on $\mathcal{R}(F_n)$ and acts properly discontinuously on $\mathcal{PS}(F_n)$. (Gelander, Minsky).

Y. Minsky, On dynamics of $\mathrm{Out}(F_n)$ on $\mathrm{PSL}(2, \mathbb{C})$ characters,
`arXiv:0906.3491`

Primitive stable

Minsky:

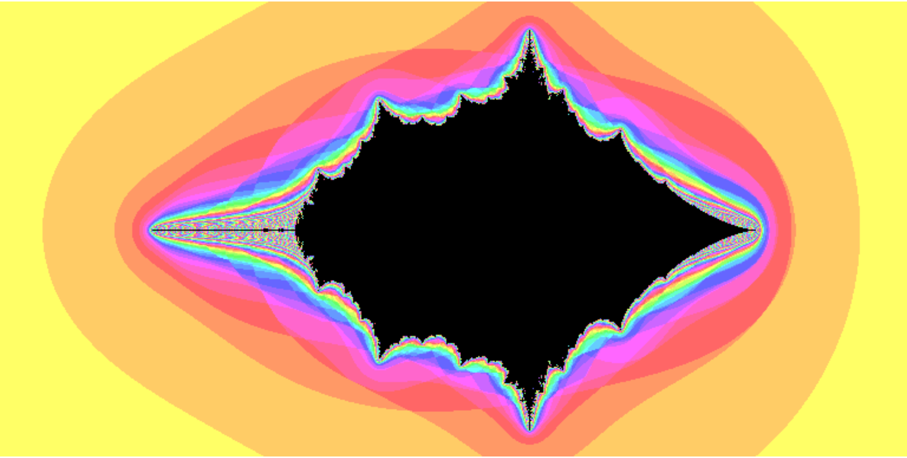
- If ρ is Schottky, then it is primitive-stable.
- Primitive-stability is an open condition in $\mathcal{X}(F_n)$.
- $\mathcal{PS}(F_n)$ contains a point on the boundary of the Schottky space.
- The action of $\mathrm{Out}(F_n)$ on $\mathcal{PS}(F_n)$ is properly discontinuous.
- $\mathcal{PS}(F_n)$ is strictly larger than the set of Schottky characters, which is $\mathrm{Out}(F_n)$ invariant, and on which $\mathrm{Out}(F_n)$ acts properly discontinuously.

Question

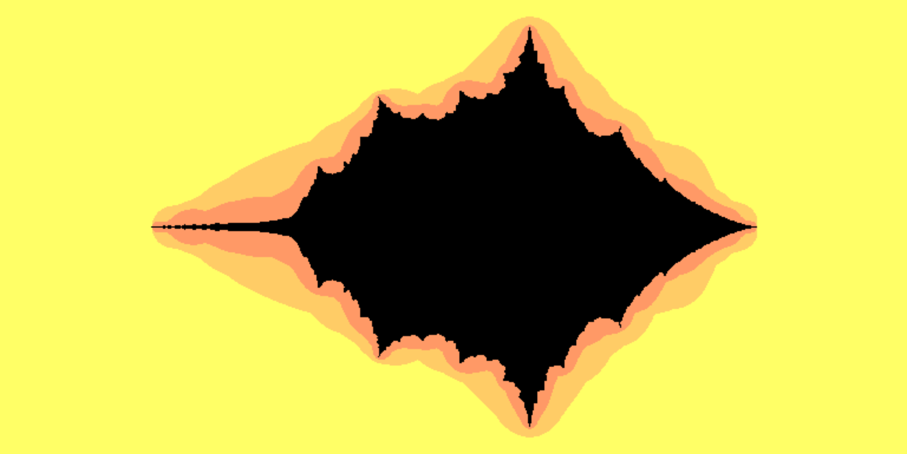
1. Is $\mathcal{PS}(F_2)$ dense in $\mathcal{X}(F_2)$?
2. $BQ = \mathcal{PS}(F_2)$?
3. How do we produce computer pictures of $\mathcal{PS}(F_n)$

We produce computer pictures of $\mathcal{PS}(F_2)$.

BQ



Primitive stability



Extended rays