Double branched covers of tunnel number one knots

Yeonhee Jang and Luisa Paoluzzi

July 3, 2020

Abstract

We provide criteria ensuring that a tunnel number one knot K is not determined by its double branched cover, in the sense that the double branched cover is also the double branched cover of a knot K' not equivalent to K.

AMS classification: Primary 57M25; Secondary 57M12, 57M50.

Keywords: tunnel number one knots, double branched covers, Heegaard splittings.

1   Introduction

A knot in the 3-sphere is said to have *tunnel number one* if its exterior admits a Heegaard splitting of genus 2. This is equivalent to say that there is a properly embedded arc in the knot exterior whose complement is a genus 2 handlebody. Torus knots, 2-bridge knots, and $(1,1)$-knots, that is knots that can be put in 1-bridge position with respect to a genus 1 Heegaard splitting of the 3-sphere, are all tunnel number one knots (see [BM, BRZ, K3, MS] for instance). There is a vast literature studying different aspects of tunnel number one knots and it appears that they are very special in many regards. For instance, this class does not contain any composite knots [N] or any *Conway reducible* knots [S]. Recall that a knot is *Conway reducible* if it admits a *Conway sphere*, that is an essential four-holed sphere properly embedded in the exterior of the knot. More generally, the exterior of a tunnel number one knot does not contain any planar meridinal essential surface (see, [GR]). Having tunnel number one seems to often be a quite constraining condition. For instance, tunnel number one knots that are satellites were classified by Morimoto and Sakuma in [MS]: these are a very restricted family of knots whose exteriors are obtained by gluing together the exteriors of a 2-bridge link and of a torus knot. Similarly, alternating knots having tunnel number one were also classified (see [L]) and, again, only very specific knots, namely 2-bridge knots and certain Montesinos knots with three tangles, are in this class. On the other hand, there are hyperbolic tunnel number one knots with arbitrarily large bridge number (see [J1]) and even genus one bridge number (see, [JT, BTZ]). Note that, since this class contains 2-bridge knots, it also contains hyperbolic knots with arbitrarily large volume.

The aim of this work is to unveil the specificity of these knots with respect to their double branched covers. We need a definition.

Definition 1. We say that a knot K is *not determined by its double branched cover* $M(K, 2)$ if there is a knot K' not equivalent to K such that the manifolds $M(K', 2)$ and $M(K, 2)$ are homeomorphic. In this case we say that K' is a *2-twin of K*.

Throughout this paper, the *double branched cover of a knot* means the double cover of the 3-sphere branched along the knot. We have the following result.

Theorem 1. *Let K be a tunnel number one knot with bridge number $b \geq 5$, then K is not determined by its double branched cover. Moreover, if the double branched cover $M = M(K, 2)$ of K has Heegaard genus 2 then either $b = 3$ (and so K is a $(1, 1)$-knot) or K is not determined by M. In particular a $(1, 1)$-knot of bridge index ≥ 4 is not determined by its double branched cover.*

An immediate consequence is that a tunnel number one knot is not determined by its double branched cover if the Heegaard genus of the cover is two and the knot is not $(1, 1)$. Surprisingly enough we know no examples of knots with this property, so it is natural to ask the following:

Question 1. Is there a tunnel number one knot that is not $(1, 1)$ but whose double branched cover has Heegaard genus 2?

The peculiarity of tunnel number one knots expressed in Theorem 1 is related to another special feature of these knots, that is the fact that they are all *strongly invertible*. Recall that a knot K is *strongly invertible* if there is an orientation-preserving involution of the 3-sphere with non-empty fixed-point set which leaves K invariant and reverses its orientation, in particular the fixed-point set of the involution meets the knot in precisely two points. Such an involution is called a *strong inversion*. Observe that the existence of a strong inversion for a tunnel number one knot is a straightforward consequence of the fact that there is a hyperelliptic involution of a genus-two Heegaard surface which extends to a global involution of the genus two splitting ([BH]).

It is well-known that the presence of symmetries may sometimes reflect the fact that the knot is not determined by its cyclic-branched covers (see, for instance [Na, S1]). For a π-hyperbolic knot K, it was initially observed by Boileau and Flapan in [BF] that if K is not determined by its double branched cover then it is either strongly invertible or admits a π-*rotation*, that is an orientation-preserving involution of the 3-sphere with non-empty fixed-point set which leaves K invariant and preserves its orientation. There are, however, knots that are not determined by their double covers and admit no symmetry: this is the case for instance of certain Conway reducible knots which admit 2-twins obtained by Conway mutation; note that for some Conway reducible knots, like Montesinos knots with at least four tangles, the presence of symmetries is not related to the existence of 2-twins (see [V, M, C, KT] and [P, MW] for other types of examples).

Observe that, in the case of tunnel number one knots, the trivial knot is determined by its double branched cover by a result of Waldhausen [W] (and more generally by the positive solution to the Smith conjecture [MB]) as are 2-bridge knots, by a work of Hodgson and Rubinstein [HR].

The only cases that remain to be considered are those of tunnel number one knots of bridge number 3 and 4. Here we will consider several classes of tunnel number one knots and show that, even if the bridge index is 3 or 4, these

knots are often not determined by their double branched covers. The following is mostly well-known and covers all non π-hyperbolic tunnel number one knots.

Proposition 1.
- *A torus knot $T(p,q)$, $p < q$, is determined by its double branched cover if and only if either $p = 2$, that is it is a 2-bridge knot, or $(p,q) \in \{(3,4),(3,5)\}$.*

- *A tunnel number one Montesinos knot is not determined by its double branched cover if and only if it is not a 2-bridge knot or a torus knot, and admits a torus knot as a 2-twin.*

- *A satellite tunnel number one knot K is never determined by its double branched cover. K admits a 2-twin K' which is hyperbolic and Conway reducible, in particular K' is not a tunnel number one knot. K does not admit any tunnel number one 2-twins.*

Taking into account the classification provided by Klimenko and Sakuma [KS], the second part of the result above can be made more precise. The statement is however slightly involved and will be given in Section 3.

Using Lackenby's classification of alternating tunnel number one knots [L] we get the following.

Corollary 1. *Tunnel number one alternating knots are determined by their double branched covers.*

It follows that alternating tunnel number one knots trivially fulfill Greene's conjecture that the 2-twin of an alternating prime knot is alternating [G].

Unfortunately, for π-hyperbolic tunnel number one knots providing a complete answer seems hard. Nonetheless, we will see in Section 4 that the knots belonging to two families, namely twisted torus knots and knots obtained by Dehn surgery on a component of a hyperbolic 2-bridge link, are "generically" not determined by their double branched covers. Note that the arguments follow the same lines as those used by Reni and Zimmermann in [RZ] to provide examples of knots not determined by their double branched covers.

More interestingly, we show that a π-hyperbolic tunnel number one knot of bridge index three or four can only be determined by its double branched cover if its minimal genus Heegaard splittings have small Hempel distance.

Theorem 2. *Let K be a π-hyperbolic tunnel number one knot with bridge number $b \in \{3,4\}$ and let M be its double branched cover. Let $g \in \{2,3\}$ be the Heegaard genus of M and assume that any minimal genus Heegaard splitting for M has Hempel distance at least $2g + 1$. Then K cannot be determined by M.*

To be more precise, we only require that a specific minimal genus Heegaard splitting induced either by a $(1,1)$-presentation for the knot or a genus-two splitting of its exterior satisfies the given condition on the Hempel distance.

We remark that Theorem 2 does not mean that a (π-hyperbolic tunnel number one) knot is determined by its double branched cover M if M admits a minimal genus Heegaard splitting with small Hempel distance. See Remark 2 for examples of knots which admit 2-twins and whose double branched covers admit genus-2 Heegaard splittings with Hempel distance at most 2.
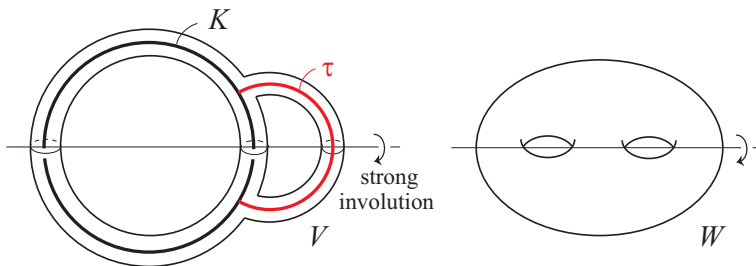
Figure 1: V, W, and a strong involution.

The paper is organised as follows: In Section 2 we prove Theorem 1. In Section 3 we provide a characterisation of the non π-hyperbolic tunnel number one knots that are not determined by their double branched covers. In the last two sections we discuss the case of π-hyperbolic tunnel number one knots: in Section 4 we show that knots of this type belonging to two specific classes are generically not determined by their double branched covers and in Section 5 we prove Theorem 2.

2   Proof of Theorem 1

Let K be a tunnel number one knot and $M = M(K, 2)$ its double branched cover. Let V be a tubular neighborhood of the union of K and an unknotting tunnel τ, and W be the closure of $\mathbf{S}^3 \setminus V$ (see Figure 1). Then $\mathrm{Cl}(V \setminus N(K)) \cup W$ is a Heegaard splitting of genus 2 of the exterior of K. Consider the lift to M of the Heegaard splitting $V \cup W$: it is a Heegaard splitting of genus 3 of M. This fact can be seen as follows. The 2-fold cover of V branched along K is the union of a solid torus which is the neighborhood of the lift of K and two 1-handles corresponding to the lifts of τ. Remark that the cover is unbranched on the complement. It follows that the Heegaard genus of M is at most 3. Denote by s the involution of M that generates the group of deck transformations of the cover, so that we have $(M, Fix(s))/\langle s \rangle = (\mathbf{S}^3, K)$.

Consider now the strong inversion of K that acts as a hyperelliptic involution of each handlebody of the genus-two splitting and whose fixed-point set meets the chosen tunnel for K in one point. Let u and su be the two lifts of the strong inversion to M. We claim that one of these two lifts, say u, acts as a hyperelliptic involution of the genus-three Heegaard splitting of M. First of all note that both $Fix(u)$ and $Fix(su)$ are non empty and, since M is a $\mathbb{Z}/2\mathbb{Z}$-homology sphere, connected. Since s commutes with both u and su it must leave both their fixed-point sets invariant. Now the union of $Fix(u)$ and $Fix(su)$ must meet each of the lifts of the tunnel. Since s exchanges these two lifts, both handles must meet the fixed-point set of the same element u. It now
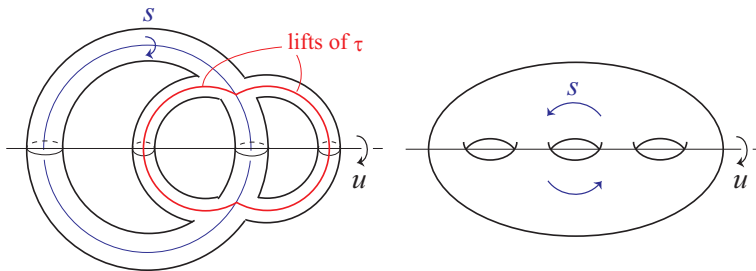
4

Figure 2: u acts as a hyperelliptic involution.

follows that u acts as a hyperelliptic involution (see Figure 2, where the lifts of V and W are shown).

Since u acts as a hyperelliptic involution, it follows that $(M, Fix(u))/\langle u \rangle = (\mathbf{S}^3, K_u)$, where K_u is a knot admitting a 4-bridge presentation. It follows that the bridge index of K_u is at most 4. If the bridge index of K is at least 5, then K_u is a 2-twin of K, and K is not determined by its double branched cover.

Assume now that the Heegaard genus of M is two. Any minimal genus Heegaard splitting of M admits a hyperelliptic involution t. We can now repeat the same argument as before. The involution t is a deck transformation of a double branched cover of a (necessarily) 3-bridge knot K_t. It follows that either the bridge index of K is 3 or K is not determined by its double branched cover. Note that by [K] a tunnel number one knot of bridge index 3 is a $(1, 1)$-knot.

Assume now that K is a $(1, 1)$-knot. The genus-one Heegaard splitting of $\mathbf{S}^3$ with respect to which K is in 1-bridge position lifts to a genus-two Heegaard splitting for M, which has thus Heegaard genus at most 2. It follows from the discussion above that a $(1, 1)$-knot of bridge index ≥ 4 is not determined by its double branched cover. □

3   Non π-hyperbolic tunnel number one knots

In this section we shall characterise the non π-hyperbolic tunnel number one knots that are determined by their double branched covers. Recall that tunnel number one knots are necessarily prime. A prime knot is either simple or toroidal. Toroidal, i.e. satellite, tunnel number one knots were classified independently by Morimoto and Sakuma [MS], and Eudave-Muñoz [E-M]. Simple knots are either torus knots, which are all tunnel number one, or hyperbolic. Since tunnel number one knots are Conway irreducible, the double cover of a hyperbolic tunnel number one knot is atoroidal and admits a geometric structure. Such structure can be spherical if the knot is a 2-bridge knot, Seifert fibred if the knot is a Montesinos knot or a torus knot, or hyperbolic if the knot is π-hyperbolic. Knots of bridge index at most 2 (including the trivial knot) have

all tunnel number one and it is well-known that they are determined by their double branched covers. Tunnel number one Montesinos knots were classified by Klimenko and Sakuma [KS].

3.1 Torus knots

Let $K = T(p, q)$ be the torus knot of type (p, q), with $2 \leq p < q$, and p and q coprime. Recall that the bridge index of K is p, in particular if $p = 2$, K is a 2-bridge knot and it is determined by its double branched cover. We can thus assume that $p \geq 3$. The double branched cover M of K is a Seifert fibred manifold with base the 2-sphere and three exceptional fibres of orders $(2, p, q)$ if pq is odd, $(p/2, q, q)$ if p is even, or $(p, p, q/2)$ if q is even. Note that, since $p > 2$ the fibration of M is unique. Now M admits an involution of Montesinos type, so that M is the double branched cover of a Montesinos knot K' with three rational tangles which may or may not be equivalent to K. In fact, K' is hyperbolic most of the time and so, in this case it is a 2-twin of K. Remark that since the bridge index of K is equal to p and that of K' is 3 by [BZ2], we can conclude that K is not determined by its double branched cover whenever $p > 3$. We only need to consider the case where $p = 3$. In this case it is known that K' is hyperbolic for all $q > 5$, and hence K is not determined by its double branched cover. If $q \in \{4, 5\}$ the knot is determined by its double branched cover. Indeed, in these two cases K' must be a simple, non hyperbolic knot. It follows that K' is a torus knot. Now since fibrations of double branched covers of torus knots are unique we conclude that in this case K' is equivalent to K. To conclude we only need to observe that, by the orbifold theorem, any involution of M that is the deck transformation of a double branched cover of a knot must preserve the Seifert fibration. If the involution induces an orientation preserving map of the base of the fibration, it must act as the covering involution for the torus knot. If the involution reverses the orientation of the base and fixes each exceptional fiber, it must act as a Montesinos involution. For the case where $(p, q) = (3, 4)$, there is also a possibility that the involution reverses the orientation of the base, fixes the fiber of order 2 and exchanges the other two of order 3. In this case, it can be seen that the quotient of M by the involution is a lens space of type $(3, 1)$ and not $\mathbf{S}^3$ by using an argument similar to that in [BZ, Proof of Lemma 3.3] for instance.

3.2 Montesinos knots

We will exploit the classification of these knots obtained by Klimenko and Sakuma.

Theorem 3 (Klimenko-Sakuma [KS]). *Let K be a Montesinos knot with invariants of the form*

$$(b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots (\alpha_r, \beta_r))$$

where, for each i, we can assume that α_i and β_i are coprime and $0 < \beta_i < \alpha_i$. Then K is a tunnel number one knot if and only if one of the following condition is satisfied:

1. K is a 2-bridge knot (that is $r \leq 2$);

2. $r = 3$, $\alpha_1 \leq \alpha_2 \leq \alpha_3$ and either

    (a) $\alpha_1 = 2$ and $\alpha_2\alpha_3$ is odd; or

    (b) $\alpha_1 = \alpha_2 = 3$, $\beta_1 = \beta_2$, $\alpha_3 \geq 4$ is not divisible by 3, and the Euler number is of the form $e = \pm 1/(3\alpha_3)$.

As in the previous subsection, we can assume that K is not a 2-bridge knot. As a consequence its double branched cover can only be the double branched cover of another Montesinos knot or a torus knot. Since a Seifert fibred space with base $\mathbf{S}^2$ and (at most) three exceptional fibers is the double branched cover of a unique Montesinos link, K is not determined by its double branched cover if and only if its double branched cover is also the double branched cover of a torus knot. In order to determine which of the knots listed in the theorem above admit torus knots as 2-twins, it is necessary to understand the Seifert invariants of the double covers of torus knots. These were determined in [NR-L, Theorem 1, page 13], in fact for branched covers of any order.

Proposition 2 ([NR-L])**.** *Let K be the torus knot $T(p,q)$, with $p < q$ coprime. Assume that K is not a 2-bridge knot, i.e. $p > 2$. Choose integers x and y so that $yp + xq = -1$. The double branched cover M of K is a Seifert fibred manifold with base $\mathbf{S}^2$ and precisely three exceptional fibres. More precisely, the Seifert invariants of M are:*

(1) *If p and q are both odd, then the fibres are of type $(2,d)$, (p,kx), and (q,ky), where d is any odd number and k is determined by the condition $pqd = 1 - 2k$; the Euler number is $1/(2pq)$.*

(2) *If $p = 2k$ is even, then the fibres are of type (k,x), (q,y), and (q,y); the Euler number is $1/(kq)$.*

(3) *If $q = 2k$ is even, then the fibres are of type (p,x), (p,x), and (k,y); the Euler number is $1/(kp)$.*

Note that the above proposition only takes into account right-handed torus knots: the covers of their mirror images are obtained by changing the orientation of the manifold, resulting in a change of sign of all Seifert invariants.

Comparing the lists in the above results we have:

Theorem 4. *Let K be a tunnel number one Montesinos knot. Assume that K is a knot of type (2a) in the Klimenko-Sakuma classification, then K is not determined by its double branched cover if and only if, up to changing the signs of its Seifert invariants, one has*

(2a-1) *α_2 and α_3 are coprime, $(\alpha_2, \alpha_3) \neq (3,5)$, $2\alpha_3\beta_2 \equiv -1$ mod α_2, $2\alpha_2\beta_3 \equiv -1$ mod α_3, and the Euler number is $1/(2\alpha_2\alpha_3)$;*

(2a-2) *$\alpha_2 = \alpha_3 > 4$ is odd, $4\beta_2 = 4\beta_3 \equiv -1$ mod $\alpha_2 = \alpha_3$, and the Euler number is $1/(2\alpha_2)$.*

Assume that K is a knot of type (2b) in the Klimenko-Sakuma classification, then K is not determined by its double branched cover if and only if, up to changing the signs of its Seifert invariants one has

(2b-1) *$\alpha_3\beta_1 \equiv 1$ mod 3, $3\beta_3 \equiv -1$ mod α_3, and the Euler number is $1/(3\alpha_3)$.*

Proof. Assume that K is a knot of type (2a) in the Klimenko-Sakuma classification. Then the double branched cover M of K is a Seifert fibered space over $\mathbf{S}^2$ with three exceptional fibers of type $(2,1)$, (α_2, β_2) and (α_3, β_3) where $\alpha_2\alpha_3$ is odd. Let e be the Euler number of M. If K admits a 2-twin K', then by arguments in the previous subsection on involutions of M it can be seen that K' must be a torus knot $T(p, q)$ for some p, q. (We may assume $p < q$.)

Assume that p and q are both odd. By Proposition 2 (1), putting $d = 1$, we have

$$\alpha_2 = p, \ \alpha_3 = q, \ \beta_2 \equiv kx \pmod{\alpha_2}, \ \beta_3 \equiv ky \pmod{\alpha_3}, \ e = 1/(2\alpha_2\alpha_3)$$

where x, y satisfy $y\alpha_2 + x\alpha_3 = -1$ and k satisfies $\alpha_2\alpha_3 = 1 - 2k$. By the first two equalities, α_2 and α_3 must be coprime. By the third equality and the conditions on x, y, k, we have

$$2\alpha_3\beta_2 \equiv 2\alpha_3 \cdot kx = 2k \cdot x\alpha_3 \equiv 1 \cdot (-1) = -1 \pmod{\alpha_2},$$

and similarly we obtain $2\alpha_2\beta_3 \equiv -1 \pmod{\alpha_3}$ by the fourth equality and the conditions on x, y, k.

Conversely, if the condition (2a-1) is satisfied, then it can be seen that K admits $T(\alpha_2, \alpha_3)$ as a 2-twin since K is hyperbolic when $(\alpha_2, \alpha_3) \neq (3, 5)$ in this case. When $(\alpha_2, \alpha_3) = (3, 5)$, it can be seen that K is a Montesinos knot with invariants of the form $(0; (2, 1), (3, -1), (5, -1))$ and is equivalent to $T(3, 5)$.

Assume that $p = 2k$ is even. Note that $q > p \geq 4$ and must be odd. By Proposition 2 (2), we have $k = 2$ and

$$\alpha_2 = \alpha_3 = q, \ \beta_2 = \beta_3 \equiv y \pmod{\alpha_2 = \alpha_3}, \ e = 1/(2\alpha_2),$$

where y satisfies $4y \equiv -1 \pmod{\alpha_2 = \alpha_3}$. Thus we obtain the condition (2a-2).

Conversely, if the condition (2a-2) is satisfied, then it can be seen that K admits a 2-twin $T(4, \alpha_2)$ since K is hyperbolic in this case.

Assume that $q = 2k$ is even. Note that p must be odd. By Proposition 2 (3), we have $k = 2$, that is $q = 4$, and

$$\alpha_2 = \alpha_3 = p = 3, \ \beta_2 = \beta_3 \equiv x \pmod{3}, \ e = 1/2p = 1/6,$$

where x satisfies $4x \equiv -1 \pmod 3$ and hence $x \equiv -1 \pmod 3$. In this case, K is a Montesinos knot with invariants of the form $(0; (2, 1), (3, -1), (3, -1))$ and is equivalent to $T(3, 4)$.

Assume that K is a knot of type (2b) in the Klimenko-Sakuma classification. Then the double branched cover M of K is a Seifert fibered space over $\mathbf{S}^2$ with three exceptional fibers of type $(3, \beta_1)$, $(3, \beta_1)$ and (α_3, β_3), where $\alpha_3 \geq 4$ is not divisible by 3, and the Euler number $e = \pm 1/(3\alpha_3)$. If K admits a 2-twin K', then again K' must be a torus knot $T(p, q)$ for some p, q. (We may assume $p < q$.) By comparing the Seifert invariants with those in Proposition 2, we can see that $p = 3$ and $q = 2k$ is even, and have

$$\beta_1 \equiv x \pmod 3, \ \alpha_3 = k, \ \beta_3 \equiv y \pmod{\alpha_3}, \ e = 1/(3k),$$

where x and y satisfy $3y + 2\alpha_3 x = -1$. The first and the third equalities together with the above condition on x and y imply $2\alpha_3\beta_1 \equiv -1 \pmod 3$, or equivalently $\alpha_3\beta_1 \equiv 1 \pmod 3$, and $3\beta_3 \equiv -1 \pmod{\alpha_3}$.

Conversely, if the condition (2b-1) is satisfied, then it can be seen that K admits a 2-twin $T(3, 2\alpha_3)$ since K is hyperbolic (because $\alpha_3 \geq 4$) in this case.

We also note that K admits $T(p, q)$ as a 2-twin if and only if the mirror of K admits the mirror of $T(p, q)$ as a 2-twin. $\qquad \square$

Proof of Corollary 1. Recall that the Euler number e is defined as $b - \sum_{i=1}^{r} \beta_i/\alpha_i$. Because of the value of e given in Proposition 2 it is easy to see that the rational invariants cannot be all of the same sign. $\qquad \square$

3.3 Satellite knots

Satellite tunnel number one knots were classified independently by Morimoto and Sakuma, and Eudave-Muñoz. It turns out that the JSJ decomposition of the exterior of such a knot is extremely simple and consists of just two pieces. These are the exteriors of a two-component 2-bridge link L (different from the trivial link and the Hopf link) and of a non trivial torus knot $T(p, q)$ in such a way that the fibre of the fibration of the torus knot is identified with the meridian of one of the two components of L.

It was proved by Schubert that the bridge number of these knots is the product of the bridge number of $T(p, q)$ and the wrapping number of K (see [Sc] for a proof). Since both these numbers are at least 2, we know that these knots have bridge index at least four. On the other hand, these knots are known to be (1,1) by [MS, Thorem 2.1]. These facts together with the last assertion of Theorem 1 imply that these knots are never determined by their double branched covers. In the rest of this section, we give an alternative proof of this conclusion and discuss more precisely what 2-twins these knots admit.

The following fact will be useful in the sequel.

Lemma 1. *Let $L = L_1 \cup L_2$ be a 2-bridge link of type $(2\alpha, \beta)$. Consider $M(L_1, 2)$ and let L' be the lift of L_2 to $M(L_1, 2)$. Then $M(L_1, 2)$ is the 3-sphere and L' is a 2-bridge knot or a 2-component 2-bridge link according to whether the linking number of L_1 and L_2 is odd or even. Moreover L' is hyperbolic if and only if $\beta \not\equiv \pm 1 \pmod{\alpha}$.*

Proof. The components of L are both knots of bridge index 1, so they are both trivial. It follows at once that $M(L_1, 2)$ is the 3-sphere. Since L admits a symmetry (a π-rotation) that exchanges its components, we also see that the lift of L_2 to $M(L_1, 2)$ coincides with the lift of L_1 to $M(L_2, 2)$. Considering now a four-plat position for L, it is straightforward to see that L' also admits a four-plat presentation (see Figure 3), so that L' is a 2-bridge link. Note that $\beta/2\alpha$ admits a continued fraction of the form

$$\frac{\beta}{2\alpha} = \cfrac{1}{2a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{2a_n}}}},$$

for some odd number n, and that the linking number of L_1 and L_2 is $a_1 + a_3 + \cdots + a_n$. One can see from Figure 3 that L' is a knot or a 2-component link
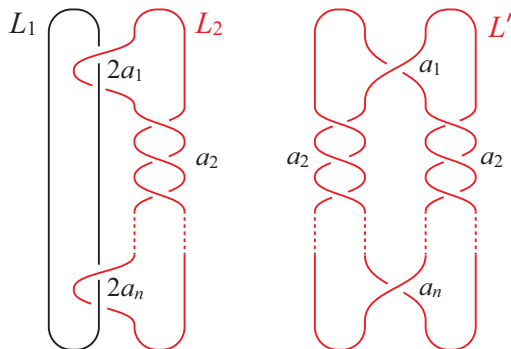
Figure 3: The lift L' of L_2 to $M(L_1, 2)$.

according to whether this linking number is odd or even. As seen in Figure 3, the invariant of L' is

$$\cfrac{1}{a_1 + \cfrac{1}{2a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}} = \frac{\beta}{\alpha}.$$

The last assertion of the lemma follows from the fact that a 2-bridge link of type (α, β) is hyperbolic if and only if $\beta \not\equiv \pm 1 \pmod{\alpha}$ (see [Me] and also [GF, Theorem A.1]). $\qquad\square$

Consider the double branched cover of K. This manifold can be obtained in the following way. First take the double cover of the exterior of a component of L branched along the other. According to the lemma above, this cover is the exterior of a 2-bridge link L' with one or two components according to the parity of the linking number of the components of L. If L' is a knot, we obtain the double branched cover of K by gluing to the exterior of L' the double (unramified) cover of the exterior of $T(p, q)$. Else, we glue on each boundary component of the exterior of L' a copy of the exterior of $T(p, q)$. It now suffices to observe that the exterior of L' admits an involution acting as a strong inversion on all components of L'; the double cover of the exterior of $T(p, q)$ admits a Montesinos type involution; $T(p, q)$ itself admits a strong inversion. In all cases, these involutions defined locally on the geometric pieces of the decomposition can be glued together to provide a global involution f such that $(M, Fix(f))/\langle f \rangle = (\mathbf{S}^3, K_f)$ where K_f is a Conway reducible (hyperbolic) knot. As such K_f is necessarily a 2-twin of K.

Observe that this shows that the 2-twin of a tunnel number one knot need not be a tunnel number one knot (the reader might have already remarked that this is also the case for some torus knots).

On the other hand, one can prove that these knots are determined by their double branched covers within the class of tunnel number one knots. Indeed, let M be the double branched cover of one of these knots and let g be the covering involution of a knot K' such that $M(K', 2)$ is diffeomorphic to M. We can assume that g preserves the JSJ decomposition of M. In particular, either $Fix(g)$ meets some torus of the JSJ decomposition so that the K' is Conway

reducible and cannot be a tunnel number one knot, or it does not meet them in which case K' is toroidal. It follows that if K' is a tunnel number one knot, it must belong to this family.

Now we want to show that the invariants determining K can be read off the double cover, so that K is determined within this class. Assume that L' is a knot, then L' is either hyperbolic or the exterior of a non trivial torus knot whose Seifert fibration has base a disc and two exceptional fibres of orders 2 and $2m + 1 > 2$. The second piece of the JSJ decomposition of M consists of a Seifert fibred manifold with base a disc and either two exceptional fibres both of odd orders (perhaps the same), or three exceptional fibres.

If L' is a link, the JSJ decomposition of M consists of three pieces, provided L' is not the Hopf link. In any case, two of the pieces of the decomposition are Seifert fibred with base a disc and two exceptional fibres of coprime orders.

Thus there is a single problematic situation, that is when L' is the Hopf link. In this case the JSJ decomposition of M consists of two copies of the same Seifert fibred piece, i.e. the exterior of $T(p, q)$. We have to make sure that this double cover cannot coincide with one of the double covers where L' is a knot. However, the two pieces of the JSJ decomposition of M in the case where L' is a knot can never be the same according to the above analysis.

4   Two families of π-hyperbolic knots

In this section we will consider some families of π-hyperbolic tunnel number one knots and show that generically they are not determined by their double branched covers.

4.1 $(1, 1)$-knots of small Hempel distance

Knots in 1-bridge position with respect to genus one Heegaard splittings with Hempel distance at most 2 were studied by Saito in [Sa]. If K is a hyperbolic knot in this class, then Saito shows that its exterior is obtained by Dehn filling a component of the exterior of a (necessarily hyperbolic) 2-bridge link L. Note that, since both components of L are trivial, infinitely many Dehn fillings give the exterior of a knot in $\mathbf{S}^3$; moreover, by Thurston's hyperbolic Dehn filling theorem, all but finitely many of these give a hyperbolic knot. As in the previous section, let L' be the link obtained by lifting one component of L to the double branched cover of the other. As we have seen in Lemma 1, L' is also hyperbolic for most choices of L: in this case, a sufficiently large surgery on a component of L will result in a π-hyperbolic knot K.

Proposition 3. *Let K be a π-hyperbolic knot constructed as above by Dehn surgery on some 2-bridge link L so that L' is hyperbolic. If K is obtained from a sufficiently large Dehn filling on a component of L, then K is not determined by its double branched cover.*

Proof. The double branched cover M of K is obtained by Dehn surgery on the components of the hyperbolic knot or link L'. If this surgery is sufficiently large, the core or cores of the surgery are the unique shortest geodesics of the hyperbolic manifold M. It follows that every hyperbolic isometry of M leaves the core or cores of the Dehn surgery invariant and thus induces a symmetry

of the 2-bridge knot or link L'; moreover different isometries induce different symmetries. Vice-versa, each symmetry of L' induces an isometry of M. Besides the covering involution s, M admits an involution r induced by a symmetry of L' acting as a strong inversion on each component of L'. It is easily seen that $(M, Fix(r))/\langle r \rangle = (\mathbf{S}^3, K_r)$. Since the actions of s and r on the shortest geodesics of M are different, these elements cannot be conjugate and we can conclude that K and K_r are not equivalent, as desired. $\qquad\square$

A similar argument to the one used in the proof of the proposition above, was used by Reni and Zimmermann to provide examples of π-hyperbolic knots with several 2-twins ([RZ], see also [VM]).

Remark 1. We remark that the Hempel distance of any (1,1)-splitting of K constructed as above is at most 4 regardless of the slope of the Dehn filling, which can be seen as follows: Let L_1 and L_2 be the two components of L, and $V_1 \cup V_2$ the genus-0 Heegaard splitting of $\mathbf{S}^3$ that gives the 2-bridge splitting of L. Let D_1 and D_2 be the essential discs in $V_1 - L$ and $V_2 - L$, respectively. Let $N(L_1)$ be a regular neighborhood of L_1 disjoint from L_2, and let $W_1 := V_1 \cup N(L_1)$ and W_2 the closure of $V_2 - N(L_1)$. Then $W_1 \cup W_2$ is a genus-1 Heegaard splitting of $\mathbf{S}^3$ which gives a (1,1)-splitting of L_2. Let D_0 be the co-core of the 1-handle $N(L_1) \cap V_2$. Then D_0 is an essential disc of $W_1 - L_2$ disjoint from $D_1 \cup D_2$, which implies that the Hempel distance of the above (1,1)-splitting of L_2 is at most 2. Let W_1' be the solid torus obtained from W_1 by applying a Dehn surgery on L_1. Then $W_1' \cup W_2$ gives a (1,1)-splitting of K. Note that D_1 and D_2 are essential discs in $W_1' - K$ and $W_2 - K$, respectively, since L_1 is disjoint from the discs, and that ∂D_0 is an essential loop on $\partial W_1' - K$ disjoint from $D_1 \cup D_2$. Hence, the Hempel distance of the above (1,1)-splitting of K is at most 2. This together with the main result of [T] implies that the Hempel distance of any other (1,1)-splitting of K is at most 4.

4.2 Twisted torus knots

Twisted torus knots were introduced by Dean in [D]. These knots depend on four parameters and are obtained by twisting s times r consecutive strands of a torus knot $T(p,q)$. As tunnel number one knots, they can be put in 0-bridge position with respect to a genus two Heegaard splitting of $\mathbf{S}^3$, by construction. Unlike tunnel number one knots, though, twisted torus knots can be composite or, more generally, admit non trivial n-tangle decompositions (see [M1, M2]).

We are interested in the twisted torus knots that are also tunnel number one knots. Infinitely many examples belong to this class: for instance all knots obtained by twisting along $r = 3$ strands are tunnel number one by work of Lee [L1]. Note that this condition is sufficient but not necessary (see [L2] or [BTZ, Lemma 5.4])

We are particularly interested in those examples that are moreover π-hyperbolic. Lee showed that these knots are hyperbolic provided $r > 1$ is not a multiple of p or q and s is sufficiently large. Indeed, a twisted torus knot can be seen as the result of $1/s$-Dehn surgery along a trivial knot C encircling r consecutive strands of a torus knot $T(p,q)$: Lee proved more precisely that $C \cup T(p,q)$ is a hyperbolic link [L2] under the aforementioned hypotheses.

Of course, these knots do not need to be π-hyperbolic, however they will be so if we can ensure that they are tunnel number one and of bridge index at least

4, thus excluding the Montesinos ones. It was proved in [BTZ] that the twisted torus knot of parameters $1 < r < p < q$ has bridge index precisely p provided that $|s| > 18p$. This ensures that knots of the type that we are interested in do exist.

Proposition 4. *Let $T(p,q)$ be a torus knot and C a trivial knot encircling r consecutive strands of $T(p,q)$. Assume that the link $C \cup T(p,q)$ is hyperbolic and moreover the lift $\tilde{C}$ of C to the double branched cover of $T(p,q)$ is also a hyperbolic knot or link. Then for sufficiently large s, the (π-hyperbolic) twisted torus knot $K = K(p,q;r,s)$ obtained by $1/s$-Dehn surgery along C is not determined by its double branched cover.*

Proof. The proof follows the exact same lines of the proof of Proposition 3. If the surgery is sufficiently large, we can assume that the core or cores of the induced surgery on $\tilde{C}$ are the shortest geodesic of the double cover. Now, the double branched cover of a torus knot admits a Montesinos involution that induces a strong inversion of $T(p,q)$. Such strong inversion can be chosen (up to isotopy) so that it acts as a strong inversion on C, too. It follows that the Montesinos involution induces an involution of the manifold obtained by surgery on $\tilde{C}$ which is a deck transformation for a double branched cover of a knot K'. Note that the covering involutions for K and K' do not act in the same way on the cores of the Dehn surgery on $\tilde{C}$ so they cannot be conjugate if the cores are the shortest geodesics. $\square$

The above proposition applies in particular to twisted torus knots that have tunnel number one. Observe that among these some have bridge index < 5 according to the discussion above.

5 Double branched covers of π-hyperbolic knots with Heegaard splittings of large Hempel distance

This section will be devoted to the proof of Theorem 2. Let K be a π-hyperbolic tunnel number one knot of bridge index $b \in \{3,4\}$. Let M denote its double branched cover: notice that M is a hyperbolic manifold.

5.1 The bridge index of K is 3

Under this hypothesis K is a $(1,1)$-knot, according to [K]. Consider a genus-one Heegaard splitting of the 3-sphere with respect to which K is in 1-bridge position. Such splitting lifts to a genus-two Heegaard splitting of M, so that $M = H_1 \cup_\Sigma H_2$, where H_1 and H_2 are genus-two handlebodies intersecting in their common boundary Σ, a Heegaard surface of genus 2. As in Section 2, let s be the involution generating the group of deck transformations of the branched cover, and let u be the lift to M of a strong inversion of K, preserving the genus-one Heegaard splitting of $\mathbf{S}^3$, chosen so that it acts as a hyperelliptic involution of Σ and of the splitting of M. We can assume that both s and u are isometries of the hyperbolic structure of M. As already remarked in Section 2, we have $(M, Fix(u))/\langle u \rangle = (\mathbf{S}^3, K_u)$. We want to show that K_u is a 2-twin of K. Let

us assume by contradiction that K_u is equivalent to K, so that s and u are conjugate in $Isom^+(M)$.

Let $h \in Isom^+(M)$ such that $u = h^{-1}sh$. Consider now $h(H_1) \cup_{h(\Sigma)} h(H_2)$: by construction it is a genus-two Heegaard splitting of M on which s acts as a hyperelliptic involution. Assume now that the Hempel distance of $H_1 \cup_\Sigma H_2$ and hence of $h(H_1) \cup_{h(\Sigma)} h(H_2)$ is at least $2g + 1 = 5$. The main result in [ST] assures that the two splittings we have are isotopic. Let f be a homeomorphism of M isotopic to the identity such that $f(h(H_i)) = H_i$, $i = 1, 2$ and consider the homeomorphism fsf^{-1}: it is a hyperelliptic involution of Σ and the splitting. Since a surface of genus two admits a unique hyperelliptic involution up to isotopy (see [FK] for instance) we deduce that fsf^{-1} and u are isotopic in M and, consequently, so are s and u. This is however absurd since u and s are distinct isometries of M. This final contradiction shows that K_u must be a 2-twin of K.

5.2 The bridge index of K is 4

The proof will follow the same lines of the previous case with slight modifications. Consider a genus-two Heegaard splitting of the exterior of K. As seen in Section 2, such splitting lifts to a genus-three Heegaard splitting of M, so that $M = H_1 \cup_\Sigma H_2$, where H_1 and H_2 are genus-three handlebodies intersecting in their common boundary Σ a Heegaard surface of genus 3. Note that if this is not a minimal genus splitting for M then K is not determined by M, according to Theorem 1. In this case there would be nothing to prove, so we can assume that the Heegaard genus of M is 3. As in Section 2, let s be the involution generating the group of deck transformations of the branched cover, and let u be the lift to M of a strong inversion of K, preserving the genus-two Heegaard splitting of the exterior of K. As remarked in Section 2, u can be chosen so that it acts as a hyperelliptic involution of Σ and of the splitting of M. We can assume that both s and u are isometries of the hyperbolic structure of M. As already remarked in Section 2, we have $(M, Fix(u))/\langle u \rangle = (\mathbf{S}^3, K_u)$. If K_u is a 2-twin of K, again there is nothing to prove, so we can assume that K_u is equivalent to K, so that s and u are conjugate in $Isom^+(M)$.

Let $h \in Isom^+(M)$ such that $u = h^{-1}sh$. Consider now $h(H_1) \cup_{h(\Sigma)} h(H_2)$: by construction it is a genus-three Heegaard splitting of M on which s acts as a hyperelliptic involution. Assume now that the Hempel distance of $H_1 \cup_\Sigma H_2$ and hence of $h(H_1) \cup_{h(\Sigma)} h(H_2)$ is at least $2g + 1 = 7$. The main result in [ST] assures that the two splittings we have are isotopic. Let f be a homeomorphism of M isotopic to the identity such that $f(h(H_i)) = H_i$, $i = 1, 2$ and consider the homeomorphism fsf^{-1}: it is a hyperelliptic involution of Σ and the splitting. Since the genus of Σ is 3, the hyperelliptic involution is not unique up to isotopy. However, since the Hempel distance of the splitting is at least 4, the mapping class group $Mod(M, \Sigma)$ of homeomorphisms of M preserving Σ up to isotopies is finite according to [J2]. Note that this group contains both u and fsf^{-1}. Now, since $Mod(M, \Sigma)$ is finite, it can be realised as a group of automorphisms of a complex structure on Σ: this is a consequence of the solution to Nielsen's realisation problem, see [K1, K2]. Such a group can contain at most one hyperelliptic element. It follows that $fsf^{-1} = u$, so that s and u are isotopic in M. This is however absurd since u and s are distinct isometries of M. This final contradiction shows that K_u must be a 2-twin of K.

Remark 2. We remark that the hypothesis in Theorem 2 is sufficient but not necessary. Indeed, let K be a knot as in Proposition 3 (or as in Remark 1), which is not determined by its double branched cover. Let $W_1' \cup W_2$ be the genus-1 Heegaard splitting of $\mathbf{S}^3$ that gives the (1,1)-splitting of K with Hempel distance at most 2 as in Remark 1, and let D_1, D_2 and D_0 be the discs in $W_1' - K$, $W_2 - K$ and $W_1 - L_2$, respectively, as in the remark. Denote by $\widetilde{W_1'}$ and $\widetilde{W_2}$ the pre-images of W_1' and W_2, respectively, in the double branched cover M of K. Then $\widetilde{W_1'} \cup \widetilde{W_2}$ is a genus-2 Heegaard splitting of M. Note that D_1 and D_2 lift to essential discs $\widetilde{D_1}$ and $\widetilde{D_2}$ in $\widetilde{W_1'}$ and $\widetilde{W_2}$, respectively, and ∂D_0 lifts to an essential loop on the Heegaard surface which is disjoint from $\widetilde{D_1} \cup \widetilde{D_2}$. Hence, the Hempel distance of $\widetilde{W_1'} \cup \widetilde{W_2}$ is at most 2.

References

[BH]   J. Birman and M. Hilden, *Heegaard splittings of branched coverings of S^3*, Trans. Amer. Math. Soc. **213**, (1975), 315-352.

[BM]   A.S. Bleiler and Y. Moriah, *Heegaard splittings and branched coverings of B^3*, Math. Ann. **281**, (1988), 531-543.

[BF]   M. Boileau and E. Flapan, *On π-hyperbolic knots which are determined by their 2-fold and 4-fold cyclic branched coverings*, Topology Appl. **61**, (1995), 229-240.

[BRZ]   M. Boileau, M. Rost and H. Zieschang, *On Heegaard decompositions of torus knot exteriors and related Seifert fibered spaces*, Math. Ann. **279**, (1988), 553-581.

[BZ]   M. Boileau and H. Zieschang, *Heegaard genus of closed orientable Seifert 3-manifolds*, Invent. Math. **76**, (1984), 455-468.

[BZ2]   M. Boileau and H. Zieschang, *Nombre de ponts et générateurs méridiens des entrelacs de Montesinos*, Comment. Math. Helv. **60** (1985), 270-279.

[BTZ]   R. S. Bowman, S. Taylor, and A. Zupan, *Bridge spectra of twisted torus knots*, Int. Math. Res. Not. IMRN **16**, (2015), 7336-7356.

[C]   J. H. Conway, *An enumeration of knots and links, and some of their algebraic properties*, Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967). Pergamon, Oxford, 1970, 329-358.

[D]   J. C. Dean, *Small Seifert-fibered Dehn surgery on hyperbolic knots*, Algebr. Geom. Topol. **3**, (2003), 435-472.

[E-M]   M. Eudave-Muñoz, *On non-simple 3-manifolds and 2-handle addition*, Topology Appl. **55**, (1994), 131-152.

[FK]   H.M. Farkas and I. Kra, *Riemann surfaces*, Graduate Texts in Mathematics **71**, Springer-Verlag, New York-Berlin, 1980.

[G]   J. E. Greene, *Lattices, graphs, and Conway mutation*, Invent. Math. **192**, (2013), 717-750.

[GR]   C. McA. Gordon and A. Reid, *Tangle decompositions of tunnel number one knots and links*, J. Knot Theory Ramifications **4**, (1995), 389-409.

[GF]   F. Gueritaud with appendix by D. Futer, *On canonical triangulations of once-punctured torus bundles and two-bridge link complements*, Geom. Topol. **10**, (2006), 1239-1284.

[HR]   C. Hodgson and J. H. Rubinstein, *Involutions and isotopies of lens spaces*, Knot theory and manifolds (Vancouver, 1983). Ed. D. Rolfsen. Lecture Notes in Math. **1144**, Springer-Verlag, Berlin, 1985, 60-96.

[J1]   J. Johnson, *Bridge number and the curve complex*, preprint (2006), arXiv:math/0603102.

[J2]   J. Johnson, *Mapping class groups of medium distance Heegaards splittings*, Proc. Amer. Math. Soc. **138**, (2010), 4529-4535.

[JT]   J. Johnson and A. Thompson, *On tunnel number one knots that are not (1,n)*, J. Knot Theory Ramifications **20**, (2011), 609-615.

[K1]   S. Kerckhoff, *The Nielsen realization problem*, Bull. Amer. Math. Soc. (N.S.) **2**, (1980), 452-454.

[K2]   S. Kerckhoff, *The Nielsen realization problem*, Ann. of Math. **117**, (1983), 235265.

[K]   S. H. Kim, *The tunnel number one knot with bridge number three is a $(1,1)$-knot*, Kyungpook Math. J. **45**, (2005), 67-71.

[KS]   E. Klimenko and M. Sakuma, *Two-generator discrete subgroups of $Isom(\mathbf{H}^2)$ containing orientation-reversing elements*, Geom. Dedicata **72**, (1998), 247-282.

[KT]   S. Kinoshita and H. Teresaka, *On unions of knots*, Osaka Math. J. **9**, (1957), 131-153.

[K3]   T. Kobayashi, *A criterion for detecting inequivalent tunnels for a knot*, Math. Proc. Camb. Phil. Soc. **107**, (1984), 483491.

[L]   M. Lackenby, *Classification of alternating knots with tunnel number one*, Comm. Anal. Geom. **13**, (2005), 151-185.

[L1]   J.H. Lee, *Twisted torus knots T(p,q,3,s) are tunnel number one*, J. Knot Theory Ramifications **20**, (2011), 807-811.

[L2]   S. Lee, *Twisted torus knots that are unknotted*, Int. Math. Res. Not. **2014**, (2014), 4958-4996.

[Me]   W. Menasco, *Closed incompressible surfaces in alternating knot and link complements*, Topology **23**, (1984), 37-44.

[M]   J. M. Montesinos, *Variedades de Seifert que son recubridores ciclicos ramificados de dos hojas*, Bol. Soc. Mat. Mexicana **18**, (1973), 1-32.

[MW]   J. M. Montesinos and W. Whitten, *Construction of two-fold branched covering spaces*, Pacific J. Math. **125**, (1986), 415-446.

[MB]   J. Morgan and H. Bass, *The Smith conjecture*, Academic Press, New York, 1984.

[M1]   K. Morimoto, *On composite twisted torus knots*, Tokyo J. Math. **35**, (2012), 499-503.

[M2]   K. Morimoto, *On tangle decompositions of twisted torus knots*, J. Knot Theory Ramifications **22**, (2013), 1-12.

[MS]   K. Morimoto and M. Sakuma, *On unknotting tunnels for knots*, Math. Ann. **289**, (1991), 143-167.

[Na]   Y. Nakanishi, *Primeness of links*, Math. Sem. Notes Kobe Univ. **9**, (1981), 415-440.

[N]   F. H. Norwood, *Every two generator knot is prime*, Proc. Amer. Math. Soc. **86**, (1982), 143-147.

[NR-L] V. Núñez and E. Ramírez-Losada, *The trefoil knot is as universal as it can be*, Topology Appl. **130**, (2003), 1-17.

[P]   L. Paoluzzi, *On hyperbolic type involutions*, Rend. Ist. Mat. Univ. Trieste **32**, suppl. 1, (2001), 221-256.

[RZ]   M. Reni and B. Zimmermann, *Isometry groups of hyperbolic 3-manifolds which are cyclic branched coverings*, Geom. Dedicata **74**, (1999), 23-35.

[S1]   M. Sakuma, *Periods of composite links*, Math. Sem. Notes Kobe Univ. **9**, (1981), 445-452.

[Sa]   T. Saito, *Genus one 1-bridge knots as viewed from the curve complex*, Osaka J. Math. **41**, (2004), 427-454.

[S]   M. Scharlemann, *Tunnel number one knots satisfy the Poenaru conjecture*, Topol. Appl. **18**, (1984), 235-258.

[ST]   M. Scharlemann and M. Tomova, *Alternate Heegaard genus bound distance*, Geom. Topol. **10**, (2006), 593-617.

[Sc]   J. Schultens, *Additivity of bridge number of knots*, Math. Proc. Cambridge Phil. Soc. **135**, (2003), 539-544.

[T]   M. Tomova, *Multiple bridge surfaces restrict knot distance*, Algebr. Geom. Topol. **7**, (2007), 957-1006.

[VM]   A. Yu. Vesnin and A. D. Mednykh, *Three-dimensional hyperbolic manifolds of small volume with three hyperelliptic involutions*, Sib. Math. J. **40**, (1999), 873-886.

[V]   O. J. Viro, *Links, two-sheeted branching coverings and braids*, Mat. Sb. **87**, (1972), 216-228.

[W]   F. Waldhausen, *Über Involutionen der 3-Sphäre*, Topology **8**, (1969), 81-91.

DEPARTMENT OF MATHEMATICS, NARA WOMEN'S UNIVERSITY, KITAUOYA NISHIMACHI, NARA 630-8506, JAPAN

yeonheejang@cc.nara-wu.ac.jp

AIX-MARSEILLE UNIV, CNRS, CENTRALE MARSEILLE, I2M, UMR 7373, 13453 MARSEILLE, FRANCE

luisa.paoluzzi@univ-amu.fr